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- da sem doktorsko disertacijo z naslovom *Fatoujeve komponente* izdelal samostojno pod mentorstvom akad. prof. dr. Franca Forstneriča in
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ABSTRACT

In this thesis we address some problems in complex dynamics and classical complex analysis of several variables.

Chapter I provides a historical background of the field of complex dynamics. Main results regarding the dynamics of complex rational functions are discussed and some motivation for generalizing this theory to higher dimensions is given.

In Chapter II we study the regularity of Fatou components for holomorphic endomorphisms of $\mathbb{P}^k$. We show that for $k = 1$ all Fatou components are regular and that this is not true in general. Next we study the difference between the Julia set and the support of the equilibrium measure. We prove that either they coincide or else the support of the equilibrium measure is nowhere dense in the Julia set. It follows from our results that if this two sets coincide, then all Fatou components are regular. We give an example of a bounded Stein domain in $\mathbb{C}^2$ whose regularization is not Stein.

Chapter III is a joint work with Han Peters and John Erik Fornæss [8]. We study invariant Fatou components for holomorphic endomorphisms in $\mathbb{P}^2$. In the recurrent case these components were classified by Fornæss and Sibony [27]. In 2008 Ueda [63] completed this classification by proving that it is not possible for the limit set to be a punctured disk. Recently Lyubich and Peters [47] classified non-recurrent invariant Fatou components, under the additional hypothesis that the limit set is unique. Again all possibilities in this classification were known to occur, except for the punctured disk. Here we show that the punctured disk can indeed occur as the limit set of a non-

recurrent Fatou component. We provide many additional examples of holomorphic and polynomial endomorphisms of $\mathbb{C}^2$ with non-recurrent Fatou components on which the orbits converge to the regular part of arbitrary analytic sets.

In Chapter IV we focus on the complex manifolds which can be exhausted by copies of $\mathbb{C}^n$, and are therefore called long $\mathbb{C}^n$'s. This manifolds have a connection to an old "union problem" which was solved in 1976 by J.E. Fornæss [22]. In 2010 Wold [68] has constructed a non-Stein long $\mathbb{C}^n$ but the question was left open whether there are more then just one non-Stein long $\mathbb{C}^n$ and does there exist one without any non-constant holomorphic functions. We answer to this questions affirmatively and we provide some more results. It is still a wide open problem whether or not $\mathbb{C}^n$ is the only Stein long $\mathbb{C}^n$. The main results were obtained in the conversation with Franc Forstnerič.

Recently Takens' Reconstruction Theorem was studied in the complex analytic setting by Fornæss and Peters [24]. They studied the real orbits of complex polynomials, and proved that for non-exceptional polynomials ergodic properties such as measure theoretic entropy are carried over to the real orbits mapping. In Chapter V we show that their results also holds for exceptional polynomials, unless the Julia set is entirely contained in an invariant vertical line, in which case the entropy is 0.

In [60] Takens proved a reconstruction theorem for endomorphisms. In this case the reconstruction map is not necessarily an embedding, but the information of the reconstruction map is sufficient to recover the $2m+1$-st image of the original map. Our main result shows an analogous statement for the iteration of generic complex polynomials and the projection onto the real axis.

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Keywords: Holomorphic function, Stein manifold, long $\mathbb{C}^n$, complex dynamics, polynomials, entropy, several complex variables Fatou–Bieberbach domain, limit sets.

POVZETEK

V disertaciji obravnavamo probleme iz kompleksne dinamike ter analize več kompleksnih spremenljivk.

V uvodnem poglavju opišemo razvoj področja kompleksne dinamike skozi zgodovino in predstavmo pomembne rezultate s področja dinamike kompleksnih racionanih funkcij. Podamo tudi motivacijo za posplošitev te teorije v višje dimenzije.

V drugem poglavju obravnavamo regularnost Fatoujevih komponent holomorfnih endomorfizmov $\mathbb{P}^k$. Pokažemo, da so v primeru $k = 1$ vse Fatoujeve komponente regularne ter da to v splošnem ne velja. V nadaljevanju primerjamo Juliajevo množico in nosilec Greenove ravnotežne mere. Dokažemo, da se nosilec Greenove ravnotežne mere ujema z Juliajevo množico natanko tedaj, kadar se ujemata na preseku z neko odprto množico. Iz dobljenih rezultatov lahko sklepamo, da so Fatoujeve komponente regularne, kadar sta si ti dve množici enaki. Podamo tudi primer omejene Steinove domene v $\mathbb{C}^2$, katere regularizacija ni več Steinova domena.

V tretjem poglavju je predstavljeno delo, ki je nastalo v sodelovanju s H. Petersom in J. E. Fornæssom [8]. V njem obravnavamo invariantne Fatoujeve komponente holomorfnih endomorfizmov $\mathbb{P}^2$. Povratne Fatoujeve komponente sta klasificirala Fornæss in Sibony [27]. Ueda [63] je dokazal, da punktiran disk ne more biti limitna množica in s tem tudi zaključil omenjeno klasificijo. Nedavno sta Lyubich in Peters [47] klasificirala nepovratne invariantne Fatoujeve komponente, pod dodatno predpostavko, da je limitna množica enolična. Tako kot pri povratnih komponentah, so bili tudi v tej

klasifikaciji znani vsi primeri razen punktiranega diska. V tem poglavju skonstruiramo preslikavo, katere limitna množica nepovratne invariantne Fatoujeve komponente je punktiran disk. S tem rezultatom tudi zaključimo klasifikacijo nepovratnih invariantnih Fatoujevih komponent v $\mathbb{P}^2$. V nadaljevanju podamo več primerov endomorfizmov $\mathbb{C}^2$ z nepovratnimi Fatoujevimi komponentami, na katerih orbite konvergirajo proti regularnemu delu poljubne analitične množice.

V četrtem poglavju obravnavamo kompleksne mnogoterosti, ki jih je mogoče izčrpati s kopijami $\mathbb{C}^n$. Take mnogotersoti imenujemo dolgi $\mathbb{C}^n$ in so povezani s starim problemom unije Steinovih domen - ang. "union problem". Slednjega je ovrgel J.E. Fornæss [22], ki je skonstruiral zaporedje krogel, katerega unija ni Steinova mnogoterost. To idejo je kasneje uporabil Wold [68], ki je s pomočjo novih tehnik skonstruiral ne-Steinov Dolgi $\mathbb{C}^2$. Iz dosedanjih rezultatov ni znano ali obstaja več različnih ne-Steinovih Dolgih $\mathbb{C}^n$ in ali morda obsajajo taki, ki nimajo nobene nekonstantne holomorfne funcije. V tem poglavju pozitivno odgovorimo na ta in še nekatera druga sorodna vprašanja. Predstavljeni rezultati so plod pogovorov s prof. Forstneričem.

Fornæss in Peters [24] sta v svojem delu preučevala Takensov rekonstrukcijski izrek za realne orbite kompleksnih polinomov. Dokazala sta, da lahko entropijo mere skoraj vseh polinomov, razberemo že iz njihovih realnih orbit. V zadnjem poglavju dokažemo, da njun rezultat velja za vsak polinom, katerega Juliajeva množica ni vsebovana v invariantni navpični premici. Kadar pa je Juliajeva množica vsebovana v invariantni navpični premici, je entropija realnih orbit enaka 0.

Glavni rezultat tega poglavja je poseben primer Takensovega rekonstrukcijskega izreka za endomorfizme [60], za primer generičnih polinomov in projekcije na realno os. Slednji nam pove, da čeprav rekonstrukcijska preslikava ni injektivna, vseeno vsebuje dovolj informacij, da lahko obnovimo $2m + 1$ sliko prvotne preslikave.

Daljši povzetek v slovenskem jeziku najdemo na koncu disertacije.

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Ključne besede: Holomorfne funkcije, Steinove mnogoterosti, dolg $\mathbb{C}^n$, kompleksna dinamika, polinomi, entropija, analiza več kompleksnih spremenljivk, Fatou–Bieberbachove domene, limitne množice.

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CHAPTER I

An Introduction to Dynamics

Complex dynamics is a field of mathematics where we study a dynamical system defined by iterates of a holomorphic self-map of a complex space. Its origins stretch back into the late 19^{th} century when the German mathematician Ernst Schröder (1841-1902) applied Newton's method to study complex roots of holomorphic polynomials in one complex variable (see Figure 1.1). He discovered the phenomenon of attracting fixed point which led him to generalize Newton's method and create a new family of root-solving algorithms.

The foundation of the contemporary study was established by the pioneering work of French mathematicians Pierre Fatou (1878-1929) and Gaston Julia (1893-1978). They studied iterates of analytic functions, in particular polynomials and rational functions and discovered a dichotomy of Riemann sphere into two dynamically diverse sets which now bear their names.

Up to the 1920's, Fatou and Julia continued to explore and expand complex dynamics, but as most open questions were successfully addressed, developments slowed. A few isolated papers have appeared in the 1940's by Carl Siegel, in the 1950's by Ivine Noel Barker and in the 1960's by Hans Brolin. The field of complex dynamics had a dramatic turn around in the 1980's when the Mandelbrot set was discovered and the computer graphics revealed beautiful fractals of the Julia set.

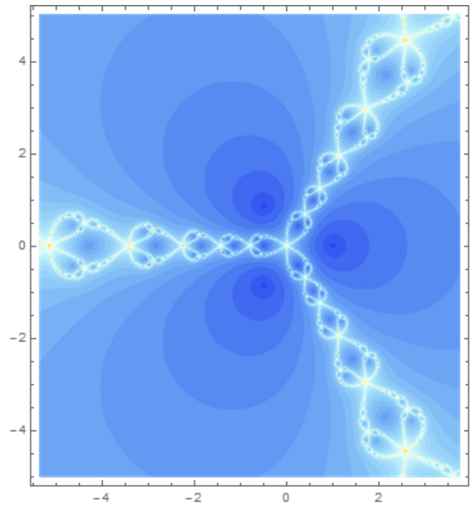

Figure 1.1: Applying Newton's method to a polynomial $P(z)$ and an initial value z_0, we obtain a sequence of values defined as $z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}$. One of the main problems is to describe those initial values for which the sequence z_n will converge to a root of P. The figure shows a dichotomy between "good" inital values (blue area) and "bad" inital values (white area) for the polynomial $P(z) = z^3 - 1$.

Afterwards many Fields medalists such as John Milnor, William Thurston, Jean-Pierre Yoccoz and Curt McMullen, as well as other eminent mathematicians entered this field. Thanks to them the dynamics of nonlinear rational functions is today very well understood.

Let $\mathbb{P}^1$ be the complex projective space (Riemann sphere) and

$$R : \mathbb{P}^1 \to \mathbb{P}^1$$

a holomorphic endomorphism, a rational function. We will denote the n^{th} iterate of R simply by

$$R^n = \overbrace{R \circ \ldots \circ R}^{n}$$

The *Fatou set* of R is the largest open set $\mathcal{F} \subset \mathbb{P}^1$ on which the family of iterates $\{R^n\}_n$ is normal (in the sense of Montel). Its complement is the *Julia set* and we denote it by $\mathcal{J}$.

It is easy to see that both sets are totally invariant i.e. $R^{-1}(\mathcal{F}) = \mathcal{F} = R(\mathcal{F})$. The dynamics on the Fatou set is *stable* and it is *chaotic* on the Julia set.

We can observe our dynamical system as two separate systems, one on the Fatou set and the other one on the Julia set, and study each one separately. Questions that could be asked regarding dynamics on these two sets are for example:

(i) **Analysis:** Find a local change of coordinates around a fixed point that conjugates R into an easy-to-describe model.

(ii) **Topology:** Connectedness, self-similarity, Hausdorff dimension, smoothness, classification of connected components.

(iii) **Measure theory:** Are there any R-preserving measures? Describe ergodic properties of such measures.

Most of these questions were successfully addressed and the reader is referred to [50] for in-depth exposition on dynamics of rational functions.

A point $z \in \mathbb{P}$ is called *periodic* if $R^n(z) = z$ for some $n \in \mathbb{N}$, moreover when $n = 1$ it is called a *fixed point.* A fixed point z is called *attractive* if $|R'(z)| < 1$, *repelling* if $|R'(z)| > 1$ and *neutral* if $|R'(z)| = 1$.

We know that for nonlinear rational functions the Julia set is a non-empty set and is either the whole $\mathbb{P}^1$ or else has an empty interior. The Julia set is either connected or else has uncountably many connected components. It is known that repelling periodic points are dense in $\mathcal{J}$ and that the forward orbit of a generic point from the Julia set is dense in it. For a nonlinear rational function R of degree d the sequence of measures

$$\mu_n = \frac{1}{d^n} \sum_{\substack{R^n(z)=z \\ z \quad \text{repelling}}} \delta_z,$$

converges weakly to the equilibrium measure μ. This measure is an R-invariant ergodic measure of maximal entropy, and it is supported on the Julia set.

A *Fatou component* is a connected component of the Fatou set. A Fatou component Ω is *invariant* if $R(\Omega) = \Omega$ and a Fatou component is called *periodic* if $R^n(\Omega) = \Omega$ for some $n \in \mathbb{N}$. A Fatou component Ω is *pre-periodic* if $R^m(\Omega)$ is periodic for some $m \in \mathbb{N}$ otherwise it is called a *wandering component.*

Invariant Fatou components were already classified by Fatou.

Theorem I.1. *(Fatou) Let Ω be an invariant Fatou component of a nonlinear rational function $R(z)$. Then we have the following possibilities:*

(i) **Attracting basin***: Ω contains an attracting fixed point z and $R^n(w) \to z$ as $n \to \infty$ for all $w \in \Omega$.*

(ii) **Parabolic basin***: There is a fixed point $z \in b\Omega$ and $R^n(w) \to z$ as $n \to \infty$ for all $w \in \Omega$.*

(iii) **Rotation domain***: Ω is conformally equivalent to a disc or an annulus and $R(z)$ is conformally conjugate to an irrational rotation.*

Fatou had examples of attracting and parabolic domains, but the existence of rotation domains was unknown at the time. The existence of rotation domains equivalent to a disk was later shown by Carl L. Siegel, and the existence of rotation domains equivalent to an annulus was shown by Michael Herman in 1979.

Nowadays we have a precise description of Fatou components for rational functions on the Riemann sphere, thanks to the following remarkable result

Theorem I.2. *(Sullivan '85) Every Fatou component of a nonlinear rational function is pre-periodic.*

An important observation is that many dynamical properties of rational functions are determined by orbits of *critical points*, i.e. $\{\{R^n(z)\}_n \mid R'(z) = 0\}$. For example, every attracting or parabolic cycle contains a critical point, hence there are only finitely many non-repelling cycles. Recall that a Herman ring and a Siegel disk are Fatou components which are conformally equivalent to an annulus and a disc respectively and $R(z)$ is conformally conjugate to an irrational rotation. The boundary of such Fatou components is contained in the closure of the orbits of the singular values. If every critical point is pre-periodic, then the Fatou set is empty. If all critical points are contained in one attracting basin, then the Julia set is a Cantor set.

The density of hyperbolicity is one of the last important conjectures in the dynamics of rational functions. It states that the set of hyperbolic rational functions of degree d (those whose critical points tend to the attracting periodic cycles under iteration) is open and dense in the space of all rational functions of degree d. Even a simple family of functions $P_c(z) = z^2 + c$ exhibits a full spectrum of dynamical behavior, reflecting many of difficulties of the general case. Moreover, we do not know if the set of $c \in \mathbb{C}$ for which P_c is hyperbolic forms an open and dense subset of $\mathbb{C}$.

Let us recall that the study of dynamics of rational functions has evolved from the study of Newton's method for finding roots of a complex polynomial. In general we are dealing with a problem of finding a solution to a system of equations which depend on more then just one variable. At the end of the 19'th century Leopold Leau was among the first who realized that Schröder's ideas and methods could be generalized and used for finding zeros of polynomial maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ (most of his studies were done in dimension 2). This was carried out by Samuel Lattes and Pierre Fatou, but at that time they were only able to study the local behaviour around an attracting fixed point. The global study of dynamics of rational functions only became possible after Montel's theorem was introduced around 1912. Similarly, the global study of iterations of rational maps only became possible after Kobayashi hyperbolicity was introduced and the generalized Montel's theorem was given by Cartan. Cartan's theorem states the following: *Let* Ω *be a domain in* $\mathbb{P}^k$ *and let* $\mathcal{G}$ *be a family of holomorphic maps from* Ω *to* $\mathbb{P}^k$*. If* $\bigcup_{f\in\mathcal{G}} f(\Omega)$ *omits* $2k+1$ *hyperplanes in general position, then* $\mathcal{G}$ *is a normal family.*

In order to be able to understand iterations of general maps, we first need to get a better understanding of the "simplest" maps which are dynamically interesting.

A polynomial skew product is a map of form

$$(z, w) \to (P(z), Q(z, w)),$$

where P and Q are complex polynomials. These maps were studied extensively by Jonsson [43] and they are very useful for producing examples of holomorphic endomorphisms of $\mathbb{P}^2$ with desired properties. This is used in Chapter 3 where we construct a holomorphic endomorphism of $\mathbb{P}^2$ with a limit set equal to a punctured disk, which completes the classification of non-wandering non-recurrent Fatou componets. Let us mention that Astorg, Buff, Dujardin, Peters and Raissy [4] recently constructed a polynomial skew product (which can be lifted to a holomorphic endomorphisms of $\mathbb{P}^2$) having a wandering Fatou component, something that is not possible in $\mathbb{P}^1$ due to Sullivan's non-wandering theorem. This shows that even "simple" maps in higher dimensions may exhibit a new dynamical behavior. Another obstacle is the lack of tools, since most of the techniques used in study of the dynamics of rational functions do not extended to higher dimensions. The following chapters will provide more background on the dynamics of holomorphic endomorphisms of $\mathbb{P}^k$.

Another motivation for generalizing this theory to several variables lies in the fact that the concept of a dynamical system has its origins in classical physics, where it describes how our observed quantities evolve over time under some fixed rules.

A famous example is the so called Lorentz system which describes fluid circulation in a shallow layer of fluid, heated uniformly from below and cooled uniformly from above. The Lorentz attractor is an attractor of Lorentz system where the dynamics is chaotic and therefore hard to understand. In the 1970's French mathematician and astronomer Michel Henon introduced a simplified model of the Poincare section of the Lorenz attractor, namely he studied the chaotic behaviour of points under iteration of a map

$$(x, y) \to (1 - ax^2 + y, bx).$$

Maps of this type are today known as Henon maps:

$$(z, w) \to (P(z) - \gamma w, z),$$

where $P(z)$ is nonlinear polynomial and $\gamma \neq 0$. These maps are polynomial automorphisms of $\mathbb{C}^2$ and were studied extensively by many authors, for example, S. Friedland, J. Milnor, E. Bedford, J. Smille, M. Lyubich, J. Hubbard, J.E. Fornæss, H. Peters etc. Rosay and Rudin [54] proved that the Fatou component of any holomorphic automorphism of $\mathbb{C}^k$ $(k > 1)$, which is the basin of an attracting fixed point (all eigenvalues have modulus less than one), is biholomorphic to the complex Euclidean space $\mathbb{C}^k$. If our automorphism has more than one fixed point, then we obtain a proper subset of $\mathbb{C}^k$ which is biholomorphic to $\mathbb{C}^k$, a new phenomenon which can not be obtained in dimension one due to Picard's theorem or the Riemann's mapping theorem. Such domains are then called Fatou-Bieberbach domains since Fatou and Bieberbach were the first to discover this phenomena. With the rise of Andersén-Lempert theory [2] we got new powerful tools which enabled us to construct many more interesting examples of Fatou-Bieberbach domains. This allows us to apply them to the various problems in complex analysis such as the "union problem" (see Chapter IV) or problem of constructing various proper embeddings of Riemann surfaces into $\mathbb{C}^2$, see [32].

CHAPTER II

Regularity of Fatou components

2.1 Introduction

The complex projective space $\mathbb{P}^k$ is a compact complex manifold of dimension k. It obtained as a quotient of $\mathbb{C}^{k+1}\backslash\{0\}$ by the natural multiplicative action of $\mathbb{C}^*$, i.e. we identify all points which lie on the same complex line passing through the origin. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a *holomorphic endomorphism of degree d*. Such a map is induced by a polynomial self-map $F = (F_0, \ldots, F_k)$ on $\mathbb{C}^{k+1}$ via the natural projection $\pi : \mathbb{C}^{k+1}\backslash\{0\} \to \mathbb{P}^k$, where F_j are homogeneous polynomials of the same degree d and $F^{-1}(0) = 0$. Note that F is unique up to a multiplicative constant and that we have $f \circ \pi = \pi \circ F$. Recall that holomorphic endomorphisms of $\mathbb{P}^1$ are exactly rational functions of $\mathbb{C}$. The interested reader is referred to [26, 28, 55, 17] for in depth survey on the dynamics of holomorphic endomorphisms and for proofs of some of the following statements.

By the Bezout theorem the number of points in a fiber of f, counted with multiplicity, is d^k. It follows that f has exactly $\frac{d^{k+1}-1}{d-1}$ fixed points, counted with multiplicities. An important property of holomorphic endomorphisms of $\mathbb{P}^k$ is that they are algebraically stable, i.e. the degree of f^n, the n^{th} iterate of f, is d^n.

Here we study some properties of a dynamical system given by the family of iterates $\{f^n\}_n$ of a holomorphic endomorphism f. As we have seen in the previous chapter, the dynamics of nonlinear rational functions is today well understood. However this is not the case for holomorphic endomorphisms of $\mathbb{P}^k$ when $k > 1$. One reason for that is the lack of tools, similar to those used in the study of rational functions. The other one is the existence of new dynamical phenomena such as wandering domains [4]. This shows that Sullivan's Non-wandering theorem for rational functions has no higher dimensional analogue.

From now on we always assume that a holomorphic endomorphism f of $\mathbb{P}^k$ is non-invertible, i.e. the degree of f is at least 2. Dynamics of an invertible map is simple to study since it takes the form $f([z]) = [Az]$, where A is an invertible $k \times k$ complex matrix.

Definition II.1. The *Fatou* set of f is the largest open set $\mathcal{F}$ of $\mathbb{P}^k$ where the sequence of iterates $\{f^n\}_{n\geq 1}$ is locally equicontinuous. A *Fatou component* is a connected component of the Fatou set. The complement $\mathcal{J}$ of $\mathcal{F}$ is called the *Julia set* of f.

Let U be an open subset of the Fatou set. A sequence of iterates f^{n_j} converges uniformly on compact subsets of U if and only if the corresponding sequence of iterates f^{n_j+1} converges uniformly on compact subsets of the open set $f^{-1}(U)$. It follows that the Fatou set is backward invariant, i.e. $\mathcal{F} = f^{-1}(\mathcal{F})$. Similarly we get forward invariance of the Fatou set, i.e. $f(\mathcal{F}) = \mathcal{F}$. Since the Julia set is the complement of the Fatou set, it is also forward and backward invariant. Ueda [62] proved that every Fatou component is:

1. *Stein* - it carries a strongly plurisubharmonic exhaustion function φ, i.e. $i\partial\bar{\partial}\varphi > 0$ and $\{\varphi(z) \leq c\}$ is holomorphically convex for all $c \in \mathbb{R}$.

2. *Kobayashi hyperbolic* - loosely speaking this means that the Fatou component contains no non-constant entire holomorphic curves.

We say that a dynamical system is *stable* at the point $x \in \mathbb{P}^k$ if and only if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that $dist_{\mathbb{P}^k}(x, y) < \delta$ implies $dist_{\mathbb{P}^k}(f^n(x), f^n(y)) < \varepsilon$

for all $n > 0$. Since the Fatou set was defined in terms of local equicontinuity, it follows immediately that the dynamics on $\mathcal{F}$ is stable and that it is unstable on $\mathcal{J}$. For example, attractive fixed points and their basins are contained in the Fatou set and repelling periodic points are always in the Julia set.

When $k > 1$ it can happen that there is a nonempty set $X \subset \mathcal{J}$, such that through any point $x \in X$ we can find a holomorphic disk $\Delta_x \subset \mathbb{P}^n$ called the Fatou disk on which the family of iterates $\{f^n_{\Delta_x}\}_n$ is normal [28, 20]. Observe that in dimension one there are no Fatou disks, since the dynamics on $\mathcal{J}$ is unstable. This suggests that, in $\mathbb{P}^n$, the right dynamical analogue of the Julia set of rational functions should contain only those points $x \in \mathcal{J}$, which are not the center of any Fatou disk. Such set can be defined using a pluri-potential methods and the reader is referred to [17] for proofs of following statements.

