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METRIC PROPERTIES OF SIERPIŃSKI GRAPHS

Doctoral thesis

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UNIVERZA V LJUBLJANI
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METRIČNE LASTNOSTI GRAFOV SIERPIŃSKEGA

Doktorska disertacija

MENTOR: prof. dr. Sandi Klavžar

Ljubljana, 2014

Vse naše sanje se lahko uresničijo -
če le imamo pogum, da gremo z njimi.
(W. Disney)

Posebna zahvala gre mentorju Sandiju Klavžarju za vso pomoč in podporo, tako pri disertaciji kot tudi pri mojem raziskovalnem udejstvovanju. Hvala za ves čas in trud ter za pestre debate ob čajankah.

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Abstract

In this thesis we study the metric properties of Sierpiński graphs. Sierpiński graphs form a two-parametric family of graphs similar to Hanoi graphs that originate in the Tower of Hanoi puzzle. Sierpiński graphs can be found in various areas of mathematics and elsewhere.

First we introduce the family of Sierpiński graphs and their variants. These families have been known under various names, and sometimes vice versa - different graphs under the same name. We therefore standardize their notations and names to avoid confusion in the future. Next we summarize what has already been studied on Sierpiński graphs.

One chapter of the thesis is completely devoted to metric properties of Sierpiński graphs, where we first list known related results, in particular we state the distance lemma and the theorem about the distance between arbitrary two vertices. Since this distance is expressed with a minimum, we give improved results on distances in Sierpiński graphs for almost-extreme vertices. Namely, the distance between an arbitrary vertex and an almost-extreme vertex in a Sierpiński graph can be expressed with a closed formula. We conclude this part with determining the metric dimension of Sierpiński graphs.

To better understand the structure of Sierpiński graphs we study various embeddings, beginning with the embeddings into Hanoi graphs. We also determine the canonical metric representation and induced embeddings. For the latter type of embeddings, we introduce the Hamming dimension and bound it for Sierpiński graphs.

We conclude with some open problems.

Math. Subj. Class. (2010): 05C12, 05C57, 05C60, 05C75, 05C76, 05C78.

Keywords: Sierpiński graph, Sierpiński-type graph, distance, almost-extreme vertex, distance of a vertex, metric dimension, Hanoi graph, Switching Tower of Hanoi, canonical metric representation, Hamming dimension, induced embedding.

Povzetek

V disertaciji preučujemo metrične lastnosti grafov Sierpińskega. Ti tvorijo 2-parametrično družino grafov, podobno grafom Hanojskega stolpa. Grafe Sierpińskega srečamo na različnih matematičnih področjih kot tudi v drugih vedah.

Najprej predstavimo družino grafov Sierpińskega in njihove različice. Te družine so poznane pod različnimi imeni, nekateri različni grafi pa si v literaturi delijo ime. V ta namen standardiziramo njihove oznake in imena, da bi se izognili zmedi pri nadaljnjem raziskovalnem delu. Naslednji korak je predstavitev znanih rezultatov o grafih Sierpińskega.

Eno poglavje disertacije v celoti namenjamo metričnim lastnostim grafov Sierpińskega, kjer najprej navedemo z metričnimi lastnostmi povezane znane rezultate. Posebno izpostavimo dobro znano lemo o razdalji in izrek o razdalji med poljubnima dvema vozliščema. Ker je ta razdalja določena z minimumom, izpeljemo izboljšane rezultate za razdalje do skoraj ekstremnih vozlišč. Natančneje povedano, razdaljo med poljubnim vozliščem in skoraj ekstremnim vozliščem na grafu Sierpińskega izrazimo z eksplicitno formulo. Poglavje zaključimo z določitvijo metrične dimenzije grafov Sierpińskega.

Da bi bolje razumeli strukturo grafov Sierpińskega, na koncu preučujemo različne vložitve. Zaradi njihove povezave s Hanojskim stolpom si najprej ogledamo vložitve v grafe Hanojskega stolpa. Prav tako določimo kanonično metrično reprezentacijo in inducirane vložitve. Za slednje vpeljemo Hammingovo dimenzijo in določimo njene meje za družino grafov Sierpińskega.

Disertacijo zaključimo z navedbo nekaterih odprtih problemov.

Math. Subj. Class. (2010): 05C12, 05C57, 05C60, 05C75, 05C76, 05C78.

Ključne besede: graf Sierpińskega, graf tipa Sierpińskega, razdalja, skoraj ekstremno vozlišče, razdalja vozlišča, metrična dimenzija, graf Hanojskega stolpa, zamenjevalni Hanojski stolp, kanonična metrična reprezentacija, Hammingova dimenzija, inducirana vložitev.

Contents

Chapter 1

Introduction

> *In the great temple at Benares, beneath the dome which marks the centre of the world, rests a brass-plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunder-clap the world will vanish.* [3, p. 92]

This is how the legend goes. The legend behind a puzzle called the Tower of Hanoi, invented by Éduard Lucas in 1883. His original puzzle consists of 3 pegs (needles in the legend) and 8 discs of different diameters which are all stacked on one of the pegs in decreasing order starting with the largest disc at the bottom of the peg. The goal of the puzzle is to transfer all discs stacked on one peg to another in such a way that we move only one disc at a time and obey the divine rule: no larger disc may be placed onto a smaller one.

Scorer, Grundy and Smith [58] were the first ones to introduce a state graph for the Tower of Hanoi puzzle (in 1944), and this is how the graph theory behind the game began to evolve. They generalized the number of the discs from the puzzle to an arbitrary number of discs but still assumed 3 pegs from the original version. It was Dudeney, however, who in his book from 1908 [10] indicated the extension of the problem to more than 3 pegs. His game, the Reve's puzzle, included 4 stools instead of pegs, and loaves of cheese instead of inedible discs, but the glove was thrown.

The extension of the original puzzle to more than 3 pegs, namely p, is the most intriguing generalization of the original game. For 3 pegs, many aspects of the puzzle were studied, starting with the minimal number of moves to transfer the entire stack of discs to another peg. For a comprehensive summary of known results see [27, Chapter 2]. When we introduce the 4th peg to the classical problem, we enter completely unfamiliar territory. The first ones to boldly cross its borders were Frame and Stewart, who, in 1941 ([11], [60], respectively), independently came to a similar solution for the minimal number of moves, assuming a generalized p-peg case. In order to transfer the entire stack of discs from one peg to another, they used different approaches to arrive to a similar conclusion, now jointly called the Frame-Stewart Conjecture (FSC). To date it still remains to be proven, however, that the solutions they presented are also the optimal ones. State graphs can also be defined for an arbitrary number of pegs ($p \in \mathbb{N}$), although, compared to those for $p = 3$, their nature is much, much more complex. These graphs are called Hanoi graphs, after the puzzle.

Graphs that are similar to Hanoi graphs, yet quite a bit simpler in their structure, are Sierpiński graphs. They play an important role in graph theory as well as in other fields of mathematics. Their value, however, extends outside the mathematical domain as they can be found in physics, psychology and elsewhere. The Sierpiński graphs were introduced in 1997 by Klavžar and Milutinović [40]. Back then, the authors presented the two-parametric family of graphs $S(n, k)$ (now denoted by S_p^n, where k was replaced by p for "pegs") the introduction of which was motivated from topological studies of the Lipscomb space as well as by the Tower of Hanoi puzzle. The name "Sierpiński graphs" was given later in 2002 [41], although the case $p = 3$ was already considered in 1990 by Hinz and Schief [32] under the name Sierpiński graph. The graph S_3^n is isomorphic to the Hanoi graph H_3^n (cf. [40, Theorem 2]). In more general terms, any S_p^n graph is isomorphic to the state graph of the Tower of Hanoi variant called the *Switching Tower of Hanoi*. There we also have p pegs, but we adjust the divine rule, so that in one move we either move the smallest disc (move of *type 0*) or, if we have a subtower of discs $1, \ldots, \delta - 1$ on one peg and disc δ lies on (top of) some other peg, we switch disc δ with the subtower of smallest discs (move of *type 1*).

Sierpiński graphs have been studied to a great extent. Many of their properties are known, but the studies are burdened with confusing names and notations. There are several types of graphs that were presented with the same name and vice versa – one name can be found in connection with different graphs. It was this mix-up that motivated us to embark upon a classification quest. We have carefully studied the known graphs among the Sierpiński-type graphs and tried to classify them once and for all.