Let X be a complex manifold of dimension n and let $\mathbb{C}T^*X = T^{*1,0}X \oplus T^{*0,1}X$ be the complexified cotangent bundle of X. The (p, q)-*form* on X is a smooth section of $(\wedge^p T^{*1,0}X) \wedge (\wedge^p T^{*0,1}X)$. In local holomorphic coordinates $z = (z_1, \dots, z_n)$ any smooth (p, q)-form $\phi(z)$ can be written as

$$\phi(z) = \sum_{\substack{|I|=p \\ |J|=q}} \phi_{IJ} dz_I \wedge d\bar{z}_J,$$

where ϕ_{IJ} are smooth complex-valued functions, $dz_I := dz_{i_1} \wedge \dots dz_{i_p}$ if $I = (i_1, \dots, i_p)$, $d\bar{z}_J := d\bar{z}_{j_1} \wedge \dots d\bar{z}_{j_q}$ if $J = (j_1, \dots, j_q)$ and $i_k, j_k \in \{1, \dots, n\}$ for all k. Let $\mathcal{E}_{p,q}$ denote the space of smooth (p, q)-forms on X and let $\mathcal{E}^*_{p,q}$ denote the space of all continuous linear forms on $\mathcal{E}_{p,q}$. A (p, q)-*current* is a continuous map form X to $\mathcal{E}^*_{n-p,n-q}$ which has a compact support. In local holomorphic coordinates a (p, q)-current can be represented as (p, q)-form with distribution coefficients.

Let d be the degree of f and ω the standard Fubini-Study form on $\mathbb{P}^k$, i.e $\pi^*(\omega) = i\partial\bar{\partial} \log ||z||^2$. Let f^* and f_* denote, respectively, the pullback and the pushforward induced by f. The sequence of positive closed $(1,1)$-currents $d^{-n}(f^n)^*(\omega)$ converges weakly to a positive closed $(1,1)$-current T of mass $\int_{\mathbb{P}^k} T \wedge \omega^{k-1} = 1$ called a Green current. This current is totally invariant, i.e. $f^*(T) = dT$ and $f_*(T) = d^{k-1}T$. By proving that $d^{-n}(f^n)^*(\omega)$ converges to 0 whenever $\{f^n\}$ is equicontinuous one can see that T is supported on the Julia set $\mathcal{J}$. Less trivial is the converse statement, that the Julia set $\mathcal{J}$ is contained in the support of T. For the proof of this statement we refer an interested reader to [17, Theorem 1.2.7]. By taking a wedge product of k copies of T (in sense of Bedford-Taylor) we obtain a Green measure $\mu := T \wedge \ldots \wedge T$, also called an equilibrium measure. The measure μ is a probability measure and it is totally invariant, i.e. $f^*(\mu) = d^k\mu$ and $f_*(\mu) = \mu$. Moreover, it is the measure of maximal entropy $k \log d$ (see Section 5.2 for the definition of entropy).

Theorem II.2. *(Briend-Duval [13]) The sequence of measures*

$$\mu_n = d^{-kn} \sum_{\substack{f^n(z)=z \\ z \quad \text{repelling}}} \delta_z$$

converges weakly to μ.

We define the *small Julia set* $\mathcal{J}_k$ as the support of the equilibrium measure μ, hence it is never an empty set. Since measure μ is totally invariant under f, the same holds for $\mathcal{J}_k$. Theorem II.2 assures us that repelling periodic points are dense in $\mathcal{J}_k$. One can see that $\mathcal{J}_1 = \mathcal{J}$ but in general we may expect $\mathcal{J}_k$ to be a proper subset of the Julia set.

Example II.3. Let us compute these sets for the map

$$[z : w : t] \xrightarrow{f} \left[z^d, w^d, t^d\right].$$

In the local chart $\{t \neq 0\}$, this map takes the form $(x, y) \to (x^d, y^d)$, where $x = z/t$ and $y = w/t$. Observe that in $\mathbb{C}$ the Julia set of the map $z \to z^d$ is equal to a circle since orbits of all other points tend to the origin or to the infinity. The unstable set of $(x, y) \to (x^d, y^d)$ is equal to $\mathbb{C} \times b\Delta \cup b\Delta \cup \mathbb{C}$. It follows that $\mathcal{J}$ contains all those points of the unstable set which have bounded orbit, i.e. points that satisfy $\{|x| \le |y| = 1\} \cup \{|y| \le |x| = 1\}$. Observe that this condition translates to $\{|z| = |t| \ge |w|\} \cup \{|t| = |w| \ge |z|\}$. To see what is happening with points $(x, y) \in \{|x| > |y| = 1\} \cup \{|y| > |x| = 1\}$ which have an unbounded orbit we need to look at the other two local charts. The set $\{|x| > |y| = 1\}$ translates to $\{|t| = |w| < |z|\}$ therefore if we write this set the local chart $\{z \ne 0\}$ we get $\{|u| = |v| < 1\}$, where $u = w/z$ and $v = t/z$. It follows that this set is contained in the Fatou set. Using same arguments for the remaining local charts we finally obtain

$$\mathcal{J} = \{|z| = |w| \ge |t|\} \cup \{|z| = |t| \ge |w|\} \cup \{|t| = |w| \ge |z|\}.$$

In order to compute $\mathcal{J}_2$ we need to find all repelling periodic points. In the local chart $\{t \ne 0\}$, periodic points of n-th iterate of f have to satisfy the following two equations: $x^{d^n} = x$ and $y^{d^n} = y$. We know that the origin is an attracting fixed point therefore all the remaining solutions are pairs (x, y), where x and y are $d^n - 1$-th roots of unity. If $(x, y) \ne (0, 0)$ is a periodic point of period n then the eigenvalues of the Jacobian matrix $Jf^n(x, y)$ are equal to d^n, hence (x, y) is a repelling periodic point of f. We know that roots of unity form a dense subset of the circle therefore by Theorem II.2

$$\mathcal{J}_2 = \{|z| = |w| = |t|\}.$$

Let f be a holomophic endomorphism of $\mathbb{P}^k$ and let $\mathcal{P}_f$ denote the closure of all repelling periodic points of f. Recall that for $k = 1$ the Julia set $\mathcal{J}$ is equal to $\mathcal{P}_f$. In general we only have the following

$$\mathcal{J}_k \subseteq \mathcal{P}_f \subseteq \mathcal{J}.$$

It turns out that the small Julia set $\mathcal{J}_k$ is better analogue of the classical Julia set from $\mathbb{P}^1$ than the closure of repelling periodic points, which may have isolated points [41].

Let us prove that $\mathcal{J}$ is always connected. Let T be the Green current and suppose that the support of T is not connected. We can write a positive closed $(1,1)$-current T a sum of two positive closed $(1,1)$-currents S_1 and S_2 with disjoint supports. Using convolution on this each of these two positive closed $(1,1)$-currents we can construct two smooth positive closed $(1,1)$-forms U_1 and U_2 with disjoint supports and hence $U_1 \wedge U_2 \equiv 0$. When $0 \leq p \leq k$, the Hodge cohomology group $H^{p,p}(\mathbb{P}^k, \mathbb{C})$ is equal to $\mathbb{C}$. Moreover, it is generated by the class of ω^p, the wedge product of p Fubini-Study forms. Since T has a positive mass, positive closed $(1,1)$-forms U_i can be chosen to be non-exact. Hence the class of U_i in $H^{1,1}(\mathbb{P}^k, \mathbb{C})$ can be represented as $[U_i] = c_i[\omega]$ for some nonzero real number c_i. On the other hand, we have a well defined bilinear operation called the *cup-product* $\smile$: $H^{p,p}(\mathbb{P}^k, \mathbb{C}) \times H^{p,p}(\mathbb{P}^k, \mathbb{C}) \to H^{2p,2p}(\mathbb{P}^k, \mathbb{C})$. The cup product sends two classes of forms to the class of their wedge product, i.e. $[U_1] \smile [U_2] := [U_1 \wedge U_2]$. In our case U_1 and U_2 have disjoint supports, hence $[U_1 \wedge U_2] = 0$. Since $\smile$ is bilinear, we also have

$$[U_1] \smile [U_2] = (c_1 \cdot c_2) \cdot ([\omega] \smile [\omega]) = (c_1 \cdot c_2)[\omega^2].$$

Now we are in the contradiction since c_i are nonzero constants and ω^2 is the generator of $H^{2,2}(\mathbb{P}^k, \mathbb{C})$.

Let T be the Green current. For every integer $l \in \{1, \ldots, k\}$ we can define the wedge product of l copies of T and we denote its support by $\mathcal{J}_l = \mathrm{supp(T^l)}$. We know already that $\mathcal{J}_1$ is our Julia set and $\mathcal{J}_k$ is the small Julia set. Since these sets form the filtration of the Julia set

$$\mathcal{J}_k \subseteq \mathcal{J}_{k-1} \subseteq \ldots \subseteq \mathcal{J}_2 \subseteq \mathcal{J},$$

it is natural to ask what can be said about sets $\mathcal{J}_l \backslash \mathcal{J}_{l+1}$ and $\mathcal{J} \backslash \mathcal{J}_k$. Some results in this direction were given by Fornæss-Sibony and Dujardin [28, 20, 17] where they have studied the number of Fatou directions in points of $\mathcal{J}_l \backslash \mathcal{J}_{l+1}$. The following theorem, which tells us that topologically there are only two possible situations, is the main result of this chapter and it will be proved in section 2.3.

Theorem II.4. *For any holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree $d \geq 2$ only one of following can occur*

1. *$\mathcal{J}_k$ is nowhere dense in $\mathcal{J}$.*

2. *$\mathcal{J} = \mathcal{J}_k$.*

We will see that the *regularity* of Fatou components is related to the above theorem, but let us first introduce some definitions.

Let U be an open set in the topological space X, and let bU denote topological boundary U. The *regularization* of U is the interior of the closure of U and we will denote it by $\widehat{U}$. The set U is called a *regular set* if and only if $U = \widehat{U}$. The topological boundary of $\widehat{U}$ will be called the *regular boundary* of U.

Lemma II.5. *(Milnor [50, p.47]) Let f be a holomorphic endomorphism of $\mathbb{P}^1$ of degree ≥ 2 and U be an open set that meets $\mathcal{J}$. Then there exist an $N \in \mathbb{N}$ such that $\mathcal{J} \subset f^N(U)$.*

We are interested in getting better topological description of Fatou components, hence we propose the following two questions:

1. Are all Fatou components regular open sets?

2. If there is a Fatou component which is not regular, is its regularization again Stein?

It doesn't take much effort to find an example which gives a negative answer to the first question. As an example observe the following map

$$F([z : w : t]) = [z^2 - 2t^2 : w^2 : t^2].$$

We know from the one-dimensional dynamics, that the Julia set of $z \to z^2 - 2$ is equal to $I = [-2, 2]$ and the Julia set of $z \to z^2$ is unit circle. Observe that F has two attractive fixed points $[1 : 0 : 0]$ and $[0 : 1 : 0]$, hence the Fatou set is nonempty and is disconnected. The basin of attraction of $[1 : 0 : 0]$ can be expressed in $\{t = 1\}$ chart as $\{\mathbb{C}\backslash I\} \times \{|w| < 1\}$ and its regularization equals to $\mathbb{C} \times \{|w| < 1\}$, hence it is not a regular open set.

What is more surprising is that under the assumption of having a disconnected Fatou set, all Fatou components on the Riemann sphere are regular. Observe that in the remaining case, when the Fatou set is connected, its regularization equals to the Riemann sphere.

Lemma II.6. *Let $f : \mathbb{P} \to \mathbb{P}$ be a holomorphic map. If the Fatou set is disconnected, then Fatou components are regular open sets.*

Proof. Suppose to the contrary, that there exists a non-regular Fatou component, i.e. $\Omega \neq \widehat{\Omega}$. Observe that $\widehat{\Omega} \cap \mathcal{J} \neq \emptyset$ and that $\widehat{\Omega}\backslash\mathcal{J} = \Omega$. Let N be as in Lemma II.5, then the set $f^N(\widehat{\Omega})\backslash\mathcal{J} \subset \mathcal{F}$ is disconnected since the Fatou set is disconnected and $f^N(\widehat{\Omega})$ is an open neighborhood of the Julia set. On the other hand we know that $\mathcal{J}$ is f-invariant and f^N is continuous, therefore $f^N(\Omega) = f^N(\widehat{\Omega}\backslash\mathcal{J})$ is connected and $f^N(\widehat{\Omega}\backslash\mathcal{J}) = f^N(\widehat{\Omega})\backslash\mathcal{J}$, hence we have a contradiction. $\square$

We have seen that for general maps, Fatou components are not regular. In the next section we give a regularity condition which ensures that a given map has only regular Fatou components.

The second question remains open, but we give an example of a bounded Stein domain in $\mathbb{C}^2$ whose regularization is not Stein.

2.2 Regularity

In this section we always assume that f is a holomorphic endomorphism of $\mathbb{P}^k$ where $\deg(f) \geq 2$ and $k \geq 2$.

When $\mathcal{F}$ is disconnected and $\mathcal{J}$ has empty interior, we can define the set $\mathcal{S}$ as the closure of the union of all regular boundaries of Fatou components, i.e.

$$\mathcal{S} := \overline{\bigcup_{\Omega} b\widehat{\Omega}} \subseteq \mathcal{J},$$

where the union is taken over all Fatou components. As we shall see later in this chapter, the set S is interesting only when the Fatou set is disconnected and the Julia set has empty interior. In this case it is obvious that all Fatou components are regular open sets if and only if $b\mathcal{F} = \mathcal{S}$.

Remark II.7. Suppose that $\mathcal{F}$ is disconnected and $\mathcal{J}$ has empty interior. By the definition of $\mathcal{S}$, a point z is contained in $\mathcal{S}$ if and only if every open neighborhood $z \in V$ has nontrivial intersection with at least two different Fatou components, hence $V \backslash \mathcal{S}$ is disconnected.

Lemma II.8. *The set $\mathcal{S}$ is totally f invariant.*

Proof. First we prove that for every $z \in \mathcal{S}$ we also have $f(z) \in \mathcal{S}$. Suppose this is not true, then we can find a Fatou component Ω such that $f(z) \in \widehat{\Omega}$. Let U be an open connected component of $f^{-1}(\widehat{\Omega})$ which contains z. Then using the fact that $\mathcal{J}$ is totally invariant, we get $U \backslash \mathcal{J}$ is connected since it is one of the connected components of $f^{-1}(\widehat{\Omega}\backslash\mathcal{J}) = f^{-1}(\Omega)$. Hence we are in the contradiction since $z \in U$ and by the definition of $\mathcal{S}$ the set U intersects least two different Fatou components.

Now suppose that $w \in f^{-1}(z) \cap \mathcal{J}\backslash\mathcal{S}$. There is a Fatou component Ω such that $w \in \widehat{\Omega}$. We can see that $f(\widehat{\Omega})\backslash\mathcal{J} = f(\widehat{\Omega}\backslash\mathcal{J}) = f(\Omega)$ is connected and this is a contradiction since $f(\widehat{\Omega})$ is an open neighborhood of z. $\square$

The following theorem is generalized version of Lemma II.5.

Theorem II.9. *(Dinh-Sibony [17]) Let U be an open set that meets $\mathcal{J}_k$. Then the set $\mathbb{P}^k\backslash\bigcup_{n\geq 0} f^n(U)$ is pluripolar, moreover it is a subset of a proper algebraic set. If $\mathcal{J}_k$ has non-empty interior, then $\mathcal{J}_k = \mathbb{P}^k$.*

Now we can state our first theorem.

Theorem II.10. *Suppose that $\mathcal{F}$ is disconnected and $\mathcal{J}$ has empty interior. Then $\mathcal{J}_k$ is a subset of $\mathcal{S}$.*

Proof. Assume that there exist a point $z_0 \in \mathcal{J}_k\backslash\mathcal{S}$. Since $\mathcal{J}_k$ and $\mathcal{S}$ are closed sets there exists U an open neighborhood of z_0 which satisfies $U\cap\mathcal{S} = \emptyset$. We have seen that both $\mathcal{J}_k$ and $\mathcal{S}$ are totally f invariant and hence $\mathcal{S} \subset \mathbb{P}^k\backslash\bigcup_{n\geq 0} f^n(U)$. By Theorem II.9 the set $\mathcal{S}$ is a pluripolar set which is in contradiction with Remark II.7. $\square$

Question II.11. *We have seen that $\mathcal{J}_k \subseteq \mathcal{S} \subseteq \mathcal{J}$ and recall that $\mathcal{J}_l = \mathrm{supp}(\mathrm{T}^l)$. For which l we can conclude that $\mathcal{J}_l \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{J}_l$?*

The following theorem by Ueda provides us with some useful information about Fatou components and the set $\mathcal{J}\backslash\mathcal{S}$.

Theorem II.12. *(Ueda [63]) Let Ω be a Fatou component. Let S be a Riemann surface and $E \subset S$ be a closed polar set. If there exists a holomorphic map $\varphi : S \to \mathbb{P}^k$ such that $\varphi(S\backslash E) \subset \Omega$, then $\varphi(S) \subset \Omega$.*

Definition II.13. We say that the regularity condition is satisfied if and only if $\mathcal{J}_k \not\subset \overline{(\mathcal{J}\backslash\mathcal{S})}$, i.e. there exist $z \in \mathcal{J}_k$ and an open neighborhood $z \in U$ such that $U \cap \mathcal{J} = U \cap \mathcal{S}$.

Theorem II.14. *Assume that $\mathcal{F}$ is disconnected and $\mathcal{J}$ has empty interior. If the regularity condition is satisfied then all Fatou components are regular open sets, i.e. $\mathcal{J} = \mathcal{S}$.*

Proof. We already know that $\mathcal{J}_k \subseteq \mathcal{S} \subseteq \mathcal{J}$. By the assumption there exist $z \in \mathcal{J}_k$ and its open neighborhood U such that $U \cap \mathcal{J} = U \cap \mathcal{S}$. By Lemma II.8 the set $\mathcal{S}$ is totally invariant and therefore $\mathcal{J}\backslash\mathcal{S} \subseteq \mathbb{P}^k \backslash \bigcup_{n\geq 0} f^n(U)$. By our assumption $U \cap \mathcal{J}_k \neq \emptyset$ and hence by Theorem II.9 the set $\mathcal{J}\backslash\mathcal{S}$ is a subset of an algebraic variety. Therefore for any given point $z_0 \in \mathcal{J}\backslash\mathcal{S}$ we can find a Fatou component Ω and a holomorphic disk $g : \Delta \to \mathbb{P}^k$ satisfying:

1. $z_0 \in \widehat{\Omega}$,

2. $g(0) = z_0$,

3. $g(\Delta^*) \subset \Omega$.

By Theorem II.12 we have $g(\Delta) \subset \Omega$ and hence $\mathcal{J}\backslash\mathcal{S} = \emptyset$. □

2.3 Proof of Theorem II.4

We divide the proof into three possible cases.

(i) Suppose that $\mathcal{J}$ has empty interior and $\mathcal{F}$ is connected. If $\mathcal{J}_k$ is somewhere dense in $\mathcal{J}$, we can find an open set U such that $U \cap \mathcal{J} = U \cap \mathcal{J}_k$, because $\mathcal{J}_k$ is closed. Since $\mathcal{J}_k$ is totally invariant we can see that $\mathcal{J}\backslash\mathcal{J}_k \subset \mathbb{P}^k \backslash \bigcup_{n\geq 0} f^n(U)$. It follows from Theorem II.9 that $\mathcal{J}\backslash\mathcal{J}_k$ is a subset of an algebraic variety. If $\mathcal{J}\backslash\mathcal{J}_k$ would be nonempty, then for any point z_0 from this set we could find a holomorphic disc centered at z_0 which avoids the rest of $\mathcal{J}$. Now Theorem II.12 states that this can happen if and only if z_0 belongs to $\mathcal{F}$ and hence $\mathcal{J} = \mathcal{J}_k$.

(*ii*) Let $\mathcal{J}$ have empty interior and let $\mathcal{F}$ be disconnected. If $\mathcal{J}_k$ is somewhere dense $\mathcal{J}$, we can find an open set U such that $U \cap \mathcal{J} = U \cap \mathcal{J}_k$. By Theorem II.10 we have $\mathcal{J}_k \subseteq \mathcal{S} \subseteq \mathcal{J}$, therefore the regularity condition is satisfied and by Theorem II.14 we have $\mathcal{J} = \mathcal{S}$. Since $\mathcal{J}_k$ is f-invariant, it follows from Theorem II.9 that $\mathcal{J} \backslash \mathcal{J}_k \subset \mathbb{P}^k \backslash \bigcup_{n \geq 0} f^n(U)$ is a subset of an algebraic variety. Suppose $\mathcal{J} \backslash \mathcal{J}_k$ is nonempty. Since $\mathcal{J}_k$ is closed we can find, for each $z \in \mathcal{J} \backslash \mathcal{J}_k$, a connected open neighborhood $z \in V$ such that $V \cap \mathcal{J}_k = \emptyset$. By making V smaller if necessary we can assume that $V \backslash \mathcal{J}$ is disconnected (see Remark II.7). Because V and $\mathcal{J}_k$ are disjoint, we can write $V \backslash \mathcal{J} = V \backslash (\mathcal{J} \backslash \mathcal{J}_k)$, which leads us to a contradiction, since we can not make open connected set disconnected only by subtracting an algebraic variety.

(*iii*) Let $\mathcal{J}$ have a non-empty interior. The following proposition is needed to complete the proof.

Proposition II.15. *If $\mathcal{J}$ has non-empty interior then $\mathcal{J} = \mathbb{P}^k$ or $\mathcal{J}_k \subset b\left(\overline{\mathcal{F}}\right)$.*

We continue with the proof of Theorem II.4. If $\mathcal{J}_k$ has nonempty interior then we can conclude from Theorem II.9 that $\mathcal{J} = \mathcal{J}_k = \mathbb{P}^k$. If $\mathcal{J}_k$ has empty interior and $\mathcal{J} = \mathbb{P}^k$ then obviously $\mathcal{J}_k$ is nowhere dense. The last case that we have to consider is when $\mathcal{J}_k \subset b\left(\overline{\mathcal{F}}\right)$. Since $b\left(\overline{\mathcal{F}}\right) = b\,(\mathrm{int}\mathcal{J})$, we see that $\mathcal{J}_k \subset \overline{\mathrm{int}\mathcal{J}}$ and hence $\mathcal{J}_k$ is nowhere dense in $\mathcal{J}$.

Proof of Proposition II.15. Observe that $b\left(\overline{\mathcal{F}}\right) = b\,(\mathrm{int}\mathcal{J})$. Suppose there exists a point $z \in \mathcal{J}_k \cap \mathrm{int}\mathcal{J}$. There exists an open set U such that $z \in U \subset \mathcal{J}$. By Theorem II.9 we know that $\mathbb{P}^k \backslash \bigcup_{n \geq 0} f^n(U)$ is pluripolar. Since the Julia set is f-invariant and the Fatou set is an open set, we have $\mathcal{F} = \emptyset$ and hence $\mathcal{J} = \mathbb{P}^k$.

Suppose on the contrary that there is a point $z \in \mathcal{J}_k \cap \widehat{\mathcal{F}}$, z lies outside the regular boundary of $\mathcal{F}$. There exists an open set U such that $z \in U \subset \widehat{\mathcal{F}}$. By Theorem II.9 we know that $\mathbb{P}^k \setminus \bigcup_{n \geq 0} f^n(U)$ is pluripolar. This is now a contradiction, since $\mathcal{J}$ has non-empty interior which is totally f-invariant. $\square$

2.4 Examples

The following three examples presented by Fornæss and Sibony [30] illustrate all possible relations between

$$\mathcal{J}_2 \subseteq \mathcal{J} \subseteq \mathbb{P}^2.$$

In all of their examples the set $\mathcal{J}_2$ is either equal to $\mathcal{J}$ or it is nowhere dense in it. Theorem II.4 says that these are the only possible relations between sets $\mathcal{J}_2$ and $\mathcal{J}$.

1. We have already seen the simple example

$$[z : w : t] \to [z^2 : w^2 : t^2]$$

   where $\mathcal{J}$ has empty interior and $\mathcal{J}_2$ is nowhere dense in $\mathcal{J}$.

2. An example where both Julia sets coincide and the Fatou set is non-empty is given by

$$[z : w : t] \to [(z - 2w)^4 : z^4 : t^4 + \lambda t z^3],$$

   provided that $|\lambda|$ is big enough. In local chart $\{t \neq 0\}$ this map takes the form

$$(x, y) \to \left(\frac{(x - 2y)^4}{1 + \lambda x^3}, \frac{x^4}{1 + \lambda x^3} \right),$$

   where $x = z/t$ and $y = w/t$. The Jacobian matrix of this map is equal to zero at the origin which is a fixed point of this map, hence it is an attracting fixed point. This proves that the Fatou set is nonempty since it contains $[0 : 0 : 1]$. Fornæss and Sibony [30] proved that for any open set U intersecting the hyperplane $\{t = 0\}$, the union of forward images of U equals $\mathbb{P}^2 \backslash \{0\}$. Therefore we can conclude that $\mathcal{J}_2 = \mathcal{J}$.

3. The map

$$[z : w : t] \to [(z - 2w)^2 : z^2 : t^2]$$

has the hyperplane $\{t = 0\}$ as an attractor. Since the Julia set of

$$[z : w] \to [(z - 2w)^2 : z^2]$$

is entire $\mathbb{P}$, we can conclude that $\mathcal{J}$ has nonempty interior. One can see that the map has an attracting fixed point at $[0 : 0 : 1]$, hence the Fatou set is nonempty. Therefore $\mathcal{J}_2$ has empty interior and it differs from $\mathcal{J}$.

4. The map

$$[z : w : t] \to [(z - 2w)^2 - ct^2 : z^2 - ct^2 : t^2]$$

also has the hyperplane $\{t = 0\}$ as an attractor but it has a small twist to it. Provided that $|c|$ is large enough, one can prove that the map is uniformly expanding on the set of points with bounded orbits, and that all other points tend towards an attractor $\{t = 0\}$ which is contained in $\mathcal{J}$. Some more effort is needed to show that $\mathcal{J}_2$ is a Cantor set if $|c|$ is large enough. In this way we get $\mathcal{J}_2 \neq \mathcal{J} = \mathbb{P}^2$.

5. Let f be a holomorphic endomorphism of $\mathbb{P}^2$ and C the set of critical points. We say that f *critically finite* if the set $\cup_{n \geq 0} f^n(C)$ is an algebraic subset of $\mathbb{P}^2$. Jonsson [42] proved that for critically finite maps the small Julia set equals to $\mathbb{P}^2$. An example of such map is

$$[z : w : t] \to [(z - 2w)^2 : (z - 2t)^2 : t^2].$$

In the introduction we raised the question whether the regularization of a Fatou component is always Stein. Unfortunately we were not able to give any positive or negative answer to this problem. Nevertheless we construct a Stein domain in $\mathbb{C}^2$ whose regularization is not Stein.

Let us define sets

$$A = \{(z,w) \in \mathbb{C}^2 \mid ||(z,w)|| \leq 1\}$$

$$B = \{(z,w) \in \mathbb{C}^2 \mid \frac{1}{2} \leq |z| \leq 1, |w| \leq \frac{1}{2}\},$$

$$C = \{(z,w) \in \mathbb{C}^2 \mid |z| = \frac{1}{2}\}.$$

Now we define an open set $U = A \backslash (B \cup C)$. Observe that U has two connected components which are Stein but $\hat{U}$ is not Stein. Let us choose two points $p = (\frac{1}{4}, \frac{\sqrt{15}}{4})$ and $q = (\frac{3}{4}, \frac{\sqrt{7}}{4})$. Next we connect these two points with a smooth real arc

$$\gamma : (0,1) \to \mathbb{C}^2 \backslash \overline{B},$$

satisfying $\gamma(0) = p$, $\gamma(1) = q$. We can also assume that γ is perpendicular to bU at p and q. For every open neighborhood V of γ there exists a strongly pseudoconvex handle-body K such that $\gamma \subset K \subset U \cup V$, see [21, 38, 34]. By taking $\Omega = U \cup K$ we have obtained a connected domain which is locally Stein (i.e. each point of $x \in b\Omega$ has a neighborhood G such that $\Omega \cap G$ is Stein). Then, by the solution of the generalized Levi problem for complex manifolds [19], Ω is also Stein. It follows that the regularization of Ω is not Stein.

CHAPTER III

Fatou components with punctured limit sets

3.1 Introduction

Let F be a holomorphic endomorphism of the complex projective space $\mathbb{P}^2$. Recall that the *Fatou set* is the largest open subset of $\mathbb{P}^2$ on which the family of iterates $\{F^n\}$ is equicontinuous. A connected component of the Fatou set is called a *Fatou component.*

In Chapter 1 we have seen that Fatou components for rational functions acting on the Riemann sphere have been precisely described. It would also be nice to have a precise description of Fatou components in higher dimensional projective spaces. This question was studied by Fornaess and Sibony, Hubbard and Papadopol, Lyubich and Peters, and many others [26, 28, 41, 44, 47, 62, 63, 65]. The following definition is due to Bedford-Smillie [5].

Definition III.1. An invariant Fatou component Ω is called *recurrent* if there exists an orbit in Ω with an accumulation point in Ω.

It follows that if an invariant Fatou component Ω is non-recurrent then all orbits in Ω converge to the boundary $\partial\Omega$. Invariant recurrent Fatou components were classified by Fornaess and Sibony [27]:

Theorem III.2 (Fornæss-Sibony). *Let F be a holomorphic endomorphism with a recurrent invariant Fatou component Ω. Then one of the following holds.*

1. *Ω is the basin of an attracting fixed point $p \in \Omega$.*

2. *All orbits in Ω converge to a closed invariant 1-dimensional submanifold $\Sigma \subset \Omega$, which is biholomorphically equivalent to either the disk, the punctured disk, or an annulus. The map F acts on Σ as an irrational rotation.*

3. *Ω is a* Siegel domain*: There exists a sequence (n_j) such that F^{n_j} converges uniformly on compact subsets of Ω to the identity map.*

The punctured disk in Case (2) was ruled out by Ueda [63].

Theorem III.3 (Ueda). *The invariant submanifold Σ in Case (2) of Theorem III.2 cannot be equivalent to a punctured disk.*

All other cases are known to occur, which makes the classification of recurrent Fatou components for holomorphic endomorphisms of $\mathbb{P}^2$ complete.

The situation is more complicated in the non-recurrent case. If Ω is a non-recurrent invariant Fatou component then by normality there exists an increasing sequence (n_j) such that the maps f^{n_j} converge to a limit map $h : \Omega \to \partial\Omega$. The main difficulty in dealing with non-recurrent Fatou components arises because it is not known whether the limit set $h(\Omega)$ is always independent of the sequence (n_j). If $h(\Omega)$ is independent of (n_j) then we say that Ω has a *unique limit set*. The following was proved by Lyubich and Peters [47].

Theorem III.4 (Lyubich-Peters). *Let F be a holomorphic endomorphism of $\mathbb{P}^2$ of degree at least 2, and let Ω be a non-recurrent invariant Fatou component with a unique limit set. Then $h(\Omega)$ is either a fixed point, or $h(\Omega)$ is equivalent to the unit disk, an annulus, or a punctured disc, and F acts on $h(\Omega)$ as an irrational rotation.*

Examples where $h(\Omega)$ is a fixed point, a rotating disk or a rotating annulus were known to exist, but whether the punctured disk could also exist remained open. In light of the aforementioned result by Ueda one might expect the punctured disk not to exist in the non-recurrent case either. In Theorem III.9 we give an explicit construction of a non-recurrent Fatou component with a unique limit set equivalent to a punctured disk.

The local dynamics of maps tangent to the identity plays an important role in our results. Let us recall the basic notions here. Let $F : (\mathbb{C}^2, p) \to (\mathbb{C}^2, p)$ be the germ of a holomorphic map. If $DF(p) = \mathrm{Id}$ then we say that F is *tangent to the identity* at the point p. In other words, after changing coordinates by a translation F takes the form

$$F = \mathrm{Id} + F_k + F_{k+1} + \dots,$$

where each F_j is a homogeneous polynomial of degree j, with *order* $k \geq 2$. Following Hakim's paper [40] we say that $v \in \mathbb{C}^2$ is a *characteristic direction* for F if there exists a $\lambda \in \mathbb{C}$ so that

$$F_k(v) = \lambda v,$$

If $\lambda = 0$ then v is said to be *degenerate*, while if $\lambda \neq 0$ then v is *non-degenerate*. An orbit $\{F^n(z)\}$ is said to converge to the origin *tangentially* to v if $F^n(z) \to 0$ and $[F^n(z)] \to [v]$ in $\mathbb{P}^1$.

A *parabolic curve* for F tangent to $[v] \in \mathbb{P}^1$ is an injective holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2 \backslash \{0\}$ satisfying the following properties:

- φ is continuous at $1 \in \partial\mathbb{D}$ and $\varphi(1) = 0$,
- $\varphi(\mathbb{D})$ is forward invariant and $(F|_{\varphi(\mathbb{D})})^n \to 0$ uniformly on compact subsets,
- $[\varphi(\zeta)] \to [v]$ as $\zeta \to 1$ in $\mathbb{D}$.

Theorem III.5 (Hakim)**.** *Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ tangent to the identity of order $k \geq 2$. Then for any non-degenerate characteristic direction v there exist (at least) $k-1$ parabolic curves for F tangent to $[v]$, each of which corresponds to a different real direction.*

Remark III.6. In Hakim's paper there is an explicit construction of $k-1$ disjoint parabolic curves and in the sequel we refer to them as the $k-1$ parabolic curves. It turns out that such explicit construction of these $k-1$ curves allows us to prove that these curves vary continuously with a base point.

In the last section of this chapter we will give a general construction of punctured limit sets.

Theorem III.7. *Let $V \in \mathbb{C}^2$ be a pure one-dimensional analytic set. Then there exists a holomorphic endomorphism F of $\mathbb{C}^2$ such that for every irreducible component V_1 of V the map F has a non-recurrent Fatou component Ω on which all orbits converge to $V_1 \setminus \mathrm{Sing}(V)$.*

The idea is the following. We will construct a map $F = \mathrm{Id} + G$, where G vanishes on the analytic set V. By our construction the map F will have a parabolic curve attached to each point $(z, w) \in \mathrm{Reg}(V)$, and these curves vary continuously with (z, w). The union of these curves will be contained in an invariant Fatou component whose orbits converge to V. The following theorem by Lyubich and Peters [47] tells us that the singular points of V are not contained in the limit set.

Theorem III.8 (Lyubich-Peters)**.** *Let X be a 2-dimensional complex manifold and $f : X \to X$ a holomorphic endomorphism. Let $\Omega \subset X$ be an invariant Fatou component and suppose that the sequence (f^{n_j}) converges uniformly on compact subsets of Ω a limit map $h : \Omega \to \partial\Omega$. Then $h(\Omega)$ is either a point or else an injectively immersed Riemann surface.*

The layout of this chapter is as follows. In Section 3.2 we construct a holomorphic endomorphism of $\mathbb{P}^2$ with an invariant Fatou component Ω where the limit set is a punctured disk in the boundary of Ω. In Section 3.3 we construct a large class of holomorphic and polynomial maps for which there exist non-recurrent Fatou components with limit sets equal to the regular parts of analytic sets.

3.2 Construction of a punctured disk

A *skew-product* ϕ is a holomoprhic endomorphism of $\mathbb{C}^2$ which can be expressed in the following form

$$\phi(z, w) = (\varphi(z), \psi(z, w)).$$

When $\varphi(z)$ and $\psi(z, w)$ are holomorphic polynomials, we call such map a polynomial skew-product [43].

Throughout this section we let f be the polynomial skew-product given by

$$f(z, w) = (\lambda z + z^3, \lambda^{-1}(w + zw^2) + w^3).$$

Here $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is chosen such that the maps $z \mapsto \lambda z + z^3$ and $w \mapsto \lambda^{-1}w + w^3$ are linearizable in a neighborhood of the origin. It is enough to assume that θ satisfies a *Diophantine condition*, i.e. there exists integer k and $\varepsilon > 0$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{\varepsilon}{q^k}$$

holds for every rational number p/q; see [50, p.129]. Observe that the polynomial map f extends to a holomorphic endomorphism of $\mathbb{P}^2$, given in homogeneous coordinates by

$$F[Z : W : T] = [\lambda ZT^2 + Z^3 : \lambda^{-1}(WT^2 + ZW^2) + W^3 : T^3]$$

Our main result is the following.

Theorem III.9. *The map F has an invariant Fatou component Ω on which all orbits converge to an embedded punctured disk $D^\star \subset \partial\Omega$, and F acts on $D^\star$ as an irrational rotation.*

The limit set will actually be a bounded simply connected set minus one point and will be contained in the z-axis, hence it will be sufficient to only consider the map f and work in Euclidean coordinates. We write $(z_n, w_n) = f^n(z_0, w_0)$ and define

$$\varphi_n(z, w) = (z, \lambda^n w),$$

and

$$\begin{aligned} G_n &= \varphi_n \circ f \circ \varphi_{n-1}^{-1} \\ &= (\lambda z + z^3, w + \lambda^{1-n} z w^2 + \lambda^{-2n+3} w^3). \end{aligned}$$

We also write

$$g_n(w) = w + \alpha_{n-1} w^2 + \lambda^{-2n+3} w^3,$$

where $\alpha_n = \lambda^{-n} z_n$. Since we have chosen λ such that the map $z \mapsto \lambda z + z^3$ is linearizable, there exists a local change of coordinates $\eta(z) = z + z^2 h(z)$, conjugating our map $z \mapsto \lambda z + z^3$ to the linear map $\zeta \mapsto \lambda\zeta$. Let $A > 0$ be such that $|h(z)| < A$ for all z sufficiently close to the origin, so that we obtain

$$|\lambda^n \eta(z) - \eta(\lambda^n z)| = |z^2 h(z) - z^2 \lambda^n h(\lambda^n z)| < 2A|z|^2.$$

Since we can also bound $|(\eta^{-1})'(z)| < B$ for z sufficiently close to 0, it follows that

$$|\alpha_n - z_0| = |\lambda^{-n} \eta^{-1}(\lambda^n \eta(z_0)) - z_0| = |\eta^{-1}(\lambda^n \eta(z_0)) - \eta^{-1}(\eta(\lambda^n z_0))| \leq C|z_0|^2$$

where $C = 2AB > 0$ does not depend on n and z_0. Hence for z_0 in a sufficiently small disk $D \subset \mathbb{C}_z$ centered at the origin we have that

$$|\alpha_n - z_0| \leq \frac{|z_0|}{2} \tag{3.1}$$

holds for all $n \in \mathbb{N}$.

Lemma III.10. *For all $z_0 \in D \setminus \{0\}$ there exists an open set $\mathcal{C}_{z_0} \subset \{z = z_0\}$ on which*

$$\|f^n(z_0, w) - f^n(z_0, 0)\| \to 0,$$

uniformly on compact subsets of $\mathcal{C}_{z_0}$.

Proof. Notice that $f^n = \varphi_n^{-1} \circ G_n \circ \ldots G_1$. Since we are interested in the set of w-values for which w_n converges to 0, and the map φ_n preserves distances to $(z_n, 0)$, it is equivalent to consider the w-values for which

$$\pi_2(G_n \circ \ldots \circ G_1(z_0, w)) = g_n \circ \ldots \circ g_1(w)$$

converges to 0. We will write $g(n)$ for the composition $g_n \circ \ldots \circ g_1$. Let us also write

$$u_n = \frac{1}{g(n)(w)}.$$

We have that

$$u_{n+1} = \frac{1}{g_{n+1}(\frac{1}{u_n})} = u_n - \alpha_n + O\left(\frac{1}{|u_n|}\right).$$

Having chosen z_0 sufficiently close to the origin so that inequality (3.1) holds, it follows that $|u_n| \to \infty$ if the initial value u_0 lies in a halfplane of the form

$$\mathbb{H}_{z_0} = \{u \in \mathbb{C} \mid \mathrm{Re}(u\bar{z}_0) < -K(z_0)\}, \tag{3.2}$$

where $K > 0$ is some constant which depends on z_0. Hence for all $u \in \mathbb{H}_{z_0}$ we have

$$\|f^n(z_0, u^{-1}) - f^n(z_0, 0)\| \to 0.$$

The statement of the lemma therefore holds for the set

$$\mathcal{C}_{z_0} = \left\{ (z_0, w) \mid \frac{1}{w} \in \mathbb{H}_{z_0} \right\}.$$

$\square$

In Equation (3.2) the constant $K(z)$ can be chosen to vary continuously with z in the punctured disk $D \setminus \{0\}$, and therefore the sets $\mathcal{C}_{z_0}$ also vary continuously with $z \in D \setminus \{0\}$. Since f maps z-plane to itself and its restriction to the z-plane is linearizable around the origin, we can find an f-invariant neighborhood of the origin $U \subset D$. Define the set

$$V = \{(z_0, w_0) \mid z_0 \in U \setminus \{0\},\ w_0 \in C_{z_0}\}.$$

We then define the f-invariant open connected set Λ by

$$\Lambda = \bigcup_{n=0}^{\infty} f^n(V).$$

It follows from Lemma III.10 that $\{f^n\}_n$ is a normal family on Λ and hence Λ is contained in some invariant Fatou component Ω.

Lemma III.11. *Every orbit in Ω converges to the plane $\mathbb{C}_z$.*

Proof. Let (f^{n_j}) be a sequence that converges uniformly on compact subsets of Ω to a map h. Then $h(\Lambda) \subset \{w = 0\}$. Since h is holomorphic and Λ has the interior, it follows that $h(\Omega) \subset \{w = 0\}$. $\square$

Our map f is a polynomial endomorphism which maps z-plane to itself and

$$f(z,0) = (\lambda z + z^3, 0).$$

With slight abuse of notation we can identify z-plane with $\mathbb{C}$ and we can think of f as a polynomial on $\mathbb{C}$. The polynomial f has a fixed point at the origin which is contained in the Fatou component V. This component is conformally equivalent to a disk and on V our polynomial $f(z)$ is conformally conjugate to an irrational rotation. We call such Fatou component a *Siegel disk.*

Lemma III.12. *Let (f^{n_j}) be a convergent subsequence with limit $h : \Omega \to \partial\Omega$. Then $h(\Omega)$ is contained in the Siegel disk in the z-plane, and is independent of the sequence (n_j).*

Proof. Lemma 13 of [47] states that the restriction of the maps $\{f^n\}$ to the set $h(\Omega)$ must also form a normal family. Hence by Lemma III.10 we have that $h(\Omega)$ is contained in the Siegel disk in the z-plane centered at the origin, which we call V. Suppose that $k : \Omega \to \bar{\Omega}$ is any other limit map of the sequence (f^n). By the same argument as above we have that $k(\Omega)$ lies in the Siegel disk V. Our map f is a skew-product, therefore $\pi_1(\Omega)$, the projection of Ω to the z-plane is contained in V. Since Ω is f invariant, the same must holds for $\pi_1(\Omega)$. Let n_j and m_j be two sequences satisfying $f^{n_j} \to h$ and $f^{m_j} \to k$. We may assume that the sequence $f^{n_j - m_j}$ converges to the map $\rho : V \to V$. Let us write

$$f^{n_j} = f^{n_j - m_j} \circ f^{m_j}.$$

In the limit we obtain $h = \rho \circ k$ and by previous observation $h(\Omega) = k(\Omega)$.

$\square$

Lemma III.13. *The Fatou component Ω is non-recurrent and the limit set $h(\Omega)$ is a punctured disk.*

Proof. It follows from the skew-product structure of f, and the fact that the restriction of f to $\{z = 0\}$ is linearizable in a neighborhood of the origin, that no orbits in Ω converge to $(0,0)$. But $h(\Omega)$ does contain every point $(z,0)$ with $z \neq 0$ sufficiently small. Hence $h(\Omega)$ is a 1-dimensional submanifold of $\mathbb{C}^2$ that is not equivalent to either the unit disk or to an annulus. Therefore it follows from Theorems III.2 and III.3 that Ω must be a non-recurrent Fatou component, and Theorem III.4 implies that $h(\Omega)$ is an embedded punctured disk. $\square$

With this lemma we have completed the proof of Theorem III.9.

Remark III.14. One easily sees that the Siegel disk centered at 0 in the $z = 0$ plane lies in the Julia set. Indeed, suppose that a point $(0,w)$ from this disk lies in the Fatou set, and let U be a neighborhood of $(0,w)$ on which the family $\{f^n\}$ is normal. Since

$$\inf_{n\geq 0}\left\{\left|\frac{\partial f_2^n}{\partial z}(0,w)\right|, \left|\frac{\partial f_2^n}{\partial w}(0,w)\right|\right\} > \delta > 0,$$

where f_2^n is the second coordinate function of f^n, and by our assumption that the iterates (f^n) form a normal family, the union of the forward images of U contains a tubular neighborhood of the set $\{(x,y) \in \mathbb{C}^2 \mid \exists \{n_j\}_j \quad \lim_{j\to\infty} f^{n_j}(0,w) = (x,y)\}$, which is a Jordan curve in the $z = 0$ plane whose bounded connected component of its complement in $z = 0$ plane contains the origin. Hence for $|c|$ sufficiently small the intersection of this tubular neighborhood with the fiber $\{z = c\}$ contains an annulus. But then the family of iterates of f restricted to the area enclosed by this annulus must be bounded, and thus a normal family. But this area includes the origin, which leads to a contradiction.

3.3 Regular limit sets

Let $V \subset \mathbb{C}^2$ be any analytic set of pure dimension one. There exist an open cover $U_n \subset \mathbb{C}^2$ of V and a collection of minimal defining functions $g_n \in \mathcal{O}(U_n)$ for sets $V \cap U_n$, i.e. $\{g_n = 0\} = V \cap U_n$ and $\{g_n = dg_n = 0\} = \mathrm{Sing}(V \cap U_n)$;
see [16, p.27]. Note that $\{(g_n, U_n)\}$ is a Cousin II distribution, i.e. g_n/g_k is does not vanish on $U_n \cap U_k$ for $n \neq k$. Since the Cousin II problem always has a solution in $\mathbb{C}^2$, there exists a holomorhic function $g : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying

$$g|_{U_n} = e^{\varphi_n} g_n,$$

where φ_n is a holomorphic function. It follows that $\{g = 0\} = V$ and

$$dg|_{U_n} = e^{\varphi_n} dg_n + g_n e^{\varphi_n} d\varphi_n.$$

Since g_n were minimal defining functions we finally obtain $\{g = dg = 0\} = \mathrm{Sing}(V)$ Let us define the following map

$$F = (z, w) + (g(z, w))^k (P(z, w), Q(z, w)), \qquad (k \geq 2), \tag{3.3}$$

where P and Q are holomorphic functions on $\mathbb{C}^2$ and g is a minimal defining function of V. Observe that F is tangent to the identity on V. If we can find P and Q such that for each point in $\mathrm{Reg}(V)$ there exists a non-degenerate characteristic direction, then by Hakim's Theorem III.5 there will be a parabolic curve attached to each point (z, w) from the regular part of V. These curves vary continuously with the base point. The union of these curves will be contained in an invariant Fatou component whose orbits converge to V. The fact that the singular points of V are not contained in the limit set then follows from Theorem III.8.

Let us pick a point $(z_0, w_0) \in \mathrm{Reg}(V)$. After conjugating F with the right translation, our map takes the form

$$(z, w) \to (z, w) + (g(z + z_0, w + w_0))^k (P(z + z_0, w + w_0), Q(z + z_0, w + w_0)).$$

Using the fact that $g(z_0, w_0) = 0$ we can rewrite the map as

$$(z, w) \to (z, w) + (zg_z(z_0, w_0) + wg_w(z_0, w_0))^k (P(z_0, w_0), Q(z_0, w_0)) + \text{h.o.t.},$$

where g_z and g_w are the partial derivatives with respect to z and w. Note that the existence of a non-degenerate characteristic direction corresponds to

$$g_z(z_0, w_0)P(z_0, w_0) + g_w(z_0, w_0)Q(z_0, w_0) \neq 0. \tag{3.4}$$

We see immediately that in order to have non-characteristic directions it is necessary that the gradient of g does not vanish on $\mathrm{Reg}(V)$. Observe that this condition is satisfied if and only if g is a minimal defining function of set V. In order to prove Theorem III.7 we first have to prove the existence of holomorphic functions P and Q for which (3.4) holds.