After the classification, we will give a survey of known results on Sierpiński graphs. A lot is known about their properties although some studies related to these graphs might still remain unexplored. Our focus were hamiltonicity and planarity, colorings, codes and labelings, as well as some other properties. In Chapter 3, we narrowed our focus on metric properties of Sierpiński-type graphs. We first discuss known results. Two of the most important are definitely the distance lemma and the accompanying theorem [40], that already in 1997 started the chase for metric properties. The first ones to join the chase were Romik [57], with a decision automaton for shortest paths in the classical case (i.e., for $p = 3$), and Parisse [52], with numerous results such as diameter, eccentricity and other metric-related outcomes, almost a decade later. Wiesenberger pitched in with the average distance on Sierpiński graphs from his diploma thesis [68] in 2010. The study of eccentricity was further deepened by Hinz and Parisse determining the average eccentricity [31]. The latest contribution is the generalization of Romik's automaton to an arbitrary p by Hinz and Holz auf der Heide [26].

Upon realizing that no proper explicit formulas for the distance between arbitrary vertices of Sierpiński graph exist, we prove important new metric properties of the graph family – the distance to almost-extreme vertices and the metric dimension of Sierpiński graphs. As previously mentioned, a relation exists between Hanoi and Sierpiński graphs. In order to connect the newly deduced metric properties with the Hanoi graphs, we will study the embeddings of Sierpiński graphs into Hanoi graphs. In particular we deal with the question whether a Sierpiński graph S_p^n is a spanning subgraph of the Hanoi graph H_p^n. We prove that this is only possible if p is odd (or trivially if $n = 1$).

Finally, we will consider embeddings of Sierpiński graphs into Cartesian product graphs. More specifically, we will discuss their isometric and induced embeddings into Cartesian product graphs. Of course we will be interested in embeddings into as many nontrivial factors as possible. In the case of isometric embeddings there is precisely one such embedding and it is called the canonical metric representation. We will explicitly determine this representation for Sierpiński graphs.

There are various dimensions defined for product graphs, but many of them are trivial for most graph families. Therefore we introduce the Hamming dimension of a graph as the maximal number of factors of a Hamming graph into which the graph embeds as an irredundant induced subgraph. We will investigate this dimension on the Sierpiński graphs and establish some bounds on it. During the process of establishing bounds we will also derive some particular embeddings of Sierpiński graphs, for instance into the Cartesian product of Sierpiński triangle graphs.

Throughout the thesis, some of the topics here discussed presented us with further problems or, better said, motivation for additional research. This we discuss at the very end as it is, it seems, far from being just it – the end.

1.1 Basic definitions

In the thesis we will use standard notation from graph theory, where we will mainly follow West [67]. Some other definitions and notation will be provided in this section. All graphs considered will be simple and connected, unless stated otherwise.

For $n \in \mathbb{N}$ we will use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n]_0$ for $\{0, \ldots, n-1\}$. In particular we will deal with sets $B := [2]_0 = \{0, 1\}$ and $T := [3]_0 = \{0, 1, 2\}$, where B stands for binary and T for ternary.

Iverson bracket (or Iverson convention) is a conversion from a boolean value to B and is defined as

$$[X] = \begin{cases} 1\,, & \text{if } X \text{ is true,} \\ 0\,, & \text{if } X \text{ is false.} \end{cases}$$

Obviously,

$$[X] = 1 - [\neg X]\,. \tag{1.1}$$

A *clique* of a graph G is a complete subgraph of G and an n-*clique* is a clique of order n. The *clique number* $\omega(G)$ is the order of a largest clique of G.

As usual, we will denote the *open neighborhood* of a vertex u in G by $N_G(u)$, and $N_G[u] = N_G(u) \cup \{u\}$ is the *closed neighborhood* of u. For $S \subseteq V(G)$ we set $N_S(u) := \{v \in S \mid \{u, v\} \in E(G)\}$, and similarly $N_S[u] := N_S(u) \cup \{u\}$.

Throughout the thesis we will often deal with isomorphic and induced subgraphs. We say that a subgraph H of a graph G is an *induced subgraph* of G (or just *induced in* G), if it is induced by $V(H)$. In other words, $H \subseteq G$ is induced, if for any two vertices $u, v \in V(H)$, $\{u, v\} \in E(G) \Rightarrow \{u, v\} \in E(H)$. Similarly, a subgraph H of a graph G is an *isometric subgraph* of G (or *isometric in* G, for short), if

$$d_H(u, v) = d_G(u, v)\,,$$

holds for any distinct vertices $u, v \in V(H)$.

Especially when defining some families of graphs, we will refer to the labeling of their vertices. Such a labeling will be considered as specifying a vertex set of a class of isomorphic graphs, so that we get a representative of that class. Note that this has nothing to do with the term *labeling of edges* of some graph. The latter is a mapping from the edge set to the set of labels, in our case those labels are numbers, for example elements of $[\ell]$.

1.2 Sierpiński graphs and their variants

The central theme of the thesis are Sierpiński graphs. Here we will give the definition of this family of graphs together with some of its basic properties. Later in this section we will also define its variants.

Definition 1.1. *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. The* Sierpiński graph S_p^n *is the graph, defined on the vertex set*

$$V(S_p^n) = [p]_0^n\,,$$

whose edge set is given recursively by

$$\begin{aligned} E(S_p^0) &= \emptyset \\ E(S_p^n) &= \{\{is, it\} \mid i \in [p]_0\,, \{s,t\} \in E(S_p^{n-1})\} \cup \\ &\qquad \{\{ij^{n-1}, ji^{n-1}\} \mid i,j \in [p]_0\,, i \neq j\}\,, \quad n \in \mathbb{N}\,. \end{aligned}$$

For a Sierpiński graph S_p^n, p is its *base* and n its *exponent*. We will denote its vertices by $s_n \ldots s_1$. Consecutive equal entries in a string will be abbreviated with powers, for example $0000211111 = 0^4 21^5$. Note that i^0 is the empty string.

Obviously there are p^n vertices in a Sierpiński graph S_p^n, i.e., its order is

$$|S_p^n| = |V(S_p^n)| = p^n\,.$$

Because of its recursive definition, it is also easy to determine its size, for example, one may just solve the recurrence

$$|E(S_p^n)| = p \cdot |E(S_p^{n-1})| + \binom{p}{2}\,,\ n \in \mathbb{N} \quad \text{and} \quad |E(S_p^0)| = 0\,.$$

This gives us

$$\|S_p^n\| = |E(S_p^n)| = \binom{p}{2} \sum_{d=1}^{n} p^{n-d} = \frac{p}{2}(p^n - 1)\,.$$

Sierpiński graphs are connected which can be shown by a simple induction argument. More about their connectivity is discussed later in Section 2.4.

Let us take a look at the first few Sierpiński graphs. For $n = 0$, the Sierpiński graph S_p^0 is a one-vertex graph, so $S_p^0 \cong K_1$ for every $p \in \mathbb{N}$. Similarly, $S_1^n \cong K_1$ for any $n \in \mathbb{N}_0$. Later we will discuss metric properties and embeddings of Sierpiński graphs, but since for $p = 1$ or $n = 0$ the Sierpiński graph has only one vertex, we will usually omit these cases. We get another known family of graphs for $n = 1$, because $S_p^1 \cong K_p$ for every $p \in \mathbb{N}$, and for $p = 2$, S_2^n is the path graph on 2^n vertices, $S_2^n \cong P_{2^n}$.

The Sierpiński graphs S_4^2 and S_3^3 are shown in Figure 1.1. The latter case, i.e., when $p = 3$, is one of the reasons why Klavžar and Milutinović introduced these graphs in 1997 [40]: base-3-Sierpiński graphs are isomorphic to the Hanoi graphs, i.e., $S_3^n \cong H_3^n$ for every $n \in \mathbb{N}_0$. We will return to this topic in Section 1.3.1.