Definition III.15. Let V be an analytic set and let $\mathrm{Sing}(V)$ be the set of all singular points of V. A function $f : V \setminus \mathrm{Sing}(V) \to \mathbb{C}$ is said to be *weakly holomorphic* on V, if f is holomorphic on $V \setminus \mathrm{Sing}(V)$ and locally bounded along V. The sheaf of weakly holomorphic functions on V is denoted by $\tilde{\mathcal{O}}_V$.

Definition III.16. Let V be an analytic set in some open set U in $\mathbb{C}^n$. A holomorphic function f on U is called a *universal denominator* for V at the point $z \in V$, if $f_z \cdot \tilde{\mathcal{O}}_{V,z} \subset \mathcal{O}_{V,z}$.

Lemma III.17. *Let g be the minimal defining function of an analytic set $V \subset \mathbb{C}^2$.*

There exist holomorphic functions P and Q on $\mathbb{C}^2$ such that $g_zP + g_wQ$ does not vanish on $\mathrm{Reg}(V)$*.*

Proof. The partial derivatives of g with respect to z and w will be denoted by g_z and g_w respectively. Recall that an analytic set $V = \{g(z, w) = 0\}$ is a Stein space. Let us denote by $\mathcal{O}_V$ the coherent sheaf of holomorphic functions on V.

If V is a manifold, then the partial derivatives $g_z|_V$ and $g_w|_V$ do not have common zeros and so they generate $\mathcal{O}_V$. By Cartan's Division Theorem [32, Corollary 2.4.4] there are $p, q \in \mathcal{O}_V$ such that $p \cdot g_z|_V + q \cdot g_w|_V = 1$. Cartan's Extension Theorem [32, Corollary 2.4.3.] gives that every holomorphic function on V can be extended to a holomorphic function on $\mathbb{C}^2$, which proves the existence of the desired holomorphic functions P and Q.

In general we can not expect V to be a manifold. Let us denote the (discrete) set of singular points of V by $\{a_n\}_{n\geq 1}$. By our assumption on g we have that $\mathrm{Sing}(V) = \{g = dg = 0\}$. Since V is a one-dimension variety, it has a normalization given by a non-compact Riemann surface M together with a holomorphic map $\pi : M \to V$, see [16]. We can lift g_z and g_w to holomorphic functions on M denoted by φ and ψ respectively. Since M is a non-compact Riemann surface, by Weierstrass Theorem [31, Theorem 26.7], we can find a holomorphic function θ on M with prescribed zeros, such that φ/θ and ψ/θ are holomorphic on M and without common zeros. As before, M is a Stein manifold and φ/θ, ψ/θ generate $\mathcal{O}_M$, so we can find $\tilde{p}, \tilde{q} \in \mathcal{O}_M$ such that

$$\frac{\varphi}{\theta}\tilde{p} + \frac{\psi}{\theta}\tilde{q} = 1,$$

hence

$$\varphi\tilde{p} + \psi\tilde{q} = \theta.$$

The functions $\tilde{p}$ and $\tilde{q}$ induce weakly holomorphic functions $p, q \in \tilde{\mathcal{O}}_V$. For every $a_n \in \mathrm{Sing}(V)$ there exists an integer $m_n > 0$ such that $g_z^{m_n}$ and $g_w^{m_n}$ are universal denominators in some neighborhood $a_n \in U_n \subset \mathbb{C}^2$ [51, Corollary 3., p.59]. By expanding $(g_z p + g_w q)^{2m_n+1}$ in to series we obtain

$$(g_z p + g_w q)^{2m_n+1} = g_z P_n + g_w Q_n,$$

with

$$P_n = g_z^{m_n} \sum_{k=0}^{m_n} \binom{2m_n+1}{k} p^{2m_n+1-k} g_z^{m_n-k} (q g_w)^k, \quad \text{and}$$
$$Q_n = g_w^{m_n} \sum_{k=0}^{m_n} \binom{2m_n+1}{k} q^{2m_n+1-k} g_w^{m_n-k} (p g_z)^k.$$

Since $g_z^{m_n}$ and $g_w^{m_n}$ are universal denominators we can conclude that P_n and Q_n are holomorphic on U_n. From the construction it follows that $g_z p + g_w q$ is non-zero on $\operatorname{Reg}(V) \cap U_n$, and the same holds for the function $h_n := g_z P_n + g_w Q_n$. We can take $\{U_n\}_{n \geq 1}$ to be pairwise disjoint. Let us take one more open set $U_0 \subset \mathbb{C}^2$ such that $W_0 = U_0 \cap V \subset \operatorname{Reg}(V)$ and that $\{W_n = U_n \cap V\}_{n \geq 0}$ covers V. The function $h_0 = g_z p + g_w q$ is non-zero holomorphic on W_0. Note that $\{(h_n, W_n)\}_{n \geq 0}$ is a Cousin II distribution and the Cousin II problem is always solvable on any one-dimensional Stein space [39, p.148]. Hence there exists a holomorphic function $H \in \mathcal{O}_V$ with the property

$$H|_{W_n} = h_n \varphi_n,$$

where the holomorphic functions φ_n are non-zero on W_n. If we define $\tilde{P}_n = P_n \varphi_n$ and $\tilde{Q}_n = Q_n \varphi_n$ (where $P_0 = p$ and $Q_0 = q$) observe that

$$H|_{W_n} = \tilde{P}_n g_z + \tilde{Q}_n g_w.$$

Let $\mathcal{J}_V$ be the ideal subsheaf of $\mathcal{O}_V$ generated by g_z and g_w. Since $\mathcal{J}_V$ is a coherent analytic sheaf we are given a short exact sequence

$$0 \to \mathcal{R} \overset{i}{\hookrightarrow} \mathcal{O}_V^2 \overset{\tau}{\to} \mathcal{J}_V \to 0$$

where i is an inclusion map and τ maps (f_1, f_2) into $f_1 g_z + f_2 g_w$. Now we can form a long exact sequence of cohomology grups

$$0 \to \Gamma(V, \mathcal{R}) \overset{i^*}{\to} \Gamma(V, \mathcal{O}_V^2) \overset{\tau^*}{\to} \Gamma(V, \mathcal{J}_V) \to H^1(V, \mathcal{R}) \to \cdots$$

and since V is Stein we know that $H^1(V, \mathcal{R}) = 0$, hence τ^* is surjective. Observe that H induces a section of $\mathcal{J}_V$. Since τ^* is surjective this section can be lifted to a section of $\mathcal{O}_V^2$ [45, Section 7.2], hence there exist holomophic functions P, Q on V such that $Pg_z + Qg_w = H$ on V. By Cartan's Extension Theorem we can extend P and Q to holomorphic functions on $\mathbb{C}^2$. $\square$

From now on let F be of the form given in (3.3), where P and Q are as in Lemma III.17.

Lemma III.18. *For any $(z, w) \in \mathrm{Reg}(V)$ we can find a local change of coordinates that maps (z, w) to $(0, 0)$ so that in the new coordinates F takes the form*

$$\begin{cases} x_1 = x - x^k + x^k O(x, u), \\ u_1 = u + x^k O(x, u). \end{cases} \tag{3.5}$$

Proof. Near any regular point of V we can change coordinates so that V is given by $\{(z, w) \mid z = 0\}$. By its definition the map F now takes the form

$$\begin{cases} z_1 = z + z^k p(z, w), \\ w_1 = w + z^k q(z, w). \end{cases}$$

Note that if we write $F = Id + F_k + F_{k+1} + \ldots$, where F_j is homogeneous of degree j, then the fact that $dg \neq 0$ on V implies that F_k is not identically zero. The assumption that $g_z P + g_w Q$ does not vanish on $\mathrm{Reg}(V)$ implies that there is a non-degenerate characteristic direction at each point in $\mathrm{Reg}(V)$, in particular at $(0,0)$. This direction cannot be $(0,1)$, which is degenerate, hence we may write $(1,\lambda)$ for the non-degenerate characteristic direction. We can now find a linear change of coordinates that fixes $V = \{z = 0\}$ and changes the non-degenerate characteristic direction to the tangent vector $(1,0)$. Therefore in these new coordinates F takes on the form

$$\begin{cases} x_1 = x + \beta x^k + x^k O(x,u), \\ u_1 = u + x^k O(x,u). \end{cases}$$

By conjugating with a linear map $(x,u) \mapsto ((-\beta)^{\frac{1}{1-k}} x, u)$ we obtain the required form. $\square$

Let us rewrite Equation (3.5) as

$$\begin{cases} x_1 = x - x^k(1 + u\phi(u)) + O(x^{k+1}), \\ u_1 = u + x^k O(x,u). \end{cases} \tag{3.6}$$

We note that for a fixed value of u the real attracting directions of the map $x \mapsto x_1$ satisfy

$$\mathrm{Arg}(x) = \mathrm{Arg}(x^k(1 + u\phi(u))) = k\mathrm{Arg}(x) + \mathrm{Arg}(1 + u\phi(u)),$$

hence the $k-1$ attracting real directions satisfy

$$\mathrm{Arg}(x) = \theta_u(m) = \frac{2m\pi - \mathrm{Arg}(1 + u\phi(u))}{k-1},$$

for $m = 1, \ldots, k-1$. For $\varepsilon > 0$ sufficiently small we define the regions

$$R_\varepsilon(m) := \left\{ (x,u) \mid 0 < |x| < \varepsilon,\ |u| < 2\varepsilon,\ |\mathrm{Arg}(x) - \theta_u(m)| < \frac{\pi}{2k-2} \right\}.$$

Lemma III.19. *For $\varepsilon > 0$ sufficiently small the iterates F^n converge uniformly on each $R_\varepsilon(m)$ to the axis $\{x = 0\}$.*

Proof. For a fixed u-value the convergence of the x-coordinate to 0 follows from standard estimates. Two estimates guarantee that variation in the u-coordinate does not break this convergence. On the one hand we have for sufficiently small x, u that satisfy

$$\frac{1}{3}\frac{\pi}{2k-2} \leq |\mathrm{Arg}(x) - \theta_u(m)| \leq \frac{5}{3}\frac{\pi}{2k-2} \tag{3.7}$$

the estimate

$$|\mathrm{Arg}(x_1) - \theta_{u_1}(m)| < |\mathrm{Arg}(x) - \theta_u(m)|\,, \tag{3.8}$$

and even

$$|x|\left(|\mathrm{Arg}(x) - \theta_u(m)| - |\mathrm{Arg}(x_1) - \theta_{u_1}(m)|\right) \gg |u_1 - u|. \tag{3.9}$$

On the other hand we have for sufficiently small x, u that satisfy

$$|\mathrm{Arg}(x) - \theta_u(m)| < \frac{1}{2}\frac{\pi}{2k-2} \tag{3.10}$$

that

$$|x_1| < |x|, \tag{3.11}$$

and even

$$|x| - |x_1| \gg |u_1 - u|. \tag{3.12}$$

The statement of the lemma follows. □

Let us write

$$U_\varepsilon = \{(x, u) \mid |x| < \varepsilon,\ |u| < 2\varepsilon\}\,,$$

and define the nested sequence

$$W_0(m) = R_\varepsilon(m), \quad \text{and}$$
$$W_{n+1}(m) = (W_n(m) \cup f^{-1}W_n(m)) \cap U_\varepsilon.$$

Finally we define

$$W(m) = \bigcup_{n\in\mathbb{N}} W_n(m).$$

Since f is invertible in a neighborhood of $(0,0)$, for sufficiently small ε the sets $W(m)$ are open, connected and disjoint. Moreover we have the following.

Lemma III.20. *Suppose that x_0 and u_0 satisfy $|x_0| < \varepsilon$ and $|u_0| < \varepsilon$, and (x_0, u_0) does not lie in one of the sets $W(m)$. Then there is an $N \in \mathbb{N}$ such that the orbit $(z_n) = ((x_n, u_n))$ satisfies*

- *For $n < N$ we have $|x_n| < \varepsilon$ and $|u_n| < 2\varepsilon$, and*
- *$|x_N| \geq \varepsilon$.*

Proof. Considering Equations (3.10), (3.11) and (3.12) for F^{-1} gives for sufficiently small x, u that satisfy

$$\left|\mathrm{Arg}(x) - (\theta_u(m) + \frac{\pi}{k-1})\right| < \frac{1}{2}\frac{\pi}{2k-2}$$

the estimate

$$|x_1| > |x|,$$

and even

$$|x_1| - |x| \gg |u_1 - u|.$$

The combination with Equations (3.7), (3.8) and (3.9) implies the statements of the lemma. $\square$

We can now complete the proof of the main result of this section.

Proof of Theorem III.7. Let g be a minimal defining function of V and let P and Q be as in Lemma III.17. Let $k \geq 2$ and define

$$F = (z, w) + g^k(z, w)(P(z, w), Q(z, w)).$$

By Lemma III.18 we can find a local change of coordinates near any point in $(z, w) \in \mathrm{Reg}(V)$ to obtain the form (3.5). In particular by Theorem III.5 we obtain $k - 1$ parabolic curves on which we have attraction towards (in original coordinates) the point (z, w).

Let us first assume that $\mathrm{Reg}(V)$ is connected. By Lemma III.18 and Lemma III.19 there exist, in a neighborhood of the point (x, y), $k-1$ open regions $R_\varepsilon(m)$ on which all orbits converge uniformly to V. These regions vary smoothly with the point (x, y) and each region $R_\varepsilon(m)$ intersects at least one of the $k - 1$ parabolic curves at (x, y). Consider the union over all base points (x, y) of the regions $R_\varepsilon(m)$, and let $\tilde{\Omega}$ be one of the connected components of this union. Then F^n converges uniformly on $\tilde{\Omega}$ to $\mathrm{Reg}(V)$. Hence $\tilde{\Omega}$ is contained in a Fatou component for F, which we denote by Ω.

By the aforementioned non-empty intersection with the parabolic curves, there exist for each point $z \in \mathrm{Reg}(V)$ corresponding points in Ω whose orbits converge to z. Hence on Ω the iterates F^n converges to a holomorphic map whose image contains $\mathrm{Reg}(V)$. By Lemma III.20 it follows that $\mathrm{Reg}(V)$ cannot be contained in Ω, hence must lie in $\partial\Omega$. Thus Ω is a non-recurrent Fatou component. By Theorem III.8 it follows that the limit set must be exactly equal to $\mathrm{Reg}(V)$.

Now suppose that $\mathrm{Reg}(V)$ is not connected and let V_1 be an irreducible component of V. By the above argument there is a Fatou component Ω on which all orbits converge to $V_1 \setminus \mathrm{Sing}(V_1)$. It remains to be shown that there are no points converging to intersection points of V_1 with other irreducible components of V. Let $V_2 \neq V_1$ be an irreducible component of V and assume that there is a point $z \in V_1 \cap V_2$. Let us recall an argument that was used in the proof of Theorem III.8. Suppose for the purpose of a contradiction that there exists a point $x \in \Omega$ whose orbit converges to z. Denote the limit of the sequence (F^n) on Ω by h. Let D be a holomorphic disk through x so that $h(D) = h(U)$ for a small neighborhood U of x. Then $F^n(D)$ intersects V_2 for n large enough. But as we noted earlier, V_2 lies in the Julia set while $F^n(D)$ lies in the Fatou set, which gives a contradiction. This completes the proof. □

Let Ω be a Fatou component on which the orbits converge to the regular part of an irreducible component $V_1 \subset V$. Then by Lemmas III.19 and III.20 the orbit of any point in Ω eventually lands in one of the sets $R_\varepsilon(m)$. Vice versa, any point whose orbit eventually lands in $R_\varepsilon(m)$ is contained in Ω. In particular, the parabolic curve at z corresponding to $R_\varepsilon(m)$ must be contained in Ω.

Proposition III.21. *The orbit of any point in Ω is eventually mapped onto a parabolic curves contained in Ω.*

Proof. What remains to be shown is that any point whose orbit lands in a set $R_\varepsilon(m)$ for ε sufficiently small is contained in a parabolic curve. Hence it is sufficient to show that, in the local coordinates given by Equation (3.5), any point $(x, 0)$ with $|x|$ sufficiently small and

$$|\mathrm{Arg}(x) - \theta_u(m)| < \frac{\pi}{2k-2} \tag{3.13}$$

lies on a parabolic curve. By Hakim's proof of Theorem III.5 in [40] we know that, for $\varepsilon > 0$ sufficiently small, the parabolic curve at $(0, u)$ is a graph over

$$\{x \in \mathbb{C} \mid |x| < \varepsilon, \;\; |\mathrm{Arg}(x) - \theta_u(m)| < \frac{\pi}{2k-2}\},$$

of the form $v_u(x) = u + x^k h_u(x)$, with $\|h_u\| < 1$. By Hakim's proof these graphs vary continuously with h_u, hence for $|x|$ small enough any point $(x, 0)$ satisfying Equation

(3.13) lies on one of the parabolic curves. $\square$

If $k \geq 3$ then for each point in $\mathrm{Reg}(V)$ there exist $k-1$ parabolic curves. We investigate whether these curves lie in distinct Fatou components.

Note that the parabolic curves vary continuously with the basepoint in $\mathrm{Reg}(V)$. To each of these parabolic curves corresponds a real tangent vector α for which

$$F_k(\alpha) = -\lambda\alpha, \tag{3.14}$$

with $\lambda > 0$. The vector α is unique up to multiplication in $\mathbb{R}^+$. Let $z_0, z_1 \in \mathrm{Reg}(V)$ with corresponding parabolic curves C_0 and C_1 and real attracting tangent vectors α_0 and α_1. We say that $(z_0, C_0) \sim (z_1, C_1)$ if there exists a continuous map ϕ from $[0,1]$ to the set of pairs (z, α), with $z \in \mathrm{Reg}(V)$ and α a real attracting tangent vector to z, so that $\phi(0) = (v_0, \alpha_0)$ and $\phi(1) = (v_1, \alpha_1)$. Clearly if $(z_0, C_0) \sim (z_1, C_1)$ then C_0 and C_1 lie in the same Fatou component. The converse also holds.

Lemma III.22. *The parabolic curves C_0 and C_1 lie in the same Fatou component if and only if $(z_0, C_0) \sim (z_1, C_1)$.*

Proof. We only need to show that if C_0 and C_1 lie in the same Fatou component, then $(z_0, C_0) \sim (z_1, C_1)$. Assume therefore that $w_1 \in C_1$ and $w_2 \in C_2$, and suppose that w_1 and w_2 lie in the same Fatou component Ω. Let γ be a continuous real curve in Ω, starting at w_1 and ending at w_2. Since γ is compact, we know by Theorem III.7 that the sequence (F^n) converges uniformly on γ to a limit set $h(\gamma)$ contained in $\mathrm{Reg}(V)$.

Since $h(\gamma)$ is compact, we can find a uniform $\varepsilon > 0$, independent form $z \in h(\gamma)$, for which the statements in Lemmas III.19 and III.20 hold. Let $N \in \mathbb{N}$ be such that $\|F^n - h\|_\gamma < \varepsilon$ for all $n \geq N$. The curve $F^N(\gamma)$ lies in the invariant Fatou component Ω, and still starts at a point in C_0 and ends at a point in C_1. It follows from Lemma III.20 that $F^N(\gamma)$ must lie in the union of the open sets $W(m)$. Hence we can follow the real attracting direction of the sets $W(m)$, starting with the direction corresponding to C_0 and ending with the direction corresponding to C_1. $\square$

We give an simple family of polynomial endomorphisms of $\mathbb{C}^2$ for which we can easily determine whether the parabolic curves lie in distinct Fatou components. Let p and q be relatively prime, $k \geq 2$ be an integer, and define

$$F(z,w) = (z,w) + (z^p - w^q)^k \cdot (z,-w).$$

As noted in the proof of Theorem III.7, every point $(t^q, t^p) \in V = \{z^p - w^q = 0\}$ has a non-degenerate characteristic direction $(t^q, -t^p)$, and by the Remark III.6 there are $k-1$ parabolic curves tangent to this characteristic direction. Whether the curves $C_1, \dots C_{k-1}$ all lie in distinct Fatou components or not depends on the values of k, p and q.

Proposition III.23. *If $p \cdot q$ is divisible by $k-1$ then the curves $C_1, \dots, C_{k-1}$ all lie in distinct Fatou components. If $p \cdot q$ and $k-1$ are relatively prime then $C_1, \dots, C_{k-1}$ all lie in the same Fatou component.*

Proof. Let us pick a point $(t^q, t^p) \in \mathrm{Reg}(V)$ and define $T(z,w) = (z - t^q, w - t^p)$. Observe that k-th term of the homogeneous expansion of $G = T \circ F \circ T^{-1}$ equals to

$$G_k(z,w) = (pzt^{q(p-1)} - qwt^{p(q-1)})^k (t^q, -t^p).$$

We note that the orbits on the parabolic curves converge to (t^q, t^p) along real directions $\alpha \cdot (t^q, -t^p)$. By the equation (3.14) these real directions have to satisfy

$$G_k(\alpha \cdot (t^q, -t^p)) = \alpha^k (p+q)^k t^{kpq} (t^p, -t^q) = -\alpha \cdot (t^q, -t^p),$$

which gives

$$\alpha^{k-1} = \frac{-1}{(p+q)^k t^{kpq}}.$$

Hence we see that if $p \cdot q$ and $k-1$ are relatively prime, and therefore kpq and $k-1$ are also relatively prime, then if t moves in a full circle around the origin then the $k-1$ parabolic curves have been permuted in a full cycle. As the parabolic curves vary smoothly, this implies that the $k-1$ parabolic curves lie in the same Fatou component.

Suppose on the other hand that pq is divisible by $k-1$. Then following the parabolic curves as t makes one full circle around the origin leaves the parabolic curves invariant. However, loops around the origin generate the fundamental group of $\mathrm{Reg}(V)$, hence the monodromy group is trivial. It follows that two parabolic curves C_1 and C_2 touching V at the same point z lie in the same Fatou component if and only if $C_1 = C_2$. $\square$

CHAPTER IV

A long $\mathbb{C}^2$ without holomorphic functions

4.1 Introduction

A complex manifold X of dimension n is said to be a *long* $\mathbb{C}^n$ if it contains an increasing sequence of domains $X_1 \subset X_2 \subset X_3 \subset \cdots$ such that $X = \cup_{j=1}^{\infty} X_j$ and each X_j is biholomorphic to the complex Euclidean space $\mathbb{C}^n$. We know that every long $\mathbb{C}^n$ is homeomorphic to Euclidean space [15]. For $n = 1$, it is immediate that any long $\mathbb{C}$ is biholomorphic to $\mathbb{C}$. However, for $n > 1$, this class of complex manifolds is still very mysterious. The long standing question, whether there exists a long $\mathbb{C}^n$ which is not biholomorphic to $\mathbb{C}^n$, was answered by E. F. Wold [68] who constructed a long $\mathbb{C}^n$ that is not holomorphically convex, hence not a Stein manifold (in particular, it is not biholomorphic to $\mathbb{C}^n$). Wold's construction depends on his result from [67] on the existence of non-Runge Fatou–Bieberbach domains in $\mathbb{C}^n$ for $n > 1$. In spite of these interesting examples, the theory has really not been developed since. For instance, it remained unknown whether there exist long $\mathbb{C}^2$'s without nonconstant holomorphic functions, and whether there exist at least two non-equivalent non-Stein long $\mathbb{C}^2$'s.

In this chapter, we modify Wold's construction to obtain the following definitive result.

Theorem IV.1. *For every integer $n > 1$ there exists a long $\mathbb{C}^n$ without any non-*

constant holomorphic functions.

This is of interest in connection with the *union problem* for Stein manifolds: *is an increasing union of Stein manifolds always Stein?* For domains in $\mathbb{C}^n$, this question was raised in 1933 by Behnke and Thullen [7], and an affirmative answer was given in 1939 by Behnke and Stein [6]. Some progress on the general question was made by Stein [56] in 1956 and by Docquier-Grauert [19] in 1960. The first counterexample to the union problem in any dimension $n \geq 3$ was given by J. E. Fornæss [22] in 1976; he found an increasing union of balls whose limit is not holomorphically convex, hence not Stein. Soon after Fornæss and Stout [29] found a three-dimensional increasing union of polydiscs without any non-constant holomorphic function. Increasing unions of hyperbolic Stein manifolds were studied further by Fornæss and Sibony [25] and Fornæss [23]. For the connection with Bedford's conjecture, which says that the stable manifolds occuring in invertible holomorphic dynamical systems are biholomorphically equivalent to Euclidean space, see the survey on non-autonomous basins [1]. In dimension $n = 2$, the first counterexample to the union problem was the already mentioned result of Wold [68] from 2010, on the existence of a non-Stein long $\mathbb{C}^2$.