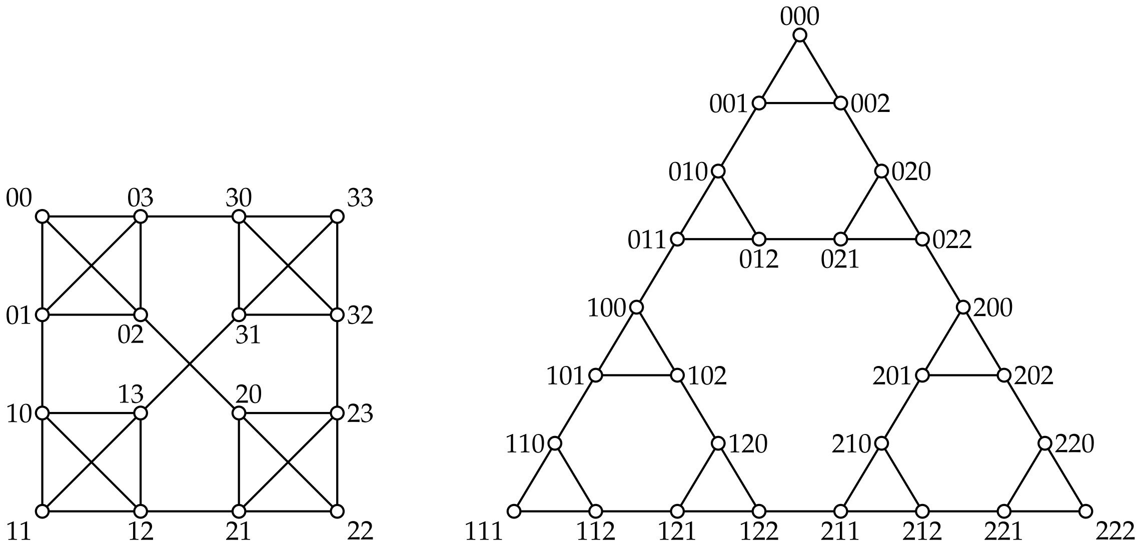


Figure 1.1: Examples of Sierpiński graphs: S_4^2 (left) and S_3^3 (right)

The edge set of Sierpiński graphs can be defined equivalently in the following way.

Proposition 1.2. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then two vertices s and t of S_p^n are adjacent if and only if they are of the form $s = \underline{s}s_\delta t_\delta^{\delta-1}$, $t = \underline{s}t_\delta s_\delta^{\delta-1}$ with $\delta \in [n]$, $\underline{s} \in [p]_0^{n-\delta}$, and $s_\delta \neq t_\delta$.*

Let us introduce some further notation for Sierpiński graphs S_p^n. The vertex of the form $i \dots i = i^n$ is called an *extreme vertex* (of S_p^n). The graph S_p^n contains p extreme vertices and these are the only vertices of degree $p-1$, all the other vertices having degree p.

A subgraph of S_p^n, whose vertices have a common prefix $\underline{s} \in [p]_0^{n-d}$, $d \in [n+1]_0$, is denoted by $\underline{s}S_p^d$ and is isomorphic to S_p^d. For $d = 1$, any $\underline{s}S_p^1$ induces a p-clique. This implies that the clique number of a Sierpiński graph S_p^n is at least p.

If $i, j \in [p]_0$ are distinct, the edge $\{ij^{n-1}, ji^{n-1}\}$ is the unique edge between subgraphs iS_p^{n-1} and jS_p^{n-1} and is denoted by $e_{ij}^{(n)}$. Note that all the edges $e_{ij}^{(n)}$ in S_p^n (for $n > 1$) are pairwise disjoint. We can generalize this by considering the edge $\underline{s}e_{ij}^{(d)}$, $d \in [n]_0$, between subgraphs $\underline{s}iS_p^{d-1}$ and $\underline{s}jS_p^{d-1}$. The edges of the form $\underline{s}e_{ij}^{(d)}$, $d > 1$, will be called *non-clique edges*, since they are included in none of the p-cliques (for $p \geq 2$). They correspond to the moves of type 1 in the Switching Tower of Hanoi. Accordingly, all the edges of a subgraph $\underline{s}S_p^1$ are *clique edges*. These edges correspond to the moves of type 0 in the Switching Tower of Hanoi. Note that for $p = 2$ the maximal cliques are of order 2. However, the non-clique edges are the unique edges between two subgraphs of smaller dimension, therefore the definition makes sense also for $p = 2$. Note that in this case we may distinguish between the clique and non-clique edges by their form, $\{\underline{s}i, \underline{s}j\}$ and $\{ij^\ell, ji^\ell\}$, $\ell \in [n-1]$, respectively.

Later we will consider Sierpiński triangle graphs, where the number of non-clique edges will be useful. It can be determined recursively from the construction of Sierpiński graphs: for a fixed $p \in \mathbb{N}$, let $f_p(n)$ denote the number of non-clique edges in S_p^n, $n \in \mathbb{N}$. Then

$$f_p(n) = p \cdot f_p(n-1) + \binom{p}{2}\,, \text{ and } f_p(1) = 0\,,$$

which gives us

$$f_p(n) = \frac{p}{2}(p^{n-1} - 1) = \|S_p^{n-1}\|\,. \tag{1.2}$$

An alternative way to determine the number of non-clique edges is through the number of the clique edges. For a fixed $p \in \mathbb{N}$ let $g_p(n)$ denote the number of clique edges in S_p^n, $n \in \mathbb{N}$. The number $g_p(n)$ can be determined either by recursion or directly: since all p-cliques in S_p^n are of the form $\underline{s}S_p^1$, we have

$$g_p(n) = p^{n-1}\binom{p}{2} = \frac{p^n}{2}(p-1)\,,$$

and $f_p(n) = \|S_p^n\| - g_p(n)$ gives us (1.2).

In Chapter 3 we will discuss distances in Sierpiński graphs, in particular we will deal with vertices that are very similar to extreme vertices; they only differ from extreme vertices in either the first or the last coordinate. For that reason they will be called almost-extreme vertices. We divide them into two classes.

Definition 1.3. *Let $n \in \mathbb{N}$, $p \in \mathbb{N}$, and $p \geq 2$. For any two distinct $i, j \in [p]_0$ the vertex of the form $i^n j$ of the graph S_p^{n+1} is called an* outer almost-extreme vertex *(of S_p^{n+1}) and the vertex ij^n of S_p^{n+1} is an* inner almost-extreme vertex *(of S_p^{n+1}).*

An outer almost-extreme vertex $i^n j$ is adjacent to the extreme vertex i^{n+1}, whereas an inner almost-extreme vertex ij^n can also be characterized as the vertex of iS_p^n, that is incident with the edge $e_{ij}^{(n+1)}$. Obviously, for $n \geq 2$ the graph S_p^{n+1} contains $p(p-1)$ outer almost-extreme vertices and $p(p-1)$ inner almost-extreme vertices. Thus, for $n \geq 2$, there are $2p(p-1)$ almost-extreme vertices in total. For $n = 1$ the vertices $i^n j$ and ij^n coincide, hence in S_p^2 there are exactly $p(p-1)$ almost-extreme vertices and any vertex is either extreme or almost-extreme. In Figure 1.2 the extreme vertices of S_5^3 are emphasized as gray circles, the outer almost-extreme vertices are red (vertices of the form ij^2) and the inner almost- extreme vertices are green (vertices of the form $i^2 j$).

We will often refer to the shortest path between two extreme vertices, therefore let $P_{ij}^{(n)}$ denote the shortest path between i^n and j^n in S_p^n (for any distinct $i, j \in [p]_0$). This path is indeed

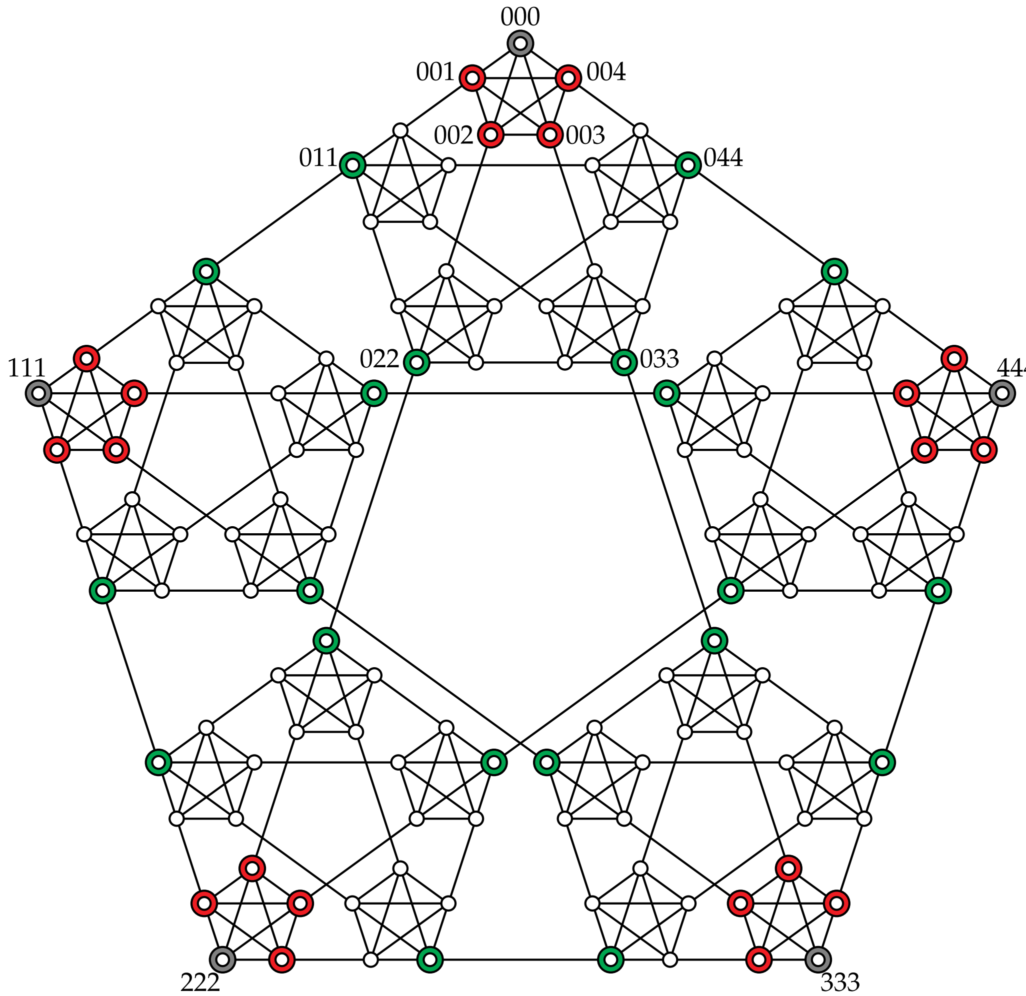


Figure 1.2: S_5^3 with its extreme and almost-extreme vertices

unique, see [40, Lemma 4]. Note also, that all the vertices of the path $P_{ij}^{(n)}$ have coordinates i or j. In other words, the vertices whose entries are i or j induce the path $P_{ij}^{(n)}$. Similarly, for pairwise distinct $i, j, \ell \in [p]_0$ let $C_{ij\ell}^{(n)}$ denote the shortest cycle in S_p^n that contains the edges $e_{ij}^{(n)}$, $e_{i\ell}^{(n)}$, and $e_{j\ell}^{(n)}$. These cycles will play an important role later, because they are isometric. For the proof of this fact we require the distance theorem (Theorem 3.6), so we will prove it in Section 3.1.

For more advanced properties of Sierpiński graphs see Chapter 2.

Now let us take a look at the variants of the Sierpiński graphs. In a similar way as we defined the family of Sierpiński graphs, we can also define Sierpiński triangle graphs and generalized Sierpiński triangle graphs. Sierpiński triangle graphs can be defined in different ways, but basically all come from the Sierpiński triangle fractal (see Section 1.3 and [51]). We will use the notation ST_3^n for the Sierpiński triangle graph, which will make sense when generalizing them to arbitrary $p \in \mathbb{N}$.

Definition 1.4. *Let $n \in \mathbb{N}_0$. Then the* class of the Sierpiński triangle graph ST_3^n *is obtained from S_3^{n+1} by contracting all non-clique edges (i.e., the edges of S_3^{n+1} that lie in no triangle).*

Beside the (ordinary) Sierpiński graphs S_p^n, these graphs have been most commonly studied in the literature. We will discuss their occurrences in the next section. Here we will first give two different labelings of their vertex set.

One way to label the Sierpiński triangle graphs is defined iteratively. We start with a complete graph on 3 vertices, $ST_3^0 \cong K_3$ and label it with $V(ST_3^0) = \hat{T} := \{\hat{0}, \hat{1}, \hat{2}\}$. Those labels will be of *length 0*. Now assume we have ST_3^n. To obtain ST_3^{n+1} we subdivide each edge of every triangle of ST_3^n and connect any two of the three new vertices of a triangle. The easiest way to explain how we label them is with the help of Sierpiński graphs. We inscribe S_3^{n+1} into the half-labeled graph and mirror the labels of the Sierpiński graph S_3^{n+1} on every unlabeled triangle. An example is shown in Figure 1.3. The underlying Sierpiński triangle graph ST_3^3 is drawn in black and the Sierpiński graph S_3^3 is red.

With this construction we get

$$V(ST_3^{n+1}) = \{\hat{0}, \hat{1}, \hat{2}\} \cup \{\, s \in T^m \mid m \in [n+1]\} \,.$$

For reasons stemming from the Tower of Hanoi puzzle, we will call this labeling the *idle peg labeling* of ST_3^n. (This will make sense later when we describe the connection between both discussed labelings.) Obviously

$$V(ST_3^{n+1}) = V(ST_3^n) \,\dot{\cup}\, V(S_3^{n+1}) \,,$$

and the edge set can be explicitly described as

$$\begin{aligned} E(ST_3^{n+1}) = &\left\{ \{\hat{k}, k^n j\} \mid k \in T, j \in T \setminus \{k\} \right\} \cup \\ &\{\{\underline{s}k, \underline{s}j\} \mid \underline{s} \in T^n, j,k \in T, j \neq k\} \cup \\ &\left\{ \{\underline{s}(3-i-j)i^{d-2}k, \underline{s}j\} \mid \underline{s} \in T^{n+1-d}, d \in [n+1] \setminus \{1\},\ i \in T, j,k \in T \setminus \{i\} \right\} \,. \end{aligned} \tag{1.3}$$

From the definition of the graphs ST_3^n we can derive another family of labeled Sierpiński triangle graphs. Denote the vertex obtained by contracting the edge $\{\underline{s}ij^d, \underline{s}i^d\} \in E(S_3^{n+1})$ by $\underline{s}\{i,j\}$. So the vertex set can be written as

$$V(ST_3^n) = \{\hat{0}, \hat{1}, \hat{2}\} \cup \left\{ \underline{s}\{i,j\} \mid \underline{s} \in T^{n-d},\ d \in [n], i,j \in T, i \neq j \right\} \,.$$

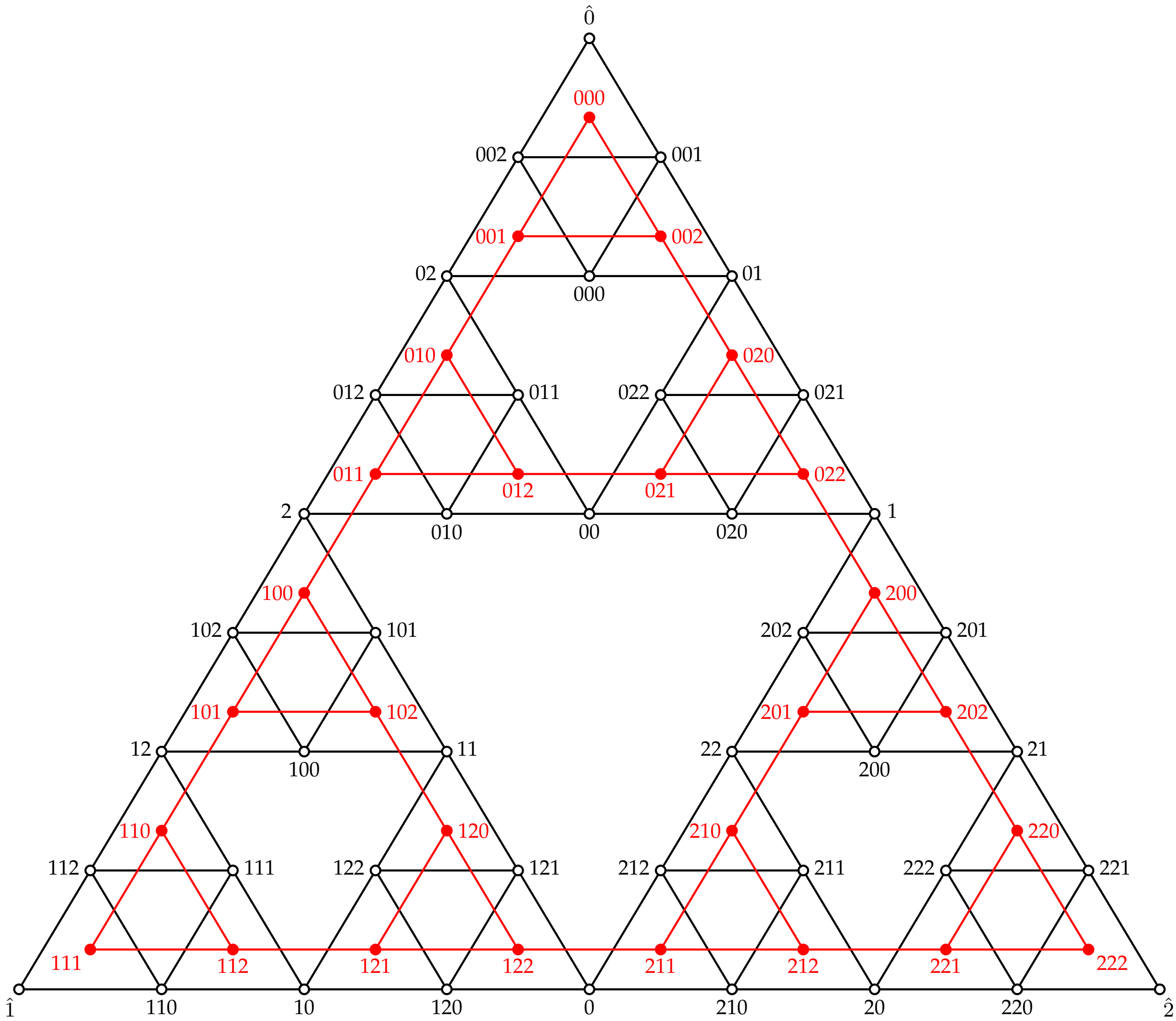


Figure 1.3: Combined Sierpiński triangle graph ST_3^3 (black) and Sierpiński graph S_3^3 (red)