By combining the technique in the proof of Theorem IV.1 with those in Forstnerič paper [33, proof of Theorem 1.1], one can easily obtain the following result which generalizes Theorem IV.1 to holomorphic families of long $\mathbb{C}^n$'s. We leave out the details since it follows the one in [33] almost verbatim.

Theorem IV.2. *Let Y be a Stein manifold of dimension p, and let $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ be disjoint sequences in Y. For every integer $n > 1$ there exists a complex manifold X of dimension $p+n$ and a surjective holomorphic submersion $\pi\colon X \to Y$ with the following properties:*

- *the fiber $X_y = \pi^{-1}(y)$ over any point $y \in Y$ is a long $\mathbb{C}^n$;*

- *each fiber X_{a_j} for $j \in \mathbb{N}$ is biholomorphic to $\mathbb{C}^n$;*

- *none of the fibers X_{b_j} for $j \in \mathbb{N}$ admits a nonconstant holomorphic functions.*

If the base Y is $\mathbb{C}^p$, then X may be chosen to be a long $\mathbb{C}^{p+n}$.

There are no restrictions on the sequences $\{a_j\}$ and $\{b_j\}$ in Theorem IV.2, other that they be disjoint. In particular, one or both of them may be chosen (everywhere) dense in Y.

Corollary IV.3. *Let (X, π, Y) be as in Theorem IV.2, and assume that the sequence $\{b_j\}_{j\in\mathbb{N}}$ is dense in Y. Then, the following hold:*

- *Every holomorphic function on X is of the form $f \circ \pi$ for some $f \in \mathcal{O}(Y)$.*

- *For every holomorphic automorphism Φ of X there is an automorphism ϕ of Y such that $\Phi(X_y) = X_{\phi(y)}$ holds for any base point $y \in Y$.*

Proof. If $F \in \mathcal{O}(X)$, then F is constant on each fiber X_{b_j}. Since these fibers are dense in X, it is constant on every fiber, and hence is of the form $F = f \circ \pi$ for some $f \in \mathcal{O}(Y)$.

Assume that $\Phi \in \mathrm{Aut}(X)$. For every holomorphic function $f \in \mathcal{O}(Y)$, the composition $f \circ \pi \circ \Phi$ is a holomorphic function on X. By the first part, this function is constant on each fiber X_y, $y \in Y$. Since the holomorphic functions on Y separate any pair of points, it follows that the map $\pi \circ \Phi\colon X \to Y$ is also constant on each fiber. This means that for every $y \in Y$, we have $\Phi(X_y) = X_{y'}$ for some point $y' = \phi(y) \in Y$. Obviously, the map $\phi\colon Y \to Y$ defined in this way is a holomorphic automorphism of Y. $\square$

Several interesting questions on long $\mathbb{C}^n$'s remain open for further research.

Problem IV.4. Does there exist a long $\mathbb{C}^2$ which admits a non-constant holomorphic function, but is not Stein?

Problem IV.5. Does there exist a long $\mathbb{C}^n$ for any $n > 1$ which is a Stein manifold different from $\mathbb{C}^n$?

Problem IV.6. Does there exist many non-biholomorphic non-Stein long $\mathbb{C}^n$'s?

In dimensions $n > 2$, affirmative answer to Problem IV.6 is provided by the product $X = \mathbb{C}^p \times X^{n-p}$ for any $p = 1, \ldots, n-2$, where X^{n-p} is a long $\mathbb{C}^{n-p}$ without nonconstant holomorphic functions furnished by Theorem IV.1. Note that $\mathcal{O}(\mathbb{C}^p \times X^{n-p}) \cong \mathcal{O}(\mathbb{C}^p)$ is the algebra of functions coming from the base. Indeed, any example furnished by Theorem IV.2, with the base $Y = \mathbb{C}^p$ ($p \geq 1$) and B dense in $\mathbb{C}^p$, is of this kind.

In Section 4.3 we make a small progress towards Problem IV.6 for the remaining case $n = 2$, by constructing two non-biholomorphic non-Stein long $\mathbb{C}^2$'s. The main idea, behind the construction of our examples, was later adopted by the author and Forstnerič [9] to create a new invariant for complex manifolds.

It was observed by Wold [68, Theorem 1.2] that, if $X = \cup_{k=1}^{\infty} X_k$ is a long $\mathbb{C}^n$ and (X_k, X_{k+1}) is a Runge pair for every $k \in \mathbb{N}$, then X is biholomorphic to $\mathbb{C}^n$. It is somewhat surprising that $\mathbb{C}^n$ for $n > 1$ can also be represented as an increasing union of non-Runge Fatou–Bieberbach domains. We present such an example in Section 4.4.

4.2 Proof of Theorem IV.1

We begin by briefly recalling the construction of a long $\mathbb{C}^n$; see [68] or [32, Section 4.20] for further details.

Every complex manifold X which is a long $\mathbb{C}^n$ is determined by a sequence of injective holomorphic maps $\phi_k : \mathbb{C}^n \to \mathbb{C}^n$ ($k = 1, 2, 3, \ldots$). The elements of X are represented by equivalence classes of infinite strings $x = (x_i, x_{i+1}, \ldots)$, where $i \in \mathbb{N}$

and for every $k = i, i+1, \ldots$ we have $x_k \in \mathbb{C}^n$ and $x_{k+1} = \phi_k(x_k)$. Another such string $y = (y_j, y_{j+1}, \ldots)$ determines the same element of X if and only if one of the following possibilities hold:

- $i = j$ and $x_i = y_i$ (and hence $x_k = y_k$ for all $k > i$);
- $i < j$ and $y_j = \phi_{j-1} \circ \cdots \circ \phi_i(x_i)$;
- $j < i$ and $x_i = \phi_{i-1} \circ \cdots \circ \phi_j(y_j)$.

For each $k \in \mathbb{N}$ we consider the canonical inclusion $\psi_k \colon \mathbb{C}^n \hookrightarrow X$ which sends a point $z \in \mathbb{C}^n$ to the equivalence class of the string $x = (z, \phi_k(z), \ldots) \in X$. Set $X_k = \psi_k(\mathbb{C}^n)$. Clearly,

$$\psi_k = \psi_{k+1} \circ \phi_k, \qquad k = 1, 2, \ldots. \tag{4.1}$$

The following lemma is the key ingredient in the construction of the sequence $(\phi_k)_{k \in \mathbb{N}}$ determining a long $\mathbb{C}^n$ as in Theorem IV.1. We shall write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Lemma IV.7. *Let K be a compact set with nonempty interior in $\mathbb{C}^n$ for some $n > 1$, and let $a \in \mathbb{C}^n$. Then, there exists an injective holomorphic map $\phi \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that the polynomial hull of the set $\phi(K)$ contains the point $\phi(a)$.*

Proof. To simplify the notation, we consider the case $n = 2$; it will be obvious that the same proof applies in any dimension $n \geq 2$. We shall follow Wold's construction from [67, 68] up to a certain point, adding a new twist at the end.

Let M be a compact set in $\subset \mathbb{C}^* \times \mathbb{C}$ with the following properties:

1. M is a disjoint union of two smooth, embedded, totally real discs;
2. M is holomorphically convex in $\mathbb{C}^* \times \mathbb{C}$;
3. the polynomial hull $\widehat{M}$ of M of contains the origin $(0, 0) \in \mathbb{C}^2$.

A set with these properties was constructed by Stolzenberg [57] in 1966; it has been reproduced in [58, pp. 392–396] and in [67, Sec. 2].

Choose an injective holomorphic map $\theta\colon \mathbb{C}^2 \to \mathbb{C}^2$ (a *Fatou–Bieberbach map*) whose image $\theta(\mathbb{C}^2)$ is contained in $\mathbb{C}^* \times \mathbb{C}$ and is Runge in $\mathbb{C}^2$. The existence of such maps is well known [54]. For example, we may take the basin of an attracting fixed point of a holomorphic automorphism of $\mathbb{C}^2$ which fixes $\{0\} \times \mathbb{C}$.

Replacing the given compact set K by its polynomial hull $\widehat{K}$, we may assume that K is polynomially convex. Since $\theta(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$, the set $\theta(K)$ is also polynomially convex, and hence $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$-convex.

Choose a closed ball $B \subset \mathbb{C}^* \times \mathbb{C}$ contained in the interior of $\theta(K)$.

By [67, Lemma 3.2], there exists a holomorphic automorphism ψ of $\mathbb{C}^* \times \mathbb{C}$ such that $\psi(M) \subset B$, and hence $\psi(M) \subset \theta(\mathring{K})$. This depends on the main result of the Andersén–Lempert theory, as formulated by Forstnerič and Rosay [36, Theorem 1.1]. The main point is to deform M through an isotopy $M_t \subset \mathbb{C}^* \times \mathbb{C}$ $(t \in [0,1])$, consisting of compact $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$-convex sets, so that $M_0 = M$ and $M_1 \subset B$. In the case at hand, it suffices to shrink each of the two discs in M within itself until they are very small, and then drag them into B. Since the manifold $\mathbb{C}^* \times \mathbb{C}$ has the holomorphic density property, see [64], each diffeomorphism $M_0 \to M_t$ in such isotopy can be approximated uniformly on M (and even in the smooth topology on M) by holomorphic automorphisms of $\mathbb{C}^* \times \mathbb{C}$. The details of this argument can be found in [68].

It follows that the injective holomorphic map $\tilde{\phi} = \psi^{-1} \circ \theta\colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ satisfies $M \subset \tilde{\phi}(\mathring{K})$. Set $K' = \tilde{\phi}(K)$ and note that K' is a compact $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$-convex set. However, its polynomial hull $\widehat{K}'$ contains $\widehat{M}$, and hence the origin $(0,0) \in \mathbb{C}^2$.

Let $a \in \mathbb{C}^2$ be as in the lemma. If $\tilde{\phi}(a) \in \widehat{K}'$, we can take $\phi = \tilde{\phi}$ and we are done. Assume that this is not the case. Choose a point $a' \in \widehat{M} \setminus K'$; such exists since $K' \subset \mathbb{C}^* \times \mathbb{C}$, while $\widehat{M}$ contains $(0,0)$, and hence it contains points in $\mathbb{C}^* \times \mathbb{C}$

arbitrarily close to $(0,0)$. We apply the Andersén-Lempert theorem, more precisely [32, Theorem 4.10.8] to find a holomorphic automorphism τ of $\mathbb{C}^* \times \mathbb{C}$ which is close to the identity map on K' and satisfies $\tau(\tilde{\phi}(a)) = a'$. Such τ exists since the union of K' with a single point of $\mathbb{C}^* \times \mathbb{C}$ is $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$-convex, so it suffices to apply the cited result to an isotopy of injective holomorphic maps which equals the identity near K' and which moves $\tilde{\phi}(a)$ to a' in $\mathbb{C}^* \times \mathbb{C} \setminus K'$. Assuming that τ is sufficiently close to the identity map on K', we have $M \subset \tau(K')$, and hence $a' \in \widehat{M} \subset \widehat{\tau(K')}$.

Clearly, the map $\phi = \tau \circ \tilde{\phi} \colon \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ satisfies the conclusion of the lemma. $\square$

Proof of Theorem IV.1. Pick a compact set $K \subset \mathbb{C}^n$ with nonempty interior and a countable dense sequence $\{a_j\}_{j \in \mathbb{N}}$ in $\mathbb{C}^n$. Set $K_1 = \widehat{K}$. We identify $\mathbb{C}^n$ with the first set X_1 in the construction of a long $\mathbb{C}^n$.

Lemma IV.7 furnishes an injective holomorphic map $\phi_1 \colon \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\phi_1(a_1) \in \widehat{\phi_1(K_1)}.$$

Set $K_2 = \widehat{\phi_1(K_1)}$. Applying Lemma IV.7 to K_2 and the point $\phi_1(a_2) \in \mathbb{C}^n$ gives an injective holomorphic map $\phi_2 \colon \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\phi_2(\phi_1(a_2)) \in \widehat{\phi_2(K_2)} =: K_3.$$

From the first step, we also have $\phi_1(a_1) \in K_2$, and hence $\phi_2(\phi_1(a_1)) \in K_3$.

Continuing inductively, we get a sequence of injective holomorphic maps $\phi_j \colon \mathbb{C}^n \to \mathbb{C}^n$ for $j = 1, 2, \ldots$ such that, setting $\Phi_k = \phi_k \circ \cdots \circ \phi_1 \colon \mathbb{C}^n \to \mathbb{C}^n$, we have

$$\Phi_k(a_j) \in \widehat{\Phi_k(K)} \quad \text{for all } j = 1, \ldots, k. \tag{4.2}$$

In the limit manifold X (the long $\mathbb{C}^n$) determined by the sequence $\{\phi_j\}$, the $\mathcal{O}(X)$-hull of the initial set $K = K_1 \subset \mathbb{C}^n = X_1 \subset X$ clearly contains the set

$K_{k+1} = \Phi_k(K) \subset X_{k+1}$ for each $k = 1, 2, \ldots$. Hence, it follows from (4.2) that the hull $\widehat{K}_{\mathcal{O}(X)}$ contains the set $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n = X_1$. Since this set is everywhere dense in $\mathbb{C}^n$ by the assumption, every holomorphic function on X is bounded on $X_1 = \mathbb{C}^n$, and hence constant. By the identity principle, it follows that the function is constant on all of X. $\square$

4.3 Two examples of non-Stein long $\mathbb{C}^2$'s

The following lemma is a simple version of a standard result whose proof follows the proof of [32, Corollary 4.12.4.] almost verbatim if we use a version of Forstnerič-Rosay theorem for complex manifolds with the density property [53, Theorem 8].

Lemma IV.8. *Let $B_1, \ldots, B_n$ be pairwise disjoint sets in $\mathbb{C}^* \times \mathbb{C}$ which are biholomorphic images of a closed ball and whose union is $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$-convex. Let $\phi_j \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ $(j = 1, \ldots, n)$, such that the images $\phi_j(B_j)$ are also pairwise disjoint and their union is polynomially convex, then for every $\varepsilon > 0$, there exists an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that*

$$||\Phi(z) - \phi_j(z)|| < \varepsilon, \qquad z \in K_j, \quad j = 1, \ldots, m.$$

We will use this lemma in the construction of our examples.

Example IV.9. Let B and U be closed disjoint unit balls in $\mathbb{C}^2$. The first example can be constructed by a sequence of Fatou-Bieberbach maps $\phi_k : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ satisfying the following conditions $(n = 1, 2, \ldots)$

1. Polynomial hull of $\phi_n \circ \ldots \circ \phi_1(B)$ is not contained in $\phi_n(\mathbb{C}^2)$,

2. The set compact set $\phi_n \circ \ldots \circ \phi_1(U)$ is polynomially convex.

Using the same notation as in previous section, we can find a Fatou-Bieberach map $\theta : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$, such that $\theta(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$. Now we can use Lemma IV.8

to obtain an automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ satisfying $M \subset \psi \circ \theta(B)$ and which is almost the identity on $\theta(U)$, hence we set $\phi_1 := \psi \circ \theta$. Now we can follow the same procedure to produce ϕ_2. In the proof we only have to replace B with a small enough ball $B' \subset \theta(B)$ and U by $\theta(U)$. Using this construction we obtain desired sequence of Fatou-Bieberbach maps $\{\phi_n\}$.

Example IV.10. The idea behind the construction of this next example is to violate the condition (2) in the previous example. More precisely we will make sure that for every compact set $K \subset X$ we will have

$$\widehat{K}_{\mathcal{O}(X_n)} \cap X_n \backslash X_{n-1} \neq \emptyset \tag{4.3}$$

for all large $n > 0$.

To begin we take any countable and dense subset $A_1 \subset \mathbb{C}^2$ and we pick $a_1^1 \in A_1$. Let $r_1 > 0$ and let $B(a_1^1, r_1)$ denote a closed ball with radius r_1 centered at a_1^1. Following Wold's construction [67] we obtain a one-to-one holomorphic map $\phi_1 : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ satisfying

$$\widehat{U^1_{(1,1)}} \cap \mathbb{C}^2 \backslash \phi_1(\mathbb{C}^2) \neq \emptyset,$$

where $U^1_{(1,1)} := \phi_1(B(a_1^1, r_1))$.

Suppose we have constructed maps $\phi_1, \ldots, \phi_k$, we have chosen countable and dense sets $A_1, \ldots, A_k \subset \mathbb{C}^2$, we have picked $r_1 > \ldots > r_k > 0$ and points

$$a_i^l \in A_i \backslash \phi_{i-1}(A_{i-1}),$$

where $1 \leq i \leq l \leq k$ and $A_1 \backslash \phi_0(A_0) := A_1$.

In $k+1$'th step we add countable many points to $\phi_k(A_k)$ to get a countable and dense subset $A_{k+1} \subset \mathbb{C}^2$. For every $i = 1, 2, \ldots, k+1$ we pick a new point (different from those we have already chosen in previous steps)

$$a_i^{k+1} \in A_i \backslash \phi_{i-1}(A_{i-1}).$$

For every $1 \leq i \leq l \leq k+1$ we denote

$$b_i^l := \phi_k \circ \ldots \circ \phi_i(a_i^l),$$

where $b_{k+1}^{k+1} := a_{k+1}^{k+1}$. Now we choose $0 < r_{k+1} < r_k$ so small that

1. closed balls $B(b_i^l, r_{k+1})$ are pairwise disjoint and their union is polynomially convex,
2. sets $(\phi_k \circ \ldots \circ \phi_i)^{-1}(B(b_i^l, r_{k+1}))$ are contained in $B(a_i^l, r_l/2^{k+1})$.

We end this induction step by choosing an one-to-one map $\phi_{k+1} : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ satisfying

$$\widehat{U_{(i,l)}^{k+1}} \cap \mathbb{C}^2 \backslash \phi_{k+1}(\mathbb{C}^2) \neq \emptyset$$

for every $1 \leq i \leq l \leq k+1$, where $U_{(i,l)}^{k+1} := \phi_{k+1}(B(b_i^l, r_{k+1}))$

Such map can be constructed by first mapping $\theta : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ such that $\theta(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$ and then use Lemma IV.8 to find an automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which moves balls $\theta(B(b_i^l, r_{k+1}))$ on prescribed places, in our case to properly selected translates of previously defined set M, i.e. such that the polynomially convex hull of $\psi \circ \theta(B(b_i^l, r_{k+1}))$ has non-empty intersection with $\{0\} \times \mathbb{C}$.

Remark IV.11. Now that we have constructed two long $\mathbb{C}^2$'s, which we denote by X and Y, we still need to argue that they are not biholomorphic. Suppose they are biholomorphic, then we can simply identify them and we only have to argue that it is not possible for a complex manifold X to have two exhaustions $\{Y_k\}$ and $\{X_k\}$ (where X_j and Y_j are biholomorphic to $\mathbb{C}^2$) satisfying:

- there exists a compact set $U \subset X$ such that U is $\mathcal{O}(X_n)$-convex for every $n > 0$,
- for every compact set $U \subset X$ and for every $n > 0$ the $\mathcal{O}(Y_{n+1})$-hull of U is not contained in Y_n.

Since X_j and Y_j are biholomorphic to $\mathbb{C}^2$ we can find large enough compact sets $K_j \subset X_j$ and $L_j \subset Y_j$ satisfying:

- both $\{K_j\}$ and $\{L_j\}$ exhaust X,
- there exists a compact set $U \subset X$ such that U is $\mathcal{O}(K_n)$-convex for every $n > 0$,
- for every compact set $U \subset X$ and for every $n > 0$ the $\mathcal{O}(L_{n+1})$-hull of U is not contained in L_n.

By choosing right elements of sequences $\{K_j\}$ and $\{L_j\}$ and by changing the indexes we may assume that

$$U \subset K_1 \subset L_1 \subset K_2 \subset L_2 \subset K_3 \subset \dots,$$

and they exhaust X (here U is the set which is $\mathcal{O}(K_n)$-convex for every $n > 0$). Now we are in the contradiction, since U is not $\mathcal{O}(L_2)$-convex but its hull is contained in the $\mathcal{O}(K_3)$-hull of U. Therefore the existence of such two exhaustions is not possible, hence the above two examples are not biholomorphic.

4.4 Non-Runge Fatou-Bieberbach domains exhaust $\mathbb{C}^2$

In this section we prove that $\mathbb{C}^2$ can be exhausted by non-Runge Fatou-Bieberbach domains. The way this is done is by constructing a long $\mathbb{C}^2$ in a way that all maps

$\phi_k : \mathbb{C}^n \to \mathbb{C}^n$ have non-Runge image but they are almost the identity on large balls centered at the origin.

In general, if we only assume that images of ϕ_k contain large enough balls centered at the origin, we get an exhaustion of a long $\mathbb{C}^n$ with Runge images of balls. By [3, Theorem 3.4.] such a long $\mathbb{C}^n$ is biholomorphic to a Stein and Runge domain in $\mathbb{C}^n$, therefore the following problem is closely related to Problem IV.5.

Problem IV.12. *If a long $\mathbb{C}^n$ is exhausted by Runge images of balls, is it then biholomorphic to the complex Euclidean space?*

In particular this problem is very interesting, since we will prove in Section 4.5 that $\mathbb{C}^n$ is the only Stein manifold with the density property having an exhaustion by Runge images of balls.

Lemma IV.13. *Let $B_n \subset \mathbb{C}^2$ be a closed ball of radius n centered at the origin and $\mathcal{B}$ a small closed ball in the complement of B_n. For every $\varepsilon > 0$ there exist a Fatou-Bieberbach map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying*

- $||\phi - id|| < \varepsilon$ *on* B_n,
- $||\phi^{-1} - id|| < \varepsilon$ *on* B_n,
- $\phi(\mathcal{B})$ *is not polynomially convex.*

Proof. Let ψ be a Fatou-Bieberbach map whose image is not Runge [67]. Let K be a holomorphically convex compact subset of $\psi(\mathbb{C}^n)$ which not polynomially convex. Let $U \subset \mathbb{C}^n$ be a closed ball which contains $\psi^{-1}(K)$ and let V be a small closed ball in the complement of U such that $\psi(V)$ is Runge. We can find linear maps L and T satisfying $L(B_n) = V$ and $T(\mathcal{B}) = U$. By [32, Corollary 4.12.4] there exists an automorphism of $\mathbb{C}^n$, denoted by F, which is close to L on B_n and close to T on $\mathcal{B}$. The map $\psi \circ F$ still has to be corrected on B_n, so we approximate the inverse map

$(\psi \circ F)^{-1}$ on $\psi(V)$ by a global automorphism G. We obtain our final map is then given by $\phi = G \circ \psi \circ F$. $\square$

We choose an integer $n_1 > 0$ and a small ball $\mathcal{B}^1$ in the complement of B_{n_1}. To start the induction let ϕ_1 be a Fatou-Bieberbach map satisfying:

- $||\phi_1 - id|| < \varepsilon_1$ on B_{n_1},
- $||\phi_1^{-1} - id|| < \varepsilon_1$ on B_{n_1},
- $\phi(\mathcal{B}^1)$ is not polynomially convex.

Suppose we have constructed $\phi_1, \ldots, \phi_k$. In $k+1$'st step choose an integer $n_{k+1} > n_k$ such that

$$\phi_k(B_{n_k+1}) \cup \phi_k(\phi_{k-1}(B_{n_{k-1}+2})) \cup \ldots \cup \phi_k(\cdots(\phi_1(B_{n_1+k}))) \cup \phi_k(\mathcal{B}^k) \subset B_{n_{k+1}}$$

and choose a ball $\mathcal{B}^{k+1}$ in the complement of $B_{n_{k+1}}$. By Lemma IV.13 there exist a ϕ_{k+1} satisfying

- $||\phi_{k+1} - id|| < \varepsilon_{k+1}$ on $B_{n_{k+1}}$,
- $||\phi_{k+1}^{-1} - id|| < \varepsilon_{k+1}$ on $B_{n_{k+1}}$,
- $\phi_{k+1}(\mathcal{B}^{k+1})$ is not polynomially convex.

Now we construct a long $\mathbb{C}^2$ from the sequence of maps ϕ_k, and we denote it by X. It follows from our construction that the sequence $\psi_k(B_{n_k})$ is a Runge exhaustion of X. By (4.1) we have

$$\psi_{k+1}^{-1} \circ \psi_k = \phi_k$$

and if the sequence ε_k has been chosen to be summable, the sequence ψ_k will be convergent on every B_{n_j}. In the terminology of [18, Theorem 5.2] we have

$$(\psi_k, B_{n_k}) \to (\Psi, \mathbb{C}^2),$$

where $\Psi(\mathbb{C}^2) = X$ and Ψ is biholomoprhic.