Let us call this labeling the *contraction labeling* of ST_3^n. Note that both definitions of Sierpiński triangle graphs give us labels of different lengths. It is also possible to pass from one labeling to the other. Let ST_3^n be labeled with the contraction labeling. The *idle peg* for i and j is defined as $k := 3 - i - j$ (see [27, p. 74]). To obtain the idle peg labeling of ST_3^n we replace each vertex $\underline{s}\{i, j\}$ with $\underline{s}k$.

Here we have just briefly explained both labelings. More details about this topic can be found in the survey paper on the Sierpiński graphs [29]. Teguia and Godbole [61] studied the basic properties of (base-3-)Sierpiński triangle graphs. They proved that their chromatic number is 3, and that the graphs ST_3^n are hamiltonian and pancyclic (i.e., they contain cycles of length ℓ, for $\ell = 3, \dots, |ST_3^n|$). In the same paper they also computed their domination number, $\gamma(ST_3^n) = 3^{n-1}$, $n \geq 2$, and $\gamma(ST_3^1) = 2$.

The definition of ST_3^n with the contraction can be generalized as follows:

Definition 1.5. *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then the* (generalized) Sierpiński triangle graph ST_p^n *is obtained by contracting all non-clique*[1] *edges of the Sierpiński graph S_p^{n+1}.*

The vertex set of the graph ST_p^n can be written similarly as in the case $n = 3$. Again, we denote the vertex obtained from $\{\underline{s}ij^d, \underline{s}i^d\} \in E(S_p^{n+1})$ by $\underline{s}\{i,j\}$. Then

$$V(ST_p^n) = \{\hat{k} \mid k \in [p]_0\} \cup \left\{\underline{s}\{i,j\} \mid \underline{s} \in [p]_0^{n-d},\ d \in [n], i,j \in [p]_0, i \neq j\right\} .$$

Writing the vertex set this way enables us to describe explicitly the edge set in a similar way as we described it for $p = 3$:

$$\begin{aligned} E(ST_p^n) = &\left\{\left\{\hat{k}, k^{n-1}\{j,k\}\right\} \mid k \in [p]_0, j \in [p]_0 \setminus \{k\}\right\} \cup \\ &\left\{\{\underline{s}\{i,j\}, \underline{s}\{i,k\}\} \mid \underline{s} \in [p]_0^{n-1}, i \in [p]_0, j,k \in [p]_0, i \neq k\right\} \cup \\ &\left\{\left\{\underline{s}ki^{d-2}\{i,j\}, \underline{s}\{i,k\}\right\} \mid \underline{s} \in [p]_0^{n-d}, d \in [n] \setminus \{1\}, i \in [p]_0, j,k \in [p]_0 \setminus \{i\}\right\} . \end{aligned} \tag{1.4}$$

The (generalized) Sierpiński triangle graph ST_4^1 is shown in Figure 1.4. Note that for $p = 3$ converting all the vertices from (1.4) into the idle peg labeling gives us the same edge set as in (1.3).

By the definition of Sierpiński triangle graphs and (1.2), we can deduce the order of Sierpiński triangle graphs, whereas their size follows directly from their construction, since we glue together complete graphs of order p.

Proposition 1.6. [34, Proposition 2.3] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then*

$$|ST_p^n| = \frac{p}{2}(p^n + 1)\,, \ \ \textit{and} \ \ \|ST_p^n\| = \frac{p-1}{2}p^{n+1}\,.$$

Directly from Definition 1.5 we can determine the degrees of the vertices in ST_p^n. An extreme vertex has obviously the same degree as the extreme vertex of S_p^{n+1}, that is $p-1$. All the other vertices have, by contraction, degree $2(p-1)$. Some other properties of the graphs ST_p^n were studied by Jakovac [34]. He proved that the Sierpiński triangle graphs are hamiltonian (for $p \geq 3$) and that their chromatic number equals p.

All non-extreme vertices of the Sierpiński graph S_p^n have degree p and the extreme vertices have degree $p-1$. So Sierpiński graphs are almost regular. This was the motivation to define two new families of Sierpiński-like graphs. Since there are p vertices of degree $p-1$ in S_p^n there

[1]Note that non-clique edges of S_p^{n+1} have the unique form, $\{ij^\ell, ji^\ell\}$, for distinct $i,j \in [p]_0$, and $\ell \in [n]$ and correspond to the move of type 1 in the Switching Tower of Hanoi.

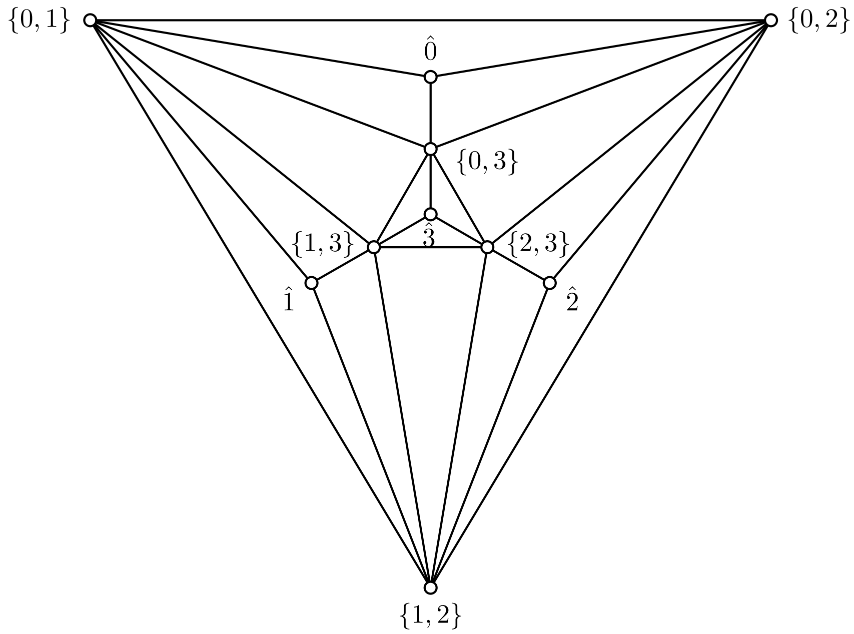


Figure 1.4: Sierpiński triangle graph ST_4^1

are two natural ways to regularize them, either we add another vertex to S_p^n and connect it with all the extreme vertices, or we add another copy of S_p^{n-1} and connect the extreme vertices of S_p^n with extreme vertices of S_p^{n-1}. To understand better the two possibilities, see Figure 1.5 for the case $p = 4$ and $n = 2$. The first possibility gives us the graph ${}^+S_p^n$, where the additional vertex w is called the *special vertex* of ${}^+S_p^n$. Formally:

Definition 1.7. *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then the graph ${}^+S_p^n$ is defined by*

$$V({}^+S_p^n) = [p]_0^n \cup \{w\}\,,$$
$$E({}^+S_p^n) = E(S_p^n) \cup \{\{w, i^n\} \mid i \in [p]_0\}\,.$$

Directly from the definition of ${}^+S_p^n$ and the size of S_p^n, we get

Proposition 1.8. *If $n, p \in \mathbb{N}_0$, then*

$$|{}^+S_p^n| = p^n + 1\,, \;\; \textit{and} \;\; \|{}^+S_p^n\| = \frac{p}{2}(p^n + 1)\,.$$

The other regularization, i.e., when adding another copy of S_p^{n-1} to S_p^n, is denoted by ${}^{++}S_p^n$. It can also be characterized as taking $p + 1$ copies of S_p^{n-1} (when building a Sierpiński graph S_p^n we take only p such copies) and joining their extreme vertices in the sense of the complete graph K_{p+1}. On the right-hand side of Figure 1.5 there are 5 copies of K_4 joined together as K_5.

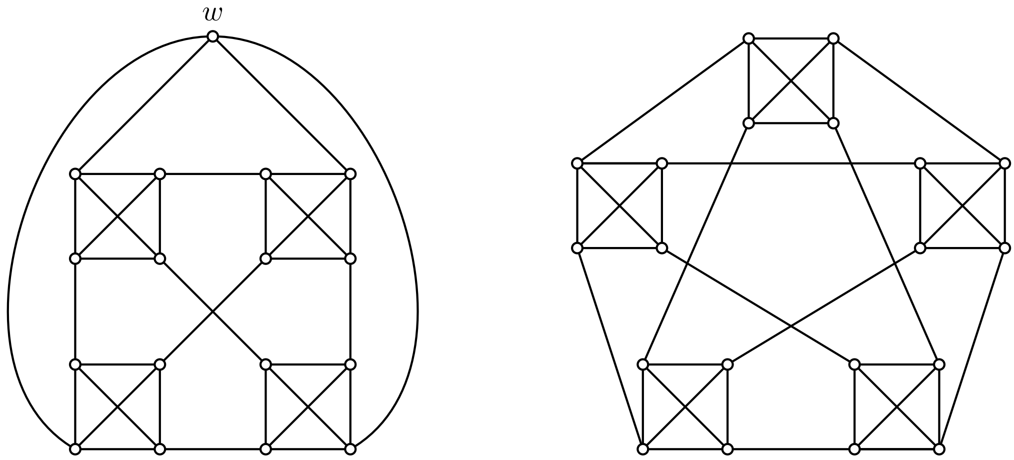


Figure 1.5: Regularizations ${}^{+}S_4^2$ (left) and ${}^{++}S_4^2$ (right)

We may think of K_4s as the vertices of K_5. This construction is similar to the construction of a Sierpiński graph, but with complete graphs of different orders. Here is a formal definition:

Definition 1.9. *Let $n, p \in \mathbb{N}$. Then the graph ${}^{++}S_p^n$ is defined by*

$$V({}^{++}S_p^n) = [p]_0^n \cup \{p\overline{s} \mid \overline{s} \in [p]_0^{n-1}\}\,,$$
$$E({}^{++}S_p^n) = E(S_p^n) \cup \{\{p\overline{s}, p\overline{t}\} \mid \{\overline{s}, \overline{t}\} \in [p]_0^{n-1}\} \cup \{\{pi^{n-1}, i^n\} \mid i \in [p]_0\}\,.$$

Similarly, we can deduce the order and the size of graphs ${}^{++}S_p^n$:

Proposition 1.10. *If $n, p \in \mathbb{N}_0$, then*

$$|{}^{++}S_p^n| = (p+1)p^{n-1}\,, \quad \textit{and} \quad \|{}^{++}S_p^n\| = \frac{p+1}{2}p^n\,.$$

1.3 Occurrences of Sierpiński-type graphs

As already mentioned before, when Klavžar and Milutinović [40] defined the graphs S_p^n, one of their main motivations was the connection to the Tower of Hanoi problem. This is also our main motivation to study metric properties on Sierpiński graphs. We will review this connection in more detail in the next subsection. The other main motivation was their connection to topology, because these graphs can also be derived from the Lipscomb spaces. For a comprehensive overview on the studies of these spaces see [51]. In particular, base-3-Sierpiński graphs S_3^n and the Sierpiński triangle graphs ST_3^n are closely related to the Sierpiński triangle fractal. For more information on the connection to topology see [27, Section 4.3].

Because of the way Sierpiński graphs are constructed, they are sometimes also called *iterated complete graphs*, denoted by K_p^n. See for instance the paper by Cull et al. [5] or any paper of his students, eg. [37, 66], which were done during the Summer Research Experiences for Undergraduates Program in Mathematics at the Oregon State University. They consider graphs isomorphic to Sierpiński graphs in relation to codes. Some variants of the Tower of Hanoi puzzle for which the corresponding graphs are the iterated complete graphs were also studied by the group of students, but only for odd p. For even values of p they generalize the idea of the spin-out puzzle. The spin-out puzzle is actually the same puzzle as Chinese rings, see [27, Chapter 1].

Another very similar structure to Sierpiński graphs is the class of *WK-recursive networks*. It was introduced by Della Vecchia and Sanges [7] in 1988 as a model for interconnection networks. In fact, $WK(p, n)$ is almost isomorphic to S_p^n. Both graphs are defined on the same vertex set $V(WK(p, n)) = [p]_0^n = V(S_p^n)$, and the edges are also the same, with the only exception that $WK(p, n)$ has additional p open edges or links, each at one of the extreme vertices. The open edges serve for further expansions. In this context various properties of these networks were studied, see for example [27, Section 4.2.3] or [29].

Even more frequent are occurrences of base-3-Sierpiński graphs. Here we will mention two of them. The first are truncations of maps, studied by Pisanski and Tucker [56]. By truncating a triangle (graph), for example S_3^1, we get a graph isomorphic to S_3^2. If T denotes the truncation operation on a graph, then $T(S_3^1) \cong S_3^2$. Repeating this, we get $T^n(S_3^n) \cong S_3^{n+1}$. Another very similar family of graphs are Schreier graphs, see [20] and [19]. As opposed to the truncated triangle, the Schreier graphs are not completely isomorphic to graphs S_3^n. To each extreme vertex a loop is attached. Schreier graphs were introduced in relation to the Hanoi Towers groups by Grigorchuk and Šunić [20] and are more closely related to Hanoi graphs, which we will define in the next subsection.

1.3.1 The Tower of Hanoi puzzle

In the introduction we presented the background of the Tower of Hanoi puzzle. Let us now consider the general version with n discs and p pegs. Keeping in mind that we may only move one disc at a time, we must also obey the divine rule, saying that no larger disc may be placed onto a smaller one. A *regular state* $s \in [p]_0$ of the puzzle is an arbitrary distribution of discs on p pegs such that no larger disc lies on a smaller one. A *perfect state* is a regular state where all discs are stacked on one peg. Finally, a *legal move* represents a move of a top disc obeying the divine rule.

There are three standard tasks:

- *P0 task* or *perfect to perfect task*, where the goal is to transfer all discs stacked on the starting peg i to the goal peg j;

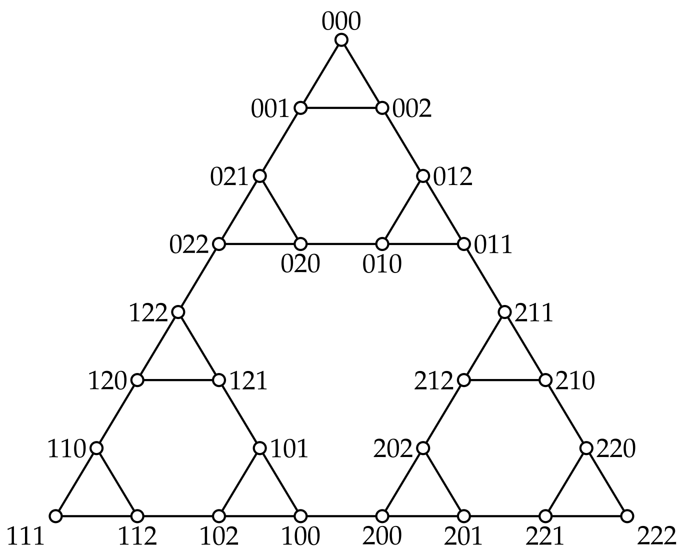
- *P1 task* or *regular to perfect task* with the goal of transferring the discs from a regular state $s \in [p]_0$ to a perfect state i^n (where $i \in [p]_0$);
- *P2 task* or *regular to regular task* where we move discs from one regular state to another.

As already mentioned, the Tower of Hanoi puzzle can be modeled with a state graph:

Definition 1.11. *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. The* Hanoi graph H_p^n *is the graph with the vertex set consisting of the regular states of the Tower of Hanoi puzzle, $V(H_p^n) = [p]_0^n$, where two vertices are adjacent if one can be obtained from the other by a legal move.*

Similarly to Sierpiński graphs, we will denote a vertex of H_p^n by $s_n \dots s_1$ meaning that a disc d lies on peg $s_d \in [p]_0$, for $d \in [n]$. A vertex of the form i^n is called a *perfect vertex* (of H_p^n), because it corresponds to the perfect state when all discs are on peg i.