4.5 A remark on Toth-Varolin conjecture

A Stein manifold is said to have the *density property* if the Lie algebra generated by all complete holomorphic vector fields is dense in the Lie algebra of all holomorphic vector fields. This class of manifolds were first introduced by Varolin [64] and they in particular contain many biholomorphic images of $\mathbb{C}^n$ (where n the dimension of the manifold). We have seen that there are nontrivial examples of complex manifolds with the density property, such as $\mathbb{C}^* \times \mathbb{C}$ and others, but the complex Euclidean space is the only contractible example that we know.

Conjecture IV.14. *(Toth-Varolin [61]) Affine spaces $\mathbb{C}^n$ are the only contractible Stein manifolds with the density property.*

This is still a wide open problem and surprisingly we can not give an answer even for some explicit manifolds. An interesting example is the Koras-Russell cubic $\mathcal{R}$, a hypersurface in $\mathbb{C}^4$ defined as a zero set of $P(x, y, z, w) = x + x^2y + z^2 + w^3$. Complex manifold $\mathcal{R}$ is diffeomorphic to $\mathbb{C}^3$ but not algebraically isomorphic to $\mathbb{C}^3$, see [48]. Recently it was proven by Leuenberger [46] that $\mathcal{R}$ has the density property, but we do not know whether $\mathcal{R}$ is biholomorphic to $\mathbb{C}^3$.

Theorem IV.15. *Let X be a Stein manifold with the density property. Then X is biholomorphic to $\mathbb{C}^n$ if and only if X can be exhausted with Runge images of the ball.*

Proof. The idea of the proof follows closely the proof of [66, Proposition 3.]. Let X be a Stein manifold with the density property and suppose that it has an exhaustion by Runge compact sets $\overline{K_j} \subset \mathring{K}_{j+1}$ which are biholomorphic to the ball. Let B_j denote the closed ball centered at the origin in $\mathbb{C}^n$ with radius j. We inductively construct a sequence of biholomorphisms that converges to a biholomorphic map $\Phi : X \to \mathbb{C}^n$.

Since K_1 is Runge and biholomorphic to a ball and X has the density property, there exist a Fatou-Bieberbach domain $\Omega_1 \subset X$ which is Runge in X and contains K_1, see [54, 35]. To start the induction, let $\varphi_1 : \Omega_1 \to \mathbb{C}^n$ be any Fatou-Bieberbach map. Assume that we have constructed biholomorphic maps $\varphi_j : \Omega_j \to \mathbb{C}^n$ and we have chosen indexes i_j satisfying:

1. $||\varphi_j - \varphi_{j-1}|| < \varepsilon_{j-1}$ on K_{j-1}, $\qquad (j > 1)$

2. $||\varphi_j^{-1} - \varphi_{j-1}^{-1}|| < \varepsilon_{j-1}$ on $B_{i_{j-1}}$, $\qquad (j > 1)$

3. $\varphi_j(K_j) \subset B_{i_j}$,

where $\varepsilon_j > 0$ and $j = 1, 2, \ldots, k$.

We need to construct a biholomorphic map φ_{k+1} that is close to φ_k on K_k and such that their inverses are close on B_{i_k}.

Since K_j exhaust X, there exist a $j > k$ such that $\varphi_k^{-1}(B_{i_k}) \subset K_j$, and we may assume by passing to a subsequence of K_j that $j = k + 1$. Since K_{k+1} is Runge and biholomorphic to a ball and X has the density property, there exists a Fatou-Bieberbach domain $\Omega_{k+1} \subset X$ which is Runge in X and contains K_{k+1}. Let $\Phi_{k+1} : \Omega_{k+1} \to \mathbb{C}^n$ be any Fatou-Bieberbach map. Since K_k and $\varphi_k^{-1}(B_{i_k})$ are Runge in Ω_{k+1}, we can first approximate $\Phi_{k+1}^{-1} \circ \varphi_k$ on K_k by an automorphism $F_{k+1} \in \mathrm{Aut}(\Omega_{k+1})$, see [2, 66], and finally we can approximate $(\Phi_{k+1} \circ F_{k+1}) \circ \varphi_k^{-1}$ on B_{i_k} by an automorphism $G_{k+1} \in \mathrm{Aut}(\mathbb{C}^n)$, to obtain $\varphi_{k+1} = G_{k+1}^{-1} \circ \Phi_{k+1} \circ F_{k+1}$ satisfying

1. $||\varphi_{k+1} - \varphi_k|| < \varepsilon_{k+1}$ on K_k

2. $||\varphi_{k+1}^{-1} - \varphi_k^{-1}|| < \varepsilon_{k+1}$ on B_{i_k}.

We finish the induction by choosing an integer $i_{k+1} > i_k$ satisfying
$\varphi_{k+1}(K_{k+1}) \subset B_{i_{k+1}}$.

In this way we get an infinite sequence of biholomorphic maps $\varphi_k : \Omega_k \to \mathbb{C}^n$. If ε_{k+1} are chosen to be summable, we have that the sequence of $\{\varphi_k\}$ converges uniformly on compact subsets of Ω to a holomorphic map $\Phi : \Omega \to \mathbb{C}^n$.

By choosing ε_{k+1}'s small enough we can guaranty that Φ is non-degenerate at every point, hence it is one-to-one. By Theorem 5.2 from [18] the sequence of $\{\varphi_k^{-1}\}$ converges uniformly to an inverse map Φ^{-1} and since this sequence converges on every closed ball in $\mathbb{C}^n$ we must have $\Phi(\Omega) = \mathbb{C}^n$. $\square$

CHAPTER V

A reconstruction theorem for complex polynomials

5.1 Introduction

Dynamical systems which arise in scientific problems can be represented, using mathematical methods, by equations which describe the evolution of the system over time. Understanding the dynamics from these equations may be difficult since they can rely on many different parameters. In order to simplify the problem we can suppress some of the parameters and obtain the results from these new equations. It is important to ask whether results obtained this way resemble the original dynamics.

In [59] Takens studied this kind of questions and proved the following theorem.

Theorem V.1. *Let M be a compact manifold of dimension m. For an open dense set of pairs (φ, y), $\varphi : M \to M$ a $\mathcal{C}^2$ diffeomorphism and $y : M \to \mathbb{R}$ a $\mathcal{C}^2$ function, the map $\Phi_{(\varphi,y)} : M \to \mathbb{R}^{2m+1}$, defined by*

$$\Phi_{(\varphi,y)}(x) = (y(x), y(\varphi(x)), \ldots, y(\varphi^{2m}(x)))$$

is an embedding.

This means that dynamical properties of the original dynamical system can be retrieved from the suppressed data. This theorem no longer holds if we do not assume that the map φ is injective. Takens also proved the following theorem.

Theorem V.2. *(Takens [60]) Let M be a compact manifold of dimension m and take $k > 2m$. For an open dense set of pairs (φ, y), $\varphi : M \to M$ a smooth endomorphism and $y : M \to \mathbb{R}$ a smooth function, there exist a map $\tau : \Phi_{(\varphi,y)}(M) \to M$ such that*

$$\tau \circ \Phi_{(\varphi,y)} = \varphi^{k-1},$$

where $\Phi_{(\varphi,y)} : M \to \mathbb{R}^k$ is defined as

$$\Phi_{(\varphi,y)}(x) = (y(x), y(\varphi(x)), \ldots, y(\varphi^{k-1}(x))).$$

Moreover if $k > 2m+1$ then the set $\Phi_{(\varphi,y)}(M)$ completely determines the deterministic structure of the time series produced by the dynamical system.

Recently Fornæss and Peters [24] studied how far the dynamical behavior of complex polynomials can be deduced from knowing only their real orbits. They proved that for an open and dense set of polynomials, measure theoretical entropy can be recovered from the real part of the orbits. In this chapter we extend their result to the set of all polynomials:

Theorem V.3. *Let $P(z)$ be a complex polynomial of degree $d \geq 2$ and $\nu = \Phi_*(\mu)$, where μ is the equilibrium measure for $P(z)$. Then the probability measure ν is invariant and ergodic. Moreover ν is the unique measure of maximal entropy* $\log d$*, except when the Julia set of P is contained in an invariant line, then its entropy equals zero.*

The main difficulty here is that in the exceptional case the induced map on the real orbits does not extend continuously to the natural compactification. The background and proof of this theorem are given in the following two sections.

The following theorem is our main result. It is in the same spirit as Theorem V.2 but for complex polynomials and the standard projection to $\mathbb{R}$.

Theorem V.4. *For a generic polynomial P of degree $d \geq 2$ there exist $M, N \in \mathbb{N}$ such that, setting $\Phi(z) = (\mathrm{Re}(z), \mathrm{Re}(P(z)), \ldots, \mathrm{Re}(P^N(z)))$, there exists a map $\tau : \Phi(\mathbb{C}) \to \mathbb{C}$ such that*

$$\tau \circ \Phi = P^M.$$

By *generic* we always mean a countable intersection of open dense sets. The constants M and N in the Theorem V.4 can be chosen to depend only on the degree of the polynomial (see Remark V.22), but their numerical values are yet to be determined. One may ask if there exists an upper bound for M, N which is independent of the degree of the polynomial (at least for generic polynomials). If such a bound exists, it would be very interesting to find the lowest possible upper bound.

5.2 Entropy

In this section we recall the definition and some basic properties of measure theoretical entropy. For further details the reader is referred to [52]. Entropy of a dynamical system is a non-negative extended real number (it can also take the value ∞) that is a measure of the complexity of the system. The notion of entropy was introduced by Kolmogorov and it was believed that it will allow to distinguish probabilistic dynamical systems and deterministic dynamical systems. There are several different ways how we can define a measure theoretical entropy. It turns out that these definitions are all equivalent if our space is a compact metric space and the dynamical system is induced by a continuous map. Choosing the right definition depends on the problem that we are trying to solve.

Let $(X, \mathcal{B}, \mu)$ be a probability space and $\mathcal{U} = \{A_i\}$ a finite partition, i.e. $\mu(X \setminus \cup_i A_i) = 0$ and $\mu(A_i \cap A_j) = 0$ for all $i \neq j$. Let $F : X \to X$ be a

μ-preserving map, i.e. $\mu(A) = \mu(F^{-1}(A))$. Let us define the n-th partition as

$$\mathcal{U}^n := \bigvee_{k=0}^{n-1} F^{-k}(\mathcal{U}).$$

By the definition every set $A \in \mathcal{U}^n$ is of the form

$$A = A_0 \cap F^{-1}(A_1) \cap \ldots \cap F^{-n+1}(A_{n-1}),$$

where $A_k \in \mathcal{U}$. The measure theoretical entropy $h_\mu(F)$ is now defined as

$$h_\mu(F) = \sup_{\mathcal{U}} \left(\lim_{n\to\infty} -\frac{1}{n} \sum_{A \in \mathcal{U}^n} \mu(A) \log \mu(A) \right). \tag{5.1}$$

This entropy is also called Kolmogorov-Sinai entropy.

Lemma V.5. *Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be probability spaces and let $T_i : X_i \to X_i$ be μ_i-preserving maps. Assume that there is a surjective map $\Phi : X_1 \to X_2$ with the properties $\Phi \circ T_1 = T_2 \circ \Phi$ and $\mu_2 = \Phi_*(\mu_1)$, then $h_{\mu_2}(T_2) \le h_{\mu_1}(T_1)$.*

Bowen [11] proposed the following slightly different definition of measure theoretic entropy.

Let X be a compact topological space, $F : X \to X$ a continuous map and μ an F-invariant ergodic probability measure. Let U be a neighborhood of the diagonal in $X \times X$, and let us define

$$B(x, n, U) = \{y \in X \mid \{(x, y), (F(x), F(y)), \ldots, (F^{n-1}(x), F^{n-1}(y))\} \subset U\}.$$

The entropy function

$$k_\mu(x, F) = \sup_{U} \left(\lim_{n\to\infty} -\frac{1}{n} \log(\mu(B(x, n, U))) \right). \tag{5.2}$$

is constant μ-almost everywhere, and the measure theoretic entropy is defined to be this constant. When X is a metric space, the sets $B(x, n, U)$ can be replaced by (n, ε)-balls

$$B(x, n, \varepsilon) = \{y \mid d(F^k(x), F^k(y)) < \varepsilon, k < n\}.$$

When X is compact, $F : X \to X$ a continuous map and μ an F-invariant ergodic probability measure we have $k_\mu(F) = h_\mu(F)$; [14].

The next three lemmas are standard results in this topic. Proofs of their slightly modified versions can be found in [24].

Let P be a self map on a measurable space X and let Q be a self map on measurable space Y such that there exist a surjective map $\Phi : X \to Y$ with the property

$$\Phi \circ P = Q \circ \Phi.$$

Lemma V.6. *If λ is invariant on X, then the push-forward ν is invariant on Y. If λ also is ergodic, then the push-forward, ν, is ergodic as well.*

Lemma V.7. *Let ν be an invariant ergodic measure on Y, $E \subset Y$ with $\nu(E) = 0$, and $N \in \mathbb{N}$. Suppose that $\#\{\Phi^{-1}(y)\} \leq N$ for any $y \in Y \backslash E$. Then ν is the push forward of an invariant ergodic measure on X.*

Lemma V.8. *Suppose we have the setting of Lemma V.7. Suppose μ is the unique invariant ergodic measure of maximal entropy on X. If $h_\mu(P) = h_\nu(Q)$, then $\nu = \Phi_*(\mu)$ and it is the unique measure of maximal entropy on Y.*

5.3 Exceptional polynomials

Given a complex polynomial P, let us define the set of real orbits as

$$\mathcal{A} := \left\{ \left(\mathrm{Re}(z), \mathrm{Re}(P(z)), \mathrm{Re}(P^2(z)), \ldots\right) | z \in \mathbb{C} \right\},$$

and observe that the map P is conjugate to the shift map

$$\rho : \mathcal{A} \to \mathcal{A}.$$

Note that there are always some points with the same real orbit.

Definition V.9. The set $\{(z, w) \in \mathbb{C}^2 \mid \mathrm{Re}(P^k(z)) = \mathrm{Re}(P^k(w)) \quad \forall k \geq 0\}$ is called the *mirrored set*. We say that w *mirrors* z if and only if (z, w) belongs to the mirrored set. A point z is called a *mirrored point* if and only if there exist a point $w \neq z$ that mirrors it.

The following lemma is the crucial result of [24].

Lemma V.10. *There exists an $N \in \mathbb{N}$, independent from z, $w \in \mathbb{C}$, with the following property: If $\mathrm{Re}(P^k(z)) = \mathrm{Re}(P^k(w))$ holds for every $0 \leq k \leq N$, then it holds for all $k \geq 0$.*

Hence $\mathcal{A}$ can be identified with a two dimensional subset of $\mathbb{R}^{N+1}$. Let us define a map $\Phi : \mathbb{C} \to \mathbb{R}^{N+1}$ as

$$\Phi(z) = \left(\mathrm{Re}(z), \mathrm{Re}(P(z)), \ldots, \mathrm{Re}(P^N(z))\right),$$

and let us denote its image by $\mathcal{S}$. The polynomial P induces a map Q on $\mathcal{S}$ with the property

$$\Phi \circ P = Q \circ \Phi.$$

Obviously the dynamics of $(\mathcal{A}, \rho)$ is completely determined by the dynamics of $(\mathcal{S}, Q)$. For a dense and open set of polynomials the map Q is continuous, but there are some cases where this is not true. See [24] for the following theorem.

Theorem V.11. *If P is exceptional, then the image $\Phi(\mathbb{C})$ is not closed in $\mathbb{R}^{N+1}$ and Q is not a globally continuous map.*

The following two types of polynomials are called exceptional.

Definition V.12. A complex polynomial $P(z)$ is strongly exceptional if it maps a vertical line to itself.

Definition V.13. A complex polynomial $P(z) = a_d z^d + \sum_{k \le d-2} a_k z^k$ is weakly exceptional if $a_d i^{d-1}$ is real but there is at least one integer k, with $0 \le k \le d-2$, for which $a_k i^{k-1}$ is not real.

We say P is non-exceptional if it is not strongly or weakly exceptional. The following theorem is the main result of [24].

Theorem V.14. *Let P be a non-exceptional complex polynomial of degree $d \ge 2$ and $\nu = \Phi_*(\mu)$, where μ is the equilibrium measure for $P(z)$. Then the probability measure ν is invariant and ergodic. Moreover it is the unique measure of maximal entropy,* $\log d$.

The problem of generalizing this theorem lies in Theorem V.11. For exceptional polynomials the set $\mathcal{S}$ is not closed and Q is not continuous. On the other hand, Q is still continuous when restricted to $\Phi(U)$, where $U \subset \mathbb{C}$ is bounded; see the proof of Theorem 2.8 in [24]. By observing that in this case $k_\nu(Q)$ is well defined and that we have an equality between $k_\nu(Q)$ and $h_\nu(Q)$, one can slightly modify the original proof of Fornæss and Peters to obtain a full result.

Lemma V.15. *Let $P(z)$ be a strongly exceptional holomorphic polynomial of degree $d \ge 2$ whose Julia set is contained in the invariant vertical line. Then the probability measure ν is invariant and ergodic. The measure ν is a Dirac measure at the origin, hence $h_\nu(Q) = 0$.*

Proof. The measure ν is invariant and ergodic by Lemma V.5. It is also clear that ν is supported only in the point $\{0\}$. We shall compute the measure theoretic entropy using the definition (5.1). Take any open cover $\mathcal{U}$ of $\mathcal{S}$. If A is contained in $\mathcal{U}^n$ and does not contain 0 then $\mu(A)\log\mu(A) = 0$ since $\mu(A) = 0$. On the other hand if $0 \in A$, then $\mu(A) = 1$ and therefore $\mu(A)\log\mu(A) = 0$. Hence we have obtained $h_\nu(Q) = 0$. □

Lemma V.16. *Let $P(z)$ be a holomorphic polynomial of degree $d \geq 2$ with an invariant vertical real line $L \supset P(L)$. If μ denotes the invariant ergodic measure, then $\mu(L) = 0$ or $\mu(L) = 1$.*

Proof. Let $A = \mathbb{C}\backslash L$. Since μ is ergodic, we need to show that $\mu(A\backslash P^{-1}(A)) = 0$ and $\mu(P^{-1}(A)\backslash A) = 0$. Observe that $P^{-1}(A) \subset A$ and hence we only need to prove the first equality. Since μ is also an invariant measure, we obtain $\mu(A) = \mu(P^{-1}(A))$. Hence, the first equality holds, and by the ergodicity $\mu(A)$ is either 0 or 1. □

5.4 Proof of Theorem V.3

Let J_P denote the Julia set of $P(z)$ and let $J_Q = \Phi(J_P)$. We can naturally obtain J_Q from $(\mathcal{S}, Q)$ in the following way. Take a generic point $x \in \mathcal{S}$ and take any point $z \in \Phi^{-1}(x)$. We know that the limit set of all preimages of z under P equals J_P. Since $\Phi \circ P = Q \circ \Phi$, we can conclude that the limit set of preimages of x under Q equals to $\Phi(J_P)$. This means that we can obtain J_Q directly from the given data, without knowing J_P and P.

The map $Q : J_Q \to J_Q$ is a continuous map (using the subspace topology in $\mathbb{R}^{N+1}$). Since ν is a push-forward of μ, it is supported on the compact set J_Q, and by Lemma V.6 it is invariant and ergodic. Under these conditions, we get the following equality

$$h_\nu(Q) = k_\nu(Q).$$

Let us assume that P is strongly exceptional with an invariant vertical line L, and that J_P is not contained in L, hence $\mu(L) = 0$. The invariance and ergodicity of ν

are given by Lemma V.6. If we prove that $h_\nu(Q) = \log d$, then Lemmas V.7, V.8 will finish the proof.

Let $X \subset \mathcal{S}$ denote the finite set of points $\Phi(z)$ where z is either an isolated mirror point, a singular point of the one dimensional mirror set, an isolated cluster point on the diagonal for mirrored points, or a critical point in either the Julia set or a parabolic basin. We include in X the full orbit for any periodic point in X.

Let Y' be the union of the vertical lines passing through the points in X. By Lemma V.16 we get $\mu(Y' \cup L) = 0$. Let us denote $Y = Y' \cup L$.

Let $k >> 1$ be an integer. Since $\mu(Y) = 0$, there exists an $\delta_0 > 0$ so that $N_{\delta_0}(Y)$, the δ_0-neighborhood of Y, satisfies $\mu(N_{\delta_0}(Y)) < \frac{1}{k}$. Let z now be a *generic point* in J_P as defined in [12]. Then it follows that

$$\lim_{n\to\infty} \frac{\#\{0 \le j < n \mid P^j(z) \in N_{\delta_0}(Y)\}}{n} < \frac{1}{k},$$

where # count the number of elements of a given set. As Y has measure 0, we may also assume that the orbit of z never hits the set Y.

Write $\overline{x} = (x_0, \dots, x_N) = \Phi(z)$, and let us estimate the entropy function for $\Phi_*(\nu)$ at $\overline{x}$.

For $\ell < n$ define balls in J_P,

$$B'(n, \ell, \epsilon) := \bigcap_{r=0}^{n-\ell} \{w \in J_P : \rho(\Phi(P^r(w)), \overline{x}_{\ell+r}) < \epsilon\}.$$

Here we use the metric $\rho(\overline{x}, \overline{y}) = \max_{0 \le i \le N} |x_i - y_i|$ in $\mathbb{R}^{N+1}$.

The idea is to show that for $\epsilon > 0$ small enough,

$$\mu(B'(n, \ell - 1, \epsilon)) \le \frac{\mu(B'(n, \ell, \epsilon))}{d}, \tag{5.3}$$

for all $\ell \le n$, except for those bad ℓ's where $N_{\delta_0}(Y) \cap B'(n, \ell, \epsilon) \ne \emptyset$. On the other hand, we now that there are at most $\frac{n}{k}$ bad ℓ's, and for those we have

$$\mu(B'(n,\ell-1,\epsilon)) \leq \mu(B'(n,\ell,\epsilon)).$$

It follows that the μ-measure of $B'(n,0,\epsilon)$ is at most $C(1/d)^{n-\frac{n}{k}}$. Therefore the metric entropy of ν is at least $\log d$. To obtain inequality (5.3) one can simply follow the proof of Lemma 4.10 from [24].

When P is weakly exceptional simply take $L=\emptyset$. Lemma V.15 now completes the proof of our theorem.

5.5 Mirrored points

In order to prove Theorem V.4, we need a better understanding of the mirrored set

$$\{(z,w)\in\mathbb{C}^2 \mid \mathrm{Re}(P^k(z)) = \mathrm{Re}(P^k(w)) \quad \forall k\geq 0\}$$

and what happens with mirrors under iteration. Suppose that z mirrors w. We say that the *mirror breaks* at time n if $P^k(z)\neq P^k(w)$ for $k<n$ and $P^n(z)=P^n(w)$. We first observe that for a polynomial P of degree $d\geq 2$, the map

$$z\to(\mathrm{Re}(z),\mathrm{Re}(P(z)),\mathrm{Re}(P^2(z)),\mathrm{Re}(P^3(z)),\ldots),$$

is not injective. This follows from the following simple observation. Take any point z close to the infinity and observe its preimages $z_k\in P^{-1}(z)$. If $\mathrm{Re}(z_k)\neq\mathrm{Re}(z_j)$ for all $j\neq k$, then by moving our initial point z along the level curve of the Green's function, we can achieve that two preimages satisfy $\mathrm{Re}(z_k)=\mathrm{Re}(z_j)$. Hence z_j mirrors z_k which proves that the set mirrored points is never empty. In this example the mirror between z_k and z_j breaks at time 1 since $P(z_j)=P(z_k)$, therefore we may ask if all mirrors break and if they do, is there an upper bound on the time. Observe that in general mirrors do not need to brake. For example, observe that for a real polynomial P a point z is mirrored by $\bar{z}$ and generic mirrors never break. Another example is a polynomial with two fixed points lying one above the other.

On the other hand, Theorem V.4 states that for a generic polynomial P all mirrors break at time M or earlier, where this constant depends only on the degree of P.