For $p = 1$ we have a one-vertex graph, as well as for $n = 0$, $H_1^n \cong K_1 \cong H_p^0$. Similarly, for $p = 2$ and $n \in \mathbb{N}$, $H_2^n \cong nK_2$, since we are only allowed to move the smallest disc from one peg to another. As with Sierpiński graphs, $H_p^1 \cong K_p$. The case when $p = 3$ is also called the *classical case* and is isomorphic to S_3^n. The isomorphism can be given by an automaton, see [27, p. 143–145] for the automaton and a nice explanation of the isomorphism. To picture the isomorphism, compare H_3^3 (drawn in Figure 1.6) with the graph S_3^3 (drawn in Figure 1.1).



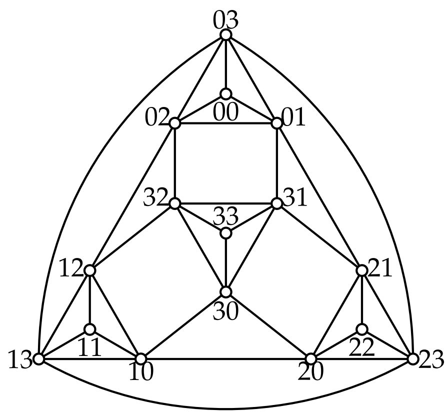


Figure 1.6: Hanoi graphs H_3^3 and H_4^2

Obviously,

$$|H_p^n| = p^n\,.$$

The size of the Hanoi graphs can be determined from their recursive construction. They are built in a similar way as Sierpiński graphs. We start with a complete graph $H_p^1 \cong K_p$ and make p copies of it. But when building a Hanoi graph H_p^n for $p > 3$, we add more edges between each two copies of H_p^{n-1} than we did with Sierpiński graphs. These edges correspond to the moves of the largest disc. So when moving the disc n from peg i to peg j, the discs $1, \ldots, n-1$ are neither on peg i nor on peg j. This means there are $(p-2)^{n-1}$ edges between iH_p^{n-1} and jH_p^{n-1}. (Note that in the case of Sierpiński graphs the edge between iS_p^{n-1} and jS_p^{n-1} is unique.) For $p \geq 3$ get

$$\begin{aligned} \|H_p^0\| &= 0\,, \\ \|H_p^n\| &= p \cdot \|H_p^{n-1}\| + \binom{p}{2}(p-2)^{n-1}\,, n \in \mathbb{N}\,, \end{aligned}$$

which gives us

$$\|H_p^n\| = \frac{p(p-1)}{4}(p^n - (p-2)^n)\,.$$

Note that the minimum number of moves for a P2 task from a state s to a state t corresponds to the distance $d_{H_p^n}(s,t)$. Therefore the study of metric properties of Hanoi graphs is both popular and important. Later (in Chapter 3) we will be studying distances in Sierpiński graphs. This might help us with distances in Hanoi graphs because of their similarity. In order to use these metric results we will also study embeddings of Sierpiński graphs into Hanoi graphs (Section 4.1). Therefore the following lemma about cliques in Hanoi graphs will be very useful.

Lemma 1.12. *If $p, n \in \mathbb{N}$, then every complete subgraph of H_p^n is induced by edges corresponding to moves of one and the same disc. In particular, $\omega(H_p^n) = p$ and the only p-cliques of H_p^n are of the form $s_n \ldots s_2 H_p^1$.*

Proof. The cases $p = 1$ and $p = 2$ are trivial. For $p \geq 3$ take any vertex s joined to two vertices s' and s'' by edges corresponding to the moves of two different discs. Then the positions of these discs differ in s' and s''. Since vertices in H_p^n can only be adjacent if they differ in precisely one coordinate, s' and s'' cannot be adjacent. This proves the first assertion. Any state s is contained in the p-clique induced by s and those states which differ from s only by the position of the smallest disc. On the other hand, a disc $d \neq 1$ can be transferred to at most $p-2$ pegs, namely those not occupied by disc 1, so that no clique larger than p exists. □

1.4 Classification of Sierpiński-type graphs

There are many graphs similar to Sierpiński graphs. In the literature we find different names for the same graphs, which can be confusing and, what is even worse, the same name for different graphs. In this section we will therefore standardize and harmonize the terms of Sierpiński graphs, Sierpiński triangle graphs etc, which we now call with one word **Sierpiński-type graphs**. We can characterize a representative of Sierpiński-type graphs as a graph which is derived from or leads to the Sierpiński triangle (fractal). The main representing classes of Sierpiński-type graphs are shown in Figure 1.7.

The first row of the diagram in Figure 1.7 represents the origins of Sierpiński-type graphs. These are the classical Hanoi graphs H_3^n. In 1990, the graphs S_3^n were used to determine the average distance on the Sierpiński triangle fractal. They were introduced with the help of the Sierpiński triangle fractal by Hinz and Schief [32]. There the name "Sierpiński graphs" was used for the first time. In [32] the authors also proved that $S_3^n \cong H_3^n$, represented in Figure 1.7 with an arrow between H_3^n and S_3^n in both directions. There is also an arrow in both directions between S_3^n and ST_3^n. The reason for the direction $S \to ST$ is the way we defined Sierpiński triangle graphs in Definition 1.4, and the other direction can be derived by taking a vertex for each (clique) triangle of ST_3^n and connecting two of them if the corresponding triangles share a vertex. Note that for the direction $S \to ST$ we actually take the graph S_3^{n+1} to obtain ST_3^n, but by the procedure we have just described we get the graph S_3^n.

Independently from the aforementioned authors, the name "Sierpiński graphs" was given to the graphs which we now call Sierpiński triangle graphs ST_3^n. The list of names for the graphs ST_3^n is hereby far from over. Mostly they were called Sierpiński gasket graphs, the name which in our opinion is not suitable, or similarly Sierpiński sieve graphs. Some authors even call graphs ST_3^n just Sierpiński gasket, without "graphs", which is actually one of the names of the Sierpiński triangle fractal and is therefore even more confusing.

Let us move to the second row of the diagram in Figure 1.7. Since $S_3^n \cong H_3^n$, in 1997 the idea arose to introduce the family of Sierpiński graphs $S(n,k)$ (in our notation S_p^n, where we replaced k by p for "pegs") as a state graph of the Switching Tower of Hanoi [40]. So the graphs S_3^n were generalized to S_p^n, where $p \in \mathbb{N}$. In a similar way that we constructed Sierpiński triangle graphs ST_3^n from graphs S_3^n we can perform this for an arbitrary $p \in \mathbb{N}$, see Definition 1.5. The family of generalized Sierpiński triangle graphs was first introduced by Jakovac in [34]. He used the notation $S[n,k]$ for the graphs which we now denote by ST_p^n (with k again replaced by p) and called them generalized Sierpiński gasket graphs. Later we decided to call them generalized Sierpiński triangle graphs, but we first used the notation $\widehat{S_k^n}$ in [44]. Since we wanted to standardize this notation we came up with ST_p^n, so that S in S_p^n stands for generic Sierpiński, and ST for Sierpiński triangle.

As mentioned at the end of Section 1.2, there are two ways to regularize Sierpiński graphs

S_p^n, i.e., the graphs ${}^+S_p^n$ and ${}^{++}S_p^n$. Other similar families are the WK-networks and Schreier graphs (for $p = 3$), which have some additional open edges and loops, respectively. All these families will be called **Sierpiński-like graphs**, since they are similar to Sierpiński graphs but not isomorphic to them. All these families are represented in the last, third row of the diagram. They can also be called *variants of Sierpiński graphs* or, even better, as *regularizations of Sierpiński graphs*. The rightmost family is a regularization of Sierpiński triangle graphs ${}^{++}ST_p^n$ and has not been introduced yet. The regularization can be done in a similar way as in the case of the graphs ${}^{++}S_p^n$. We will say more about this topic when discussing future work in Chapter 5.