Lemma V.17. *If $P(z) = a_d z^d + \ldots + a_1 z + a_0$ is non-exceptional polynomial and $a_d \notin \mathbb{R}$, then in a small neighborhood of infinity all mirrors break at time 1.*

Proof. Near infinity there exists a holomorphic change of coordinates of the form $\xi := z + \mathcal{O}(1)$ that conjugates $P(z)$ to the map $\xi \to a_d \xi^d$. It follows from Lemma 3.4 in [24] that near infinity any mirrored pair must lie on a level curve $\{|\xi| = R\}$. Observe that such level curves for P are of form $\{z = Re^{i\varphi} + \mathcal{O}(1)\}$, hence near infinity every a mirrored pair (z, w) must satisfy

$$w = \bar{z} + \mathcal{O}(1). \tag{5.4}$$

It follows that

$$P(w) = a_d w^d + \mathcal{O}(w^{d-1}) = a_d \bar{z}^d + \mathcal{O}(\bar{z}^{d-1}).$$

We also have

$$P(z) = a_d z^d + \mathcal{O}(z^{d-1}),$$

hence by 5.4 for $P(z)$ and $P(w)$ instead of z and w we have

$$P(w) - \overline{P(z)} = (a_d - \bar{a}_d)\bar{z}^d + \mathcal{O}(\bar{z}^{d-1}),$$

which converges to infinity as $z \to \infty$. Hence if z mirrors w then $P(z) = P(w)$. $\square$

In what follows, we will be using only basic results from the real algebraic set theory such as: the increasing sequence of ideals generated by real polynomials always stabilizes, algebraic set has only finitely many connected components, the projection of a real algebraic set is a semi-algebraic set, the intersection of two algebraic sets is an algebraic set. The reader is referred to the standard text [10]. We can identify the set of all complex polynomials of degree at most d with $\mathbb{C}^{d+1}$. Let us define

$$X = \{(z, w, P) \mid \mathrm{Re}(P^k(z)) = \mathrm{Re}(P^k(w)) \quad \forall k \geq 0\} \subset \mathbb{C}^2 \times \mathbb{C}^{d+1}.$$

As was observed by Fornæss and Peters [24], there exist an N such that

$$X = \{(z, w, P) \mid \mathrm{Re}(P^k(z)) = \mathrm{Re}(P^k(w)) \quad \forall k \leq N\}$$

and hence X is a real algebraic set.

From here on we will denote the intersection of a set $U \subset X$ with the P-fiber $\mathbb{C}^2 \times \{P\}$ with subscript P, for example $U_P = (\mathbb{C}^2 \times \{P\}) \cap U$.

Let us define the map $\Psi(z, w, P) = (P(z), P(w), P)$. Observe that Ψ is a well-defined map on X that maps X_P into itself. By iterating Ψ we obtain a decreasing sequence

$$X \supset \Psi(X) \supset \Psi^2(X) \supset \dots$$

Since all $\Psi^k(X_P)$ are closed sets, it follows that they are also algebraic and hence the sets $\Psi^k(X)$ are also algebraic. Therefore $\Psi^{\tilde{M}}(X) = \Psi^{\tilde{M}+1}(X)$ for all large $\tilde{M}$. For such $\tilde{M}$ we define

$$\mathcal{X} := \Psi^{\tilde{M}}(X).$$

For a given polynomial P we will need a better description of points on the diagonal $\Delta := \{(z, z) \mid z \in \mathbb{C}\}$ that can be approximated by mirrored pairs, i.e. by (z, w) where

$z \neq w$. Let us write $P(z) = a_d z^d + \ldots + a_1 z + a_0$ and observe that

$$P(z) - P(w) = (z-w)(a_d \sum_{k=0}^{d-1} z^k w^{d-1-k} + a_{d-1} \sum_{k=0}^{d-2} z^k w^{d-2-k} + \ldots + a_2(z+w) + a_1),$$

and similarly

$$P^n(z) - P^n(w) = (z-w)R_n(z,w)$$

where the polynomial R_n satisfies

$$R_n(z,z) = \frac{d}{dz}P^n(z)$$

Then the condition $\mathrm{Re}(P^n(z) - P^n(w)) = 0$ implies

$$\mathrm{Re}(R_n(z,w))\mathrm{Re}(z-w) - \mathrm{Im}(R_n(z,w))\mathrm{Im}(z-w) = 0.$$

For a given polynomial P we define

$$Q_0(z,w) = \mathrm{Re}(z-w),$$
$$Q_n(z,w) = \mathrm{Im}\left(\frac{P^n(z) - P^n(w)}{z-w}\right) = \mathrm{Im}(R_n(z,w)),$$

and

$$D_P = \bigcap_{n \geq 0} \{Q_n = 0\}.$$

Observe that D_P is a real algebraic set and that for every polynomial P we have $D_P \supseteq \overline{X_P \backslash \Delta}$. This follows from a simple fact that every mirrored pair $(z,w) \in X_P \backslash \Delta$, where $z \neq w$, satisfies equations $Q_n = 0$ for all $n \geq 1$. For points on the diagonal we obtain simplified equations

$$Q_1(z,z) = \mathrm{Im}(P'(z)) = 0,$$

$$Q_2(z,z) = \mathrm{Im}(P'(P(z)) \cdot P'(z)) = 0,$$

and for general $n \geq 1$

$$Q_n(z,z) = \mathrm{Im}\left(\frac{d}{dz}P^n(z)\right) = 0.$$

With a slight abuse of notation we consider the intersection of zero sets of polynomials Q_n as a subset of $\mathbb{C}$. These equations tells us that critical points, i.e. $P'(z) = 0$, are always contained in the intersection. In general we may also have other points in the intersection. For example if we have a polynomial with real coefficients, then the real line is always in the intersection. On the other hand, if we look for solutions of

$$\{Q_1 = Q_2 = Q_3 = 0\}$$

which are not critical points of P, then we end up with points which satisfy

$$\{\mathrm{Im}(P'(z)) = P'(P(z)) = 0\}$$

or

$$\{\mathrm{Im}(P'(z)) = \mathrm{Im}(P'(P(z))) = \mathrm{Im}(P'(P^2(z))) = 0\}.$$

One can see that there is an open and dense set polynomials for which the above two sets are empty.

Remark V.18. *For an open and dense set of polynomials P we have an equality*

$$D_P \cap \Delta = \{(z,z) \mid P'(z) = 0\}.$$

Remark V.19. *An open and dense set of polynomials has no Siegel disks*: Suppose that P has a Siegel disk. Then we can take an arbitrarily small perturbation of the polynomial for which the neutral periodic point at the center of the Siegel disk becomes attracting. As attracting periodic points are stable under sufficiently small perturbations and since the number of periodic attracting cycles is bounded by the degree of the polynomial minus 1, the set of polynomials for which the number of distinct attracting cycles is locally maximal is open and dense, and these polynomials do not have Siegel disks.

Remark V.20. *The set of non-exceptional polynomials whose leading coefficient is non-real and whose critical set does not contain any periodic cycle, is open and dense.* The only less trivial fact here is that the set of polynomials whose critical set does not contain any periodic cycle, is open and dense. This follows from the fact that $\{P'(z) = P'(P(z)) = 0\}$ is empty for open and dense set of polynomials.

Remark V.21. *For a generic polynomial P, no two periodic points are mirrored.* For any non-exceptional polynomial, a periodic point can be mirrored only by another pre-periodic point. Observe that if we conjugate a polynomial P by a rotation, i.e. $P_\varphi(z) = e^{i\varphi}P(ze^{-i\varphi})$, there are only countably many $\varphi \in [0, 2\pi]$ for which there are two periodic points lying one above the other.

5.6 Proof of Theorem V.4

With a slight abuse of notation we consider a P-fiber of $\mathcal{X}$ as a subset of $\mathbb{C}^2$, i.e. $\mathcal{X}_P \subset \mathbb{C}^2 \cong \mathbb{C}^2 \times \{P\}$.

For given polynomial P let us define

$$Y_n = \{(z, w) \mid \mathrm{Re}(P^k(z) - P^k(w)) = 0 \quad \forall k < n, \quad P^n(z) = P^n(w)\}.$$

The algebraic sets Y_n form an increasing sequence $Y_n \subset Y_{n+1} \subset X$ and satisfy $P(Y_{n+1}) \subset Y_n$, where $P(z, w) := (P(z), P(w))$. As before, the decreasing sequence of algebraic sets $\overline{\mathcal{X}_P \backslash Y_n}$ stabilizes at some $M := M(P)$, and we define

$$\mathcal{Y}_P = \overline{\mathcal{X}_P \backslash Y_M}.$$

A generic polynomial P of degree at most d satisfies the assumptions of Remarks V.18 through V.21. We will prove that under this assumptions on P the set $\mathcal{Y}_P$ is an empty set. Since $\mathcal{X}$ is Ψ invariant, it follows that $\mathcal{Y}_P$ is P invariant. It follows from Lemma 3.10 of [24] that for a non-exceptional polynomial P, which has at least one non-real coefficient, the set $\mathcal{Y}_P$ is a one dimensional real algebraic set, and by Lemma V.17 it is compact. Since $\mathcal{Y}_P$ is real algebraic, it has only finitely many connected components. Furthermore, since $\mathcal{Y}_P$ is P invariant, one can conclude that isolated points of $\mathcal{Y}_P$ can only be pairs of mirrored periodic points.

Since P satisfies the assumption of Remark V.21, the isolated points of $\mathcal{Y}_P$ can only be pairs (z, z) where z is a periodic point of P. By the assumption of Remark V.20, the set of critical points does not contain any periodic cycle. Since

$$\mathcal{Y}_P = \overline{\mathcal{X}_P \backslash Y_M} \subset \overline{X_P \backslash \Delta} \subset D_P$$

and since by Remark V.18

$$\mathcal{Y}_P \cap \Delta \subset D_P \cap \Delta = \{(z, z) \mid P'(z) = 0\},$$

we can conclude that $\mathcal{Y}_P \cap \Delta = \emptyset$, hence $\mathcal{Y}_P$ contains only mirrored pairs whose mirror never breaks.

Since $\mathcal{Y}_P$ is closed, invariant under P, and contains no periodic points, it follows that $\mathcal{Y}_P \subset J_P \times J_P$, where J_P is the Julia set of P. Indeed, if $\mathcal{Y}_P$ enters the Fatou component of an attracting or parabolic periodic point z_0, it follows that $(z_0, z_0) \in \mathcal{Y}_P$. Since $\mathcal{Y}_P$ is algebraic, it has only finitely many connected components, and since it is also invariant, there exists $k \geq 1$ such that every connected component is invariant for P^k. Let $V \subset \mathcal{Y}_P$ be any connected component and let $p_1 : \mathbb{C}^2 \to \mathbb{C}$ be a projection to the first coordinate. The projection of an algebraic set is a semi-algebraic set which is a triangulable space. Lefschetz fixed point Theorem states that every continuous self-map f of compact triangulable space X with non-zero Lefschetz number

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \mathrm{Tr}(f_* \mid H_k(X, \mathbb{Q}))$$

has a fixed point.

If $p_1(V)$ is contractible, then $H_k(p_1(V), \mathbb{Q}) = 0$ for $k \geq 1$, hence

$$\Lambda_{P^k} = \mathrm{Tr}(P^k_* \mid H_0(p_1(V), \mathbb{Q})) = 1.$$

It follows that P^k has a fixed point which is a contradiction with our earlier observation that $\mathcal{Y}_P$ has no periodic points.

Suppose that $p_1(V)$ is not contractible. Then there is a P^k-invariant bounded Fatou component Ω whose boundary is a subset of $p_1(V)$. By replacing V with $P^n(V)$ for some $n < k$ if necessary we may assume that Ω contains a critical point. A map $P^k : \partial\Omega \to \partial\Omega$ is a holomorphic self-map of a closed curve, hence the Lefschetz number is non-zero since it equals to 1 minus the winding number of $P^k(\partial\Omega)$. It follows that P^k has a fixed point in $\partial\Omega \subset p_1(V)$ which is again a contradiction.

Remark V.22. Let us argue why N and M from the Theorem V.4 can be chosen to depend only on the degree of a polynomial. Observe that N comes from the definition of the set X, and that $\tilde{M}$ comes from the definition of the set $\mathcal{X}$. It is clear from the construction of these sets that N and $\tilde{M}$ depend only on the degree of a polynomial. From the definition of the set $\mathcal{Y}_P$ we get an integer M that may depend on P, and we have seen that for a generic P the set $\mathcal{Y}_P$ is empty. We claim that for a generic

P the constant M is bounded above by $\tilde{M}$, hence $\mathcal{X}_P = \Delta$. Suppose that P is a polynomial for which $\mathcal{Y}_P = \emptyset$ and let $(z, w, P) \in \mathcal{X}_P$ such that $z \neq w$. If $M > \tilde{M}$ then by the definition of M we have $(P^M(z), P^M(w)) \in \Delta$, which contradicts the property $\Psi(\mathcal{X}) = \mathcal{X}$.

5.7 Examples and concluding remarks

The following two examples show that the set of all polynomials satisfying Theorem V.4 is not open, but it has a non-empty interior.

Example V.23. Take polynomials $P_{\varepsilon,\varphi}(z) = z + e^{i\varphi}(z^2 + \varepsilon)$, where $\varepsilon > 0$. Observe that for a generic $\varphi \in [0, 2\pi]$ the polynomial $P_{0,\varphi}(z) = z + e^{i\varphi} z^2$ satisfies all the conditions in the proof of Theorem V.4, hence there exist $P_{0,\varphi}$ for which Theorem V.4 holds. Now observe that $P_{\varepsilon,a}$ converges to $P_{0,a}$ and that every $P_{\varepsilon,a}$ has two fixed points $-i\sqrt{\varepsilon}$ and $+i\sqrt{\varepsilon}$ with the same real parts.

Example V.24. Let $d \geq 2$ and $P(z) = iz^d - e^{i\psi}c$ where $c := c(d)$ is some large positive constant. We will prove that P and any small perturbation of P satisfies Theorem V.4.

Let us define $R_c = \sqrt[d]{c + c^{\frac{3}{2d}}}$, $r_c = \sqrt[d]{c - c^{\frac{3}{2d}}}$ and for every $k \in \{0, 1, \ldots, d-1\}$ we define

$$V_k = \left\{ z \in \mathbb{C} \mid r_c < |z| < R_c, \quad \left| d \arg z - \psi - 2k\pi - \frac{\pi}{2} \right| < \left| \arccos \frac{R_c}{c} - \frac{\pi}{2} \right| \right\}.$$

Observe that $|\arccos \frac{R_c}{c} - \frac{\pi}{2}|$ tends to 0 when c is sent to infinity. Now define

$$V = \bigcup_{k=0}^{d-1} V_k.$$

We will see that for sufficiently large c our map P will map the complement of V into the complement of a ball of radius R_c centered at the origin. It is easy to verify that P maps the ball of radius r_c to the complement of the ball of radius R_c, and that P maps to the complement of the ball of radius R_c to itself. Observe that

$$|P(re^{i\varphi})| = \sqrt{r^{2d} + c^2 + 2cr^d \sin{(d\varphi - \psi)}}$$

and that one obtains a minimum when $r^d = -c\sin{(d\varphi - \psi)}$, hence

$$|P(re^{i\varphi})| \geq c|\cos d\varphi - \psi|.$$

A φ for which $|P(re^{i\varphi})| > R_c$ must satisfy $|\cos{(d\varphi - \psi)}| > \frac{R_c}{c}$.

Taking c sufficiently large we can assume that the critical point has an unbounded orbit, hence the Julia set J_P is totally disconnected and it is contained in V. Observe that every V_k contains a fixed point. By taking $\psi = \frac{\pi}{\sqrt{5}}$ and by increasing c if necessary we may assume that the projections of V_k to the real line are pairwise disjoint.

Using symbolic dynamics we can uniquely identify every point from the Julia set by elements in $\{1, \ldots, d\}^{\mathbb{N}}$. Every point $z \in J_P$ can be represented with a unique sequence $\{a_n\}_{n\geq 0} \subset \{1, \ldots, d\}^{\mathbb{N}}$ satisfying the property $P^n(z) \in V_{a_n}$. This implies that points in the Julia set can not mirror each other, moreover it follows from Lemma V.17 that points in the Julia set have no mirrors. As a consequence we obtain that the set $\mathcal{Y}$, defined in the proof of the Theorem V.4, is an empty set. Hence every mirror breaks at time $k \leq M$.

By Mañé, Sad and Sullivan [49] our polynomial P is a J-stable polynomial when c is large. Hence the Julia of slightly perturbed polynomial P resembles the Julia set of P. Moreover, their Julia sets are topologically conjugate. Since the above equations are only slightly perturbed when P is perturbed, we can conclude that there exists an open set of polynomials satisfying Theorem V.4.

Question V.25. *Is there an open and dense set of polynomials satisfying Theorem V.4?*

We have seen that for a generic P we have $(\mathcal{X}\backslash\Delta) \cap \mathbb{C}^2 \times \{P\} = \emptyset$. If one could prove that $\overline{\mathcal{X}\backslash\Delta} \cap \mathbb{C}^2 \times \{P\} = \emptyset$ for a dense set of polynomials, than the answer to this question would be positive, since the projection of $\overline{\mathcal{X}\backslash\Delta}$ to the polynomial coordinate is a semi-algebraic set.

In this chapter we have considered only complex polynomials together with a projection to the real line, so it is natural to ask to what extent these ideas could be generalized. The first thing we can try to do is to replace polynomials with rational functions, since their dynamics is also well understood. If we would again use the projection to the real line, we would quickly be forced to stop, since it is not clear what would be the proper value of the point at infinity under the real projection. This shows that a real projection is not the best choice for general treatment of these kind of problems. Instead of real orbits we can look at the Fubini-Study distances of iterates of a point from some given base point. For rational functions $f : \mathbb{C} \to \mathbb{C}$ one can simply assume that the distance from the origin has the following form

$$d_n(z) := \frac{|f^n(z)|^2}{1 + |f^n(z)|^2},$$

where the distance between the origin and the point at the infinity is equal to 1. Using the fact that this functions $d_n(z)$ are real analytic, we can achieve an analogue of Lemma V.10, saying that any two points $w \neq z$ satisfying $d_n(z) = d_n(w)$ for all $n < N$, they satisfy also $d_n(z) = d_n(w)$ for all n. The same can be said if we replace rational functions with holomorphic endomorphisms of $\mathbb{P}^k$ and we look at the Fubini-Study distances of iterates of a point from $[0 : 0 : \ldots : 0 : 1]$. The Fubini-Study distance between $[0 : 0 : \ldots : 0 : 1]$ and a point from the line at infinity $\{z_k = 0\}$ is equal to 1, therefore in order to be able to distinguish points, we would have to exclude, at least at the beginning, those holomorphic endomorphisms for which the line $\{z_k = 0\}$ is forward invariant. Observe that we did the same with strongly exceptional polynomials, since they map the imaginary axis to itself. To achieve similar theorems as Theorem V.3 and Theorem V.4 we would first need to study the mirrored set (the set of all pairs of points with the same distance orbit). This is a much harder problem then it was in the case of complex polynomials.

We hope that in the near future these ideas will enable us to generalize our results at least to rational functions or, more generally, to holomorphic endomorphisms of projective spaces.

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Slovenski povzetek

Dinamični sistem je sistem, ki opisuje kako s časom dana opazljivka prehaja iz enega stanja v drugo, npr. nihanje urnega nihala. Mnogo dinamičnih sistemov, ki jih najdemo v vsakodnevnem življenu, lahko opišemo s pomočjo diferencialnih enačb. Reševanje takih enačb in analiziranje njihovih rešitev pa je ponavadi zelo zahteven problem, saj so rešitve zvezno odvisne od časa. Problem si lahko poenostavimo tako, da najprej poiščemo nek podprostor (Poincaréjev prerez), ki je transverzalen na tokovnice našega dinamičnega sistema, nato pa opazujemo iteracije Poincaréjeve preslikave. Slednjo naravno določajo preseki tokovic s Poincaréjevim prerezom. V nadaljevanju obravnavamo zgolj diskretne dinamične sisteme.

Kompleksna dinamika je področje v matematiki, kjer proučujemo diskretne dinamične sisteme, ki so podani z iterati holomorfnih endomorfizmov kompleksnih prostorov. V nadaljevanju bomo z f^n označevali n-ti iterat preslikave f. Naš dinamični sistem vsakemu elementu x kompleksnega prostora X priredi njegovo orbito glede na preslikavo f, t.j.

$$x \to f(x) \to f^2(x) \to f^3(x) \to \dots .$$

Naravno vprašanje, ki se nam ob tem postavi je, kako se orbite odzivajo na majhne spremembe v začetnih pogojih in ali lahko prostor X smiselno razdelimo na več disjunktnih podprostorov, v katerih imajo orbite točk podobne dinamične lastnosti.

Ena od motivacij za proučevanje kompleksne dinamike izhaja iz problema iskanja ničel holomorfnega polinoma $P(z)$ s pomočjo Newtnove metode. Slednja temelji na iteraciji racionalne funkcije $R(z) = z - \frac{P(z)}{P'(z)}$ in nam, ob primernem izboru začetne vrednosti, zagotavlja konvergenco orbite $\{R^n(z)\}_n$ k eni od ničel polinoma. Katere pa so primerne začetne vrednosti? Natančnega odgovora na to vprašanje žal ne moremo dati, lahko pa povemo, da je množica primernih začetnih vrednosti vselej odprta in gosta podmnožica $\mathbb{C}$. To dejstvo sta v začetku dvajsetega stoletja odkrila francoska matematika P. Fatou in G. Julia, ki sta postavila temelje sodobne teorije dinamičnih sistemov holomorfnih preslikav. Med proučevanjem iteracij holomorfnih racionalnih funkcij na $\mathbb{C}$ (oz. holomorfnih endomorfizmov na Riemannovi sferi $\mathbb{P}^1$) sta odkrila dihotomijo Riemannove sfere, t.j. razcep na dve komplementarni množici, ki danes nosita njuno ime. Področje kompleksne dinamike racionalnih funkcij je doživelo razcvet v osemdesetih in devetdesetih letih prejšnega stoletja, tako da je danes naše razumevanje teh dinamičnih sistemov na visoki ravni; glej [50]. V višjih dimenzijah je to področje precej manj raziskano, kar je predvsem posledica pomankanja učinkovitih metod za dokazovanje in obstoja novih dinamičnih fenomenov. V nadaljevnaju bomo govorili o dinamiki holomorfnih endomorfizmov kompleksnih projektivnih prostorov in o dinamiki holomorfnih avtomorfizmov kompleksnih evklidskih prostorov. Zainteresiranemu bralcu svetujemo naslednjo literaturo [5, 17, 37, 26, 28, 41, 54].

Naj bo $\mathbb{P}^k$ k-razsežni kompleksni projektivni prostor. Slednjega dobimo tako, da identificiramo vse točke v $\mathbb{C}^{k+1}\backslash\{0\}$, ki ležijo na isti kompleksni premici skozi izhodišče. Naj bo $f : \mathbb{P}^k \to \mathbb{P}^k$ holomorfni endomorfizem stopnje d. Taka preslikava je preko naravne projekcije $\pi : \mathbb{C}^{k+1}\backslash\{0\} \to \mathbb{P}^k$ semi-konjugirana polinomski preslikavi $F = (F_0, \dots, F_k) : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$, kjer so F_j homogeni polinomi stopnje d in velja $F^{-1}(0) = 0$. V nadaljevanju vselej predpostavimo, da preslikava f ni injektivna ($d \geq 2$), saj je proučevaje takih preslikav enostavno. Spomnimo se, da je $\mathbb{P}^1$ Riemannova sfera in, da so holomorfni endomorfizmi v tem primeru "racionalne funckije".

Za dan holomorfni endomorfizem f definiramo naslednji množici. *Fatoujeva množica*

$\mathcal{F}$ je množica vseh točk iz $\mathbb{P}^k$, ki imajo odprto okolico, na kateri je družina iteratov $\{f^n\}$ enakozvezna. *Juliajeva množica* $\mathcal{J}$ pa je definirana kot komplement Fatoujeve množice. Po definiciji je Fatoujeva množica odprta, Juliajeva množica pa kompaktna. Naslednja lastnost Fatoujeve in Juliajeve množice, ki jo lahko opazimo pa je, da sta množici povsem invarianti, t.j.

$$f^{-1}(\mathcal{F}) = \mathcal{F} = f(\mathcal{F}).$$

Pogosto pravimo, da je dinamični sistem na Juliajevi množici kaotičen, t.j. občutljiv na spremembe začetnih pogojev. To sledi direktno iz definicije Juliajeve množice, saj za vsak $x \in \mathcal{J}$, $\varepsilon > 0$ in $\delta > 0$, obstaja $y \in \mathbb{P}^k$ in $N \in \mathbb{N}$ z lastnostjo $\mathrm{d}(x, y) < \delta$ in $\mathrm{d}(f^N(x), f^N(y)) > \varepsilon$, kjer smo z $\mathrm{d}(\cdot, \cdot)$ označili metriko na $\mathbb{P}^k$.

Komponente za povezanost Fatoujeve množice imenujemo *Fatoujeve komponente*. Za Fatoujevo komponento Ω pravimo, da je *periodična*, če obstaja tak $n > 0$, da velja $f^n(\Omega) = \Omega$. V primeru $n = 1$ pravimo, da je Fatoujeva komponenta *invariantna*. Fatoujeva komponenta je *predperiodična*, če je nek njen iterat periodična komponenta. V nasprotnem primeru pravimo, da je Fatoujeva komponenta *potujoča*.

Točka x je *periodična*, če obstaja tak $n > 0$, da velja $f^n(x) = x$. V primeru $n = 1$ pravimo, da je točka x *negibna točka*. Negibna točka je *privlačna* (*odbojna*) natanko tedaj, ko so vse lastne vrednosti Jacobijeve matrike f v točki x po absolutni vrednosti strogo manjše od 1 (strogo večje od 1).

Iz definicij lahko hitro sklepamo, da so vse odbojne periodične točke vsebovane v Juliajevi množici, vse privlačne periodične točke pa v Fatoujevi množici.

Proučevanje dinamičnega sistema lahko razdelimo na dva dela, saj lahko neodvisno obravnavamo dinamiko, ki se odvija na Juliajevi množici ali pa na Fatoujevi množici. Vprašanja, ki si jih ob tem lahko zastavimo so:

(*i*) Ali lahko v okolici negibne točke najdemo take lokalne koordinate, v katerih bo naša preslikava f imela tako obliko, da jo bomo lahko enostavneje analizirali? Ali lahko najdemo klasifikacijo povezanih komponent glede na dinamične lastnosti?

(ii) Kaj lahko povemo o povezanosti, samopodobnosti, Hausdorffovi dimenziji, gladkosti in topologiji povezanih komponent naših množic?

(iii) Ali obstajajo kakšne f invariantne mere? Kakšne so ergodične lastnost takih mer?

Skoraj vsa vprašanja iz druge in tretje točke se nanašajo na Juliajevo množico, tista iz prve pa na Fatoujevo množico. Omenimo še, da v primeru racionalnih funkcij poznamo odgovre na večino zgornjih vprašanj.

V drugem poglavju si podrobneje ogledamo Juliajevo množico. Na Riemanovi sferi je Juliajeva množica lahko povezana ali pa povsem nepovezana množica. Odbojne periodične točke racionalne funkcije tvorijo gosto podmnožico Juliajeve množice. V višjih dimenzijah pa je $\mathcal{J}$ vselej povezana množica in v večini primerov, množica odbojnih periodičnih točk ni gosta v Juliajevi množici. Zaprtje množice odbojnih periodičnih točk ima lahko celo kakšne izolirane točke; glej [41]. Definirajmo množico $\mathcal{J}_k$ kot nosilec mere μ, ki jo dobimo kot šibko limito mer

$$\mu_n = d^{-kn} \sum_{\substack{f^n(z)=z \\ z \text{ odbojna}}} \delta_z.$$

Množico $\mathcal{J}_k$ imenujemo mala Juliajeva množica. Zanjo velja, da je vselej neprazna, povsem invariantna, kompaktna podmnožica Juliajeve množice in, da so odbojne periodične točke v njej goste [17]. To pomeni, da predstavlja nek dinamični analog Juliajeve množice v dimenziji 1.

V poglavju II skušamo opisati razliko $\mathcal{J}\backslash\mathcal{J}_k$. Naj omenimo, da so se s podobnimi vprašanji ukvarjali že Fornæss, Sibony in Dujardin; glej [28, 20, 17]. Glavni rezultat tega poglavja je Izrek II.4:

Kadar se množici $\mathcal{J}_k$ in $\mathcal{J}$ ne ujemata, je množica $\mathcal{J} \backslash \mathcal{J}_k$ gosta v $\mathcal{J}$.

Izrek dokažemo s preprostimi metodami, kjer imata osrednjo vlogo popolna invariantnost Fatoujeve in Juliajeve množice ter Izrek II.9. Slednji nam pove, da je za vsako odprto množico U, ki ima neprazen presek z $\mathcal{J}_k$, množica $\mathbb{P}^k \backslash \bigcup_{n \geq 0} f^n(U)$ pluripolarna.

V nadaljevnaju za dan holomorfni endomorfizem $\mathbb{P}^k$ definiramo množico

$$\mathcal{S} := \overline{\bigcup_{\Omega} b\overline{\Omega}} \subseteq \mathcal{J},$$

kjer smo vzeli unijo po vseh Fatoujevih komponentah. Ta definicija ima smisel izključno le, kadar je Fatoujeva množica nepovezana in velja $\overline{\mathcal{F}} = \mathbb{P}^k$. Invariantnost množice $\mathcal{S}$ nam zagotavlja Lema II.8:

Množica $\mathcal{S}$ je povsem invariantna.

S pomočjo te leme nato dokažemo Izrek II.10:

Mala Juliajeva množica $\mathcal{J}_k$ je vselej vsebovana v $\mathcal{S}$.

V nadaljevanju poglavja II se sprašujemo po nekaterih lastnostih Fatoujevih komponent, ki se odražajo v razliki $\mathcal{J} \backslash \mathcal{S}$. Ueda [62] je dokazal, da so vse Fatoujeve komponente hiperbolične Steinove mnogoterosti. Nekaj o topologiji Fatoujevih komponent pove tudi njegov Izrek II.12. Ta pravi, da za vsako preslikavo $\varphi : \Delta \to \mathbb{P}^k$, za katero je slika $\varphi(\Delta^*)$ vsebovana v Fatoujevi komponenti Ω, velja $\varphi(\Delta) \subset \Omega$. Od tod sledi, da Fatoujeva komponenta lokalno ne more biti oblike $\Delta \times \Delta^*$. Opazimo, da je ta množica primer neregularne množice, t.j. množica ni enaka notranjosti svojega zaprtja. Zato si zastavimo vprašanje: ali so vse Fatoujeve komponente regularne množice? Ob predpostavki, da je Fatoujeva množica nepovezana, najprej dokažemo Lemo II.6:

Fatoujeve komponente racionalnih funkcij so regularne množice.

Nato s preprostim primerom v $\mathbb{P}^2$ pokažemo, da v splošnem tega ne moremo trditi. Iz definicije množice $\mathcal{S}$ tudi sledi, da so vse Fatoujeve komponente regularne natanko tedaj, kadar velja $\mathcal{S} = \mathcal{J}$. Zgoraj omenjeni rezultati nam med drugim povedo, da so vse Fatoujeve komponente regularne, kadar je množica odbojnih periodičnih točk gosta v Juliajevi množici. Že omenjeni Izrek II.12 pa nam pove, da množica $\mathcal{J}\backslash\mathcal{S}$ ne more biti pluripolarna oz. velja, da za nobeno odprto množico U, množica $U \cap \mathcal{J}\backslash\mathcal{S}$ ni pluripolarna.

Invariantne Fatoujeve komponente nelinearnih racionalnih funkcij je uspešno klasificiral že Fatou ter kasneje s primeri še Siegel in Herman. Iz njihovih rezultatov sledi, da je invariantna Fatoujeva komponenta Ω lahko le:

(i) *območje privlaka*: Orbita poljubne točke iz Ω konvergira proti privlačni fiksni točki $z \in \Omega$,

(ii) *parabolična domena*: Orbita poljubne točke iz Ω konvergira proti fiksni točki $z \in \mathrm{b}\Omega$,

(iii) *rotacijska domena*: Ω je konformo ekvivalentna disku ali kolobarju, racionalna funkcija pa je konformno konjugirana iracionalni rotaciji. Prvemu primeru pravimo Siegelov disk drugemu pa Hermanov kolobar.

Klasifikacijo Fatoujevih komponent je leta 1985 zaključil Sullivan, ki je dokazal, da so vse Fatoujeve komponente racionalnih funkcij predperiodične.

V splošnem ($k \geq 2$) je klasifikacija Fatoujevih komponent holomorfnih endomorfizmov $\mathbb{P}^k$ dosti bolj zaheven problem, kar dokazuje tudi nedavno odkritje holomorfnega endomorfizema $\mathbb{P}^2$, ki ima potujočo Fatoujevo komponento [4]. V nadaljevanaju se bomo osredotočili na posebne tipe Fatoujevih komponent.

Invariantne Fatoujeve komponente so lahko *povratne* ali pa *nepovratne*. Pravimo, da je Ω povratna invariantna Fatoujeva komponenta, če orbita poljubne točke ostane znotraj kompaktne podmnožice Ω. V nasprotnem primeru, t.j. kadar orbita poljubne točke zapusti vsak kompakt pravimo, da je Fatoujeva komponenta nepovratna.

Povratne invariantne komponente v $\mathbb{P}^2$ sta klasificirala Fornæss in Sibony [27]. Dokazala sta, da so povratne invariantne Fatoujeve komponente lahko le naslednjega tipa:

(i) *Območje privlaka negibne točke*: Orbita poljubne točke iz Ω konvergira proti privlačni fiksni točki $z \in \Omega$,

(ii) Obstaja taka zaprta invariantna Riemannova ploskva $\Sigma \subset \Omega$, da orbita poljubne točke iz Ω konvergira proti Σ. Riemannova ploskva Σ je lahko biholomorfna disku, punktiranemu disku ali kolobarju, preslikava $f|_\Sigma$ pa je konjugirana iracionalni rotaciji.

(iii) *Sieglova domena*: Obstaja zaporedje iteratov f^{n_j}, ki enakomerno po kompaktih v Ω konvergira proti identični preslikavi.

Ta rezultat močno spominja na klasifikacijo invariantnih Fatoujevih komponent racionalnih funkcij, zato tudi ni presenetljivo, da sta že Fornæss in Sibony v svojem delu predstavila primere endomorfizmov $\mathbb{P}^2$, pri katerih dobimo območje privlaka negibne točke, Sieglovo domeno, ter v primeru invariantne Riemannove ploskve, disk in kolobar. Vse do leta 2008 ni bilo znano ali obstaja primer preslikave, pri kateri bi dobili punktiran disk. Tega leta je namreč Ueda [63] dokazal, da take preslikave ni in s tem tudi zaključil klasifikacijo povratnih invariantnih Fatoujevih komponent.

Naslednji korak v smeri klasifikacije sta naredila Lyubich in Peters [47], ki sta klasificirala nepovratne invariantne Fatoujeve komponente v $\mathbb{P}^2$ ob dodatni predpostavki, da je limitna množica $h(\Omega) \subset b\Omega$ neodvisna od izbire konvergentnega zaporedja $f^{n_j} \to h$. Kot zanimivost naj dodamo, da ni jasno ali je ta predpostavka že trivialno izpolnjena. Lyubich in Peters sta dokazala, da je limitna množica $\Sigma = h(\Omega)$ lahko le:

(i) negibna točka,

(ii) biholomorfna disku, punktiranemu disku ali kolobaru, preslikava $f|_\Sigma$ pa je konjugirana iracionalni rotaciji.

Tudi v tem primeru smo do sedaj poznali le preslikave, ki so imele za limitno množico točko, disk ali pa kolobar. V poglavju III, ki je skupno delo Fornæssa, Petersa in avtorja, dokažemo Izrek III.9, ki pravi:

Preslikava $f([z:w:t]) = [\lambda z t^2 + z^3 : \lambda^{-1}(w t^2 + z w^2) + w^3 : t^3]$ *ima za primerno izrbran* λ*, nepovratno invariantno Fatoujevo komponento, katere limitna množica je biholomorfna punktiranemu disku.*

Glavno vlogo v poglavju III ima lokalna dinamika preslikav, ki so *tangentne identiteti*. To so zarodki holomorfnih preslikav $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, za katere velja $DF(0) = \mathrm{Id}$, oz.

$$F = \mathrm{Id} + F_k + F_{k+1} + \ldots,$$

kjer so F_j homogeni polinomi stopnje $j \geq 2$. Najmanjšo stopnjo k imenujemo *red*. Pravimo, da je $v \in \mathbb{C}^2$ *karakteristična smer* za F, če obstaja tak $\lambda \in \mathbb{C}$, da velja

$$F_k(v) = \lambda v.$$

Če je $\lambda = 0$ pravimo, da je v *degenerirana*, v nasprotnem primeru $\lambda \neq 0$ pa pravimo, da je v *nedegenerirana*. Pravimo, da orbita $\{F^n(z)\}$ konvergira k izhodišču *tangentno* v smeri v, če velja $F^n(z) \to 0$ in $[F^n(z)] \to [v]$ v $\mathbb{P}^1$.

Parabolična krivulja preslikave F, ki tangentna $[v] \in \mathbb{P}^1$, je injektivna holomorfna preslikava $\varphi : \mathbb{D} \to \mathbb{C}^2 \backslash \{0\}$, ki zadošča naslednjim pogojem:

(i) φ je zvezna v $1 \in \partial\mathbb{D}$ in velja $\varphi(1) = 0$,

(ii) $\varphi(\mathbb{D})$ je invariantna za F in zaporedje $(F|_{\varphi(\mathbb{D})})^n \to 0$ konvergira enakomerno po kompaktih, *(iii)* $[\varphi(\zeta)] \to [v]$, ko gre $\zeta \to 1$ v $\mathbb{D}$.

Hakim je dokazala Izrek III.5, ki pravi naslednje: Naj bo $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ holomorfen zarodek, ki je tangenten identiteti in je reda $k \geq 2$. Tedaj za vsako nedegenerirano karakteristično smer v obstaja vsaj $k-1$ paraboličnih krivulj za F, ki so tangentne $[v]$.

Njen rezultat uporabimo v zadnjem delu poglavja III, kjer podamo splošno konstrukcijo punktiranih limitnih množic v $\mathbb{C}^2$. Izrek III.7 pravi:

Naj bo $V \subset \mathbb{C}^2$ analitična množica dimenzije 1 (brez izoliranih točk). Obstaja tak holomorfni endomorfizem $F : \mathbb{C}^2 \to \mathbb{C}^2$, da ima preslikava F za vsako nerazcepno komponento V_1 množice V nepovratno invariantno Fatoujevo komponento Ω, na kateri orbite konvergirajo proti $V_1 \setminus \mathrm{Sing}(V)$.

Izrek dokažemo tako, da skonstruiramo preslikavo $F = \mathrm{Id} + G$, kjer je preslikava G na analitični množici V identično enaka 0. Iz konstrucije preslikave F nato sledi, da imamo v vsaki točki $(z, w) \in \mathrm{Reg}(V)$ parabolično krivuljo, te pa so zvezno odvisne od (z, w). Unija teh krivulj je nato vsebovana v neki Fatoujevi komponenti, kjer vse orbite konvergirajo proti V. Glavni argument, ki ga na koncu uporabimo, je Izrek III.8, ki sta ga dokazala Lyubich in Peters. Ta nam pove, da je limitna množica nepovratne invariantne Fatoujeve komponente za poljubno kompleksno mnogoterost dimenzije 2 in holomorfni endomorfizem $f : X \to X$ lahko le točka ali pa injektivno imerzirana Riemannova ploskev.

Omenili smo že, da lahko mnoge dinamične sisteme opišemo z matematičnimi enačbami, ki nam opisujejo spremembo sistema skozi čas. Razumevanje dinamičnega sistema zgolj na podlagi teh enačb je zelo zahtevno, saj so odvisne od mnogo spremenljivk. Problem si lahko poenostavimo tako, da enostavno odstranimo nekaj parametrov in iz tako poenostavljenih enačb izluščimo rešitev. Naravno vprašanje, ki temu sledi je, ali ta rešitev dovolj dobro oponaša originalen dinamičen sistem.

S podobnimi problemi se je ukvarjal že Takens [59, 60], ki je dokazal Izrek V.2. Ta nam pove, da za poljubno mnogoterost M dimenzije m velja naslednje. Za odprto in gosto množico parov (φ, y), kjer je $\varphi : M \to M$ gladek endomorfizem in $y : M \to \mathbb{R}$ gladka funkcija, obstaja preslikava $\tau : \Phi_{(\varphi,y)}(M) \to M$, za katero velja $\tau \circ \Phi_{(\varphi,y)} = \varphi^{2m+1}$, rekonstrukcijska preslikava $\Phi_{(\varphi,y)} : M \to \mathbb{R}^{2m+2}$ pa je definirana kot $\Phi_{(\varphi,y)}(x) = (y(x), y(\varphi(x)), \ldots, y(\varphi^{2m+1}(x)))$. Ta izrek nam pove, da čeprav rekonstrukcijska preslikava ni injektivna, vseeno vsebuje dovolj informacij, da lahko obnovimo $2m+1$ sliko prvotne preslikave.

Povedali smo že, da je dinamični sistem neke preslikave P določen z orbitami točk

$$x \to P(x) \to P^2(x) \to P^3(x) \to \ldots.$$

V poglavju V se ukvarjamo z dinamiko polinomov. Zanima nas, katere dinamične lastnosti kompleksnega polinoma P lahko razberemo iz realnih delov orbit

$$\mathrm{Re}(z) \to \mathrm{Re}(P(z)) \to \mathrm{Re}(P^2(z)) \to \mathrm{Re}(P^3(z)) \to \ldots.$$

Glavni problem s katerim se srečamo je ta, da preslikava iz orbit v realne dele orbit ni injektivna. To pomeni, da je poznavanje vlaken te preslikave ključnega pomena pri raziskovanju zgornjega problema. V večini primerov so vlakna diskretna, število točk v vlaknu pa je navzgor omejeno s kvadratom stopnje polinoma. Ena od naravnih lastnosti, ki bi jo lahko preverili, je *entropija*. Entropija je količina, izražena kot nenegativno realno število, ki meri kaotičnost dinamičnega sistema, t.j. občutljivost na majhne spremembe začetnih pogojev (glej razdelek 5.2). Znano je, da je entropija kompleksnega polinoma stopnje d enaka $\log \mathrm{d}$. S to tematiko sta se ukvarjala že Fornæss in Peters [24], ki sta preučevala Takensov rekonstrukcijski izrek za realne orbite kompleksnih polinomov. Dokazala sta, da entropijo neizjemnih polinomov razberemo že iz realnih orbit, pri čemer pravimo, da je polinom

$P(z) = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_0$ izjemen, kadar je $a_d i^{d-1}$ realno število. V tem poglavju posplošimo njun rezultat na vse polinome, tako da dokažemo Izrek V.3:

Naj bo $P(z)$ kompleksen polinom stopje $d \geq 2$ in $\nu = \Phi_*(\mu)$, kjer je μ ravnotežna mera $P(z)$. Verjetnostna mera ν je invariantna in ergodična. Mera ν je mera maksimalne entropije $\log d$, *razen v primeru, kadar je Juliajeva množica polinoma P(z) vsebovana v invariantni vertikalni premici. V tem primeru je njena entropija enaka* 0.

Glavni rezultat tega poglavja je Izrek V.4, ki je posebna verzija Takensovega izreka za realne orbite kompleksnih polinomov:

Za generičen kompleksen polinom P stopnje $d \geq 2$ obstajata taki naravni števili M in N, da za poljubni točki x in y iz $\mathbb{C}$, za kateri pri vsakem $0 \leq k \leq N$ velja $\mathrm{Re}(P^k(x)) = \mathrm{Re}(P^k(y))$, *velja tudi $P^M(x) = P^M(y)$. Konstanti M in N lahko izberemo tako, da sta odvisni le od stopnje polinoma.*

Za dokaz izreka je potrebna natančnejša analiza množice zrcalnih točk $X = \{(x, y) \in \mathbb{C}^2 \mid \mathrm{Re}(P^k(x)) = \mathrm{Re}(P^k(y)) \quad \forall k\}$. Velik korak v tej smeri, sta naredila že Fornæss in Peters [24]. V dokazu porabljamo osnovne rezultate iz teorije realno algebraičnih množic ter rezultate dinamike kompleksnih polinomov. Jasno je, da izrek ne velja za vse polinome saj se lahko zgodi, da ima polinom dve negibni točki, ki ležita na isti vertikalni premici. Dokaz zato nadaljujemo s spretnim izključevanjem potencialno problematičnih polinomov, kjer poskrbimo, da periodične točke preostalih polinomov zrcalijo le same sebe. Dokaz zaključimo s protislovjem tako, da znotraj množice zrcalnih točk, za katere pri vsakem k velja $P^k(x) \neq P^k(y)$, najdemo zrcalni par periodičnih točk.

V poglavju IV osrednjo vlogo igrajo Fatou-Bieberbachove domene. To so prave odprte podmnožice $\Omega \subsetneq \mathbb{C}^n$ $(n \geq 2)$, ki so biholomorfne $\mathbb{C}^n$. Rosay in Rudin [54] sta dokazala, da je območje privlaka privlačne negibne točke holomorfnega avtomorfizma

$\mathbb{C}^n$ vselej biholomorfno $\mathbb{C}^n$. Fatou-Bieberbachovo domeno lahko dobimo tako, da vzamemo avtomorfizem, ki ima dve negibni točki, od tega vsaj eno privlačno. Kasneje je Wold [66] dokazal, da lahko vsako Rungejevo Fatou-Bieberbachovo domeno Ω dobimo kot območje privlaka privlačne negibne točke w nekega zaporedja f_n holomorfnih avtomorfizmov $\mathbb{C}^n$, t.j. $\Omega = \{z \in \mathbb{C}^n \mid f_n \circ \ldots \circ f_1(z) \to w, \quad n \to \infty\}$. Leta 2008 je Wold [67] podal še konstrukcijo Fatou-Bieberbachove domene, ki ni Rungejeva.

Kompleksno mnogoterost X dimenzije n imenujemo *dolg* $\mathbb{C}^n$, če vsebuje naraščajoče zaporedje domen $X_1 \subset X_2 \subset X_3 \subset \cdots$, za katero velja $X = \cup_{j=1}^{\infty} X_j$, domene X_j pa so biholomorfne kompleksnemu evklidskemu prostoru $\mathbb{C}^n$. Vsak dolg $\mathbb{C}^n$ je homeomorfen evklidskemu prostoru. Za $n = 1$ očitno velja, da je vsak dolg $\mathbb{C}$ biholomorfen $\mathbb{C}$, v splošnem pa so te mnogoterost precej slabo raziskane. Naravno vprašanje, ki se nam ob tem porodi je, ali je evklidski prostor edini primeri dolgega $\mathbb{C}^n$. Na to vprašanje je negativno odgovoril Wold [68], ki je s pomočjo svoje konstrucije ne-Rungejeve Fatou-Bieberbachove domene, dokazal obstoj dolgega $\mathbb{C}^n$, ki ni Steinova mnogoterost. Dokazal je tudi, da kadar so vsi pari (X_j, X_{j+1}) Rungejevi, je dolg $\mathbb{C}^n$, ki ga dobimo kot unijo domen X_j, vselej biholomorfen evklidskemu prostoru. Navkljub vsem tem rezultatom, pa je na tem področju še vedno precej odprtih vprašanj:

(A) Ali obstaja več različnih ne-Steinovih dolgh $\mathbb{C}^n$?

(B) Ali lahko evklidski prostor izčrpamo z ne-Rungejevimi Fatou-Bieberbachovimi domenami?

(C) Ali obstaja tak dolg $\mathbb{C}^n$, ki nima nekonstantnih holomorfnih funkcij?

(C') Ali obstaja tak ne-Steinov dolg $\mathbb{C}^n$, ki ima kakšno holomorfno funkcijo?

(D) Ali obstja tak Steinov dolg $\mathbb{C}^n$, ki ni biholomorfen evklidskemu prostoru?

V poglavju IV pozitivno odgovorimo na prva tri vprašanja; glej razdelek 4.3, razdelek 4.4 in Izrek IV.1. Dokazi temeljijo na uporabi Andersén-Lepertove teorije in na metodah, ki jih je v svojem delu razvil Wold. Glavna novost je v konstrukciji dveh različnih ne-Steinovih dolgh $\mathbb{C}^2$. Idejo, ki se skriva za konstrukcijo, sta kasneje v svojem delu uporabila Forstnerič in avtor [9], kjer sta definirala novo invarianto za kompleksne mnogoterosti.