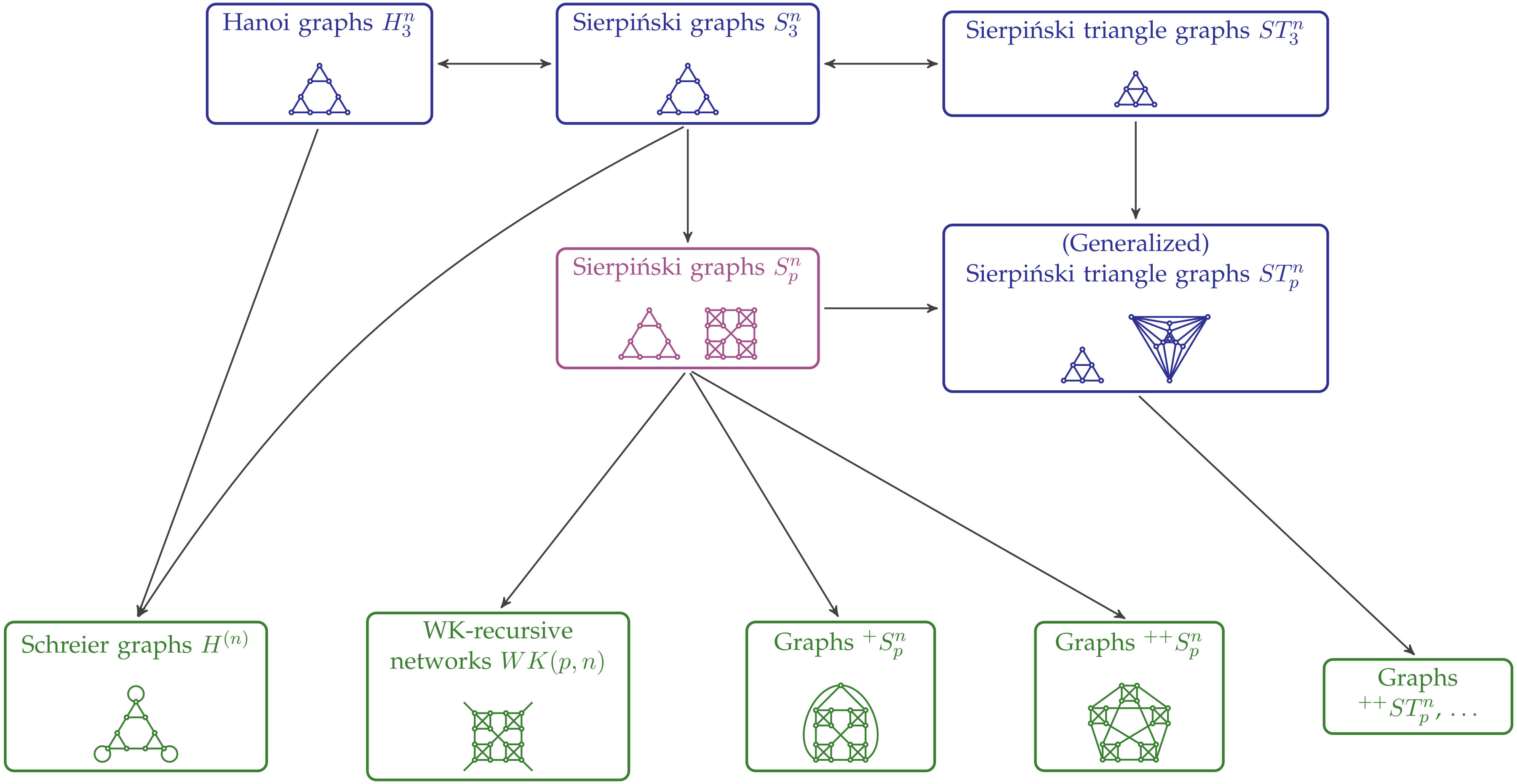


Figure 1.7: A diagram of the Sierpiński-type graphs

Chapter 2

A survey of known results on Sierpiński graphs

Ever since the family of Sierpiński graphs was introduced, it has been studied in different fields of mathematics and elsewhere. These studies were mostly motivated by the relation of the Sierpiński graphs to the Tower of Hanoi puzzle and also by their nice recursive structure. Although the recursive structure of these graphs is simple and similar to the structure of complete graphs, it is sometimes very difficult to prove their properties.

In this chapter we present known results on Sierpiński graphs. Sections that follow are organised into groups of similar properties. In the first section we discuss some standard properties of Sierpiński graphs, such as hamiltonicity or planarity. Next we devote a section to colorings of Sierpiński graphs, since many different colorings have been studied on this family. Another topic that has been studied extensively on these graphs is the theory of codes, domination and related problems. Known results from this area are given in the third section. In the last section we gather miscellaneous properties that have been observed on Sierpiński graphs.

The relation to the Tower of Hanoi is why metric properties of Sierpiński graphs play a very important role. Since the entire Chapter 3 is devoted to metric properties of Sierpiński graphs, we postpone a presentation of known results on this topic to Section 3.1.

2.1 Hamiltonicity and planarity

Already in 1997, when Klavžar and Milutinović introduced the family of Sierpiński graphs, they proved the following result about hamiltonicity of Sierpiński graphs.

Theorem 2.1. [40, Proposition 3] *If $n, p \in \mathbb{N}$ and $p \geq 3$, then the graph S_p^n is hamiltonian.*

A hamiltonian cycle of S_p^n can be constructed as follows. Let $iQ_{j,k}^{(n-1)}$ be a path in iS_p^{n-1}

between vertices ij^{n-1} and ik^{n-1}, such that it includes all the vertices from iS_p^{n-1} (such a path exists, for example use induction to prove it). Then we can build a hamiltonian cycle with

$$0Q^{(n-1)}_{(p-1),1} \cup e^{(n)}_{01} \cup 1Q^{(n-1)}_{0,2} \cup e^{(n)}_{12} \cup \cdots \cup e^{(n)}_{(p-2)(p-1)} \cup (p-1)Q^{(n-1)}_{(p-2),0} \cup e^{(n)}_{(p-1)0}\,.$$

Later Klavžar also showed [39] that in the case $p = 3$, the Sierpiński graphs contain a unique hamiltonian cycle. Xue et al. [70] deepened the study of hamiltonicity of Sierpiński graphs. They proved the following result.

Proposition 2.2. [70, Theorem 3.1] *If $n, p \in \mathbb{N}$ and $p \geq 2$, then S_p^n can be decomposed into an edge-disjoint union of $\lfloor \frac{p}{2} \rfloor$ hamiltonian paths the end vertices of which are extreme vertices.*

They also determined the number of edge-disjoint hamiltonian cycles of S_p^n.

Theorem 2.3. [70, Theorem 3.2] *If $n, p \in \mathbb{N}$ and $p \geq 3$, then S_p^n contains $\lceil \frac{p}{2} \rceil - 1$ edge-disjoint hamiltonian cycles.*

Another interesting and standard property to study on graphs is planarity. Let us establish which Sierpiński graphs are planar by determining for which values $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, S_p^n is planar. We have already seen that $S_1^n \cong K_1$ and $S_2^n \cong P_{2^n}$ and thus Sierpiński graph S_p^n is obviously planar for $p = 1, 2$ and arbitrary $n \in \mathbb{N}_0$. The graph S_3^n is planar for every $n \in \mathbb{N}_0$ as well. This for instance can be proved by induction. The graphs $S_4^1 \cong K_4$ and S_4^2 are planar, but S_4^3 is not planar. A planar drawing of S_4^2 is shown on the left side of Figure 2.1 and a K_5- subdivision of S_4^3 is depicted with gray vertices on the right side of Figure 2.1. Since S_4^3 is contained in any S_4^n for $n \geq 3$, none of these graphs is planar. Any S_p^n contains K_p as a subgraph and is thus not planar for $p \geq 5$.

Because most of the Sierpiński graphs are not planar, it is natural to study the crossing numbers of Sierpiński graphs. The crossing number, $\mathrm{cr}(G)$, of a graph G, is the minimum number of (edge) crossings of a drawing of G in the plane. In the case of Sierpiński graphs this was first studied by Klavžar and Mohar in 2005 [42]. The main result of the paper is an estimate of the crossing number of S_4^n.

Theorem 2.4. [42, Proposition 3.2] *If $n \in \mathbb{N}$, $n \geq 3$, then*

$$\frac{3}{16}4^n \;\leq\; \mathrm{cr}(S_4^n) \;\leq\; \frac{1}{3}4^n - \frac{12n-8}{3}\,.$$

In case $n = 3$ this asserts $\mathrm{cr}(S_4^3) = 12$ and a drawing of S_4^3 with 12 crossings is shown in Figure 2.1.

An upper bound on the crossing number of S_p^n for arbitrary $p \geq 5$ was also discussed in [42],

$$\mathrm{cr}(S_p^n) \leq \frac{p(p^{n-1}-1)}{p-1} \cdot \mathrm{cr}(K_{p+1}) + \mathrm{cr}(K_p)\,.$$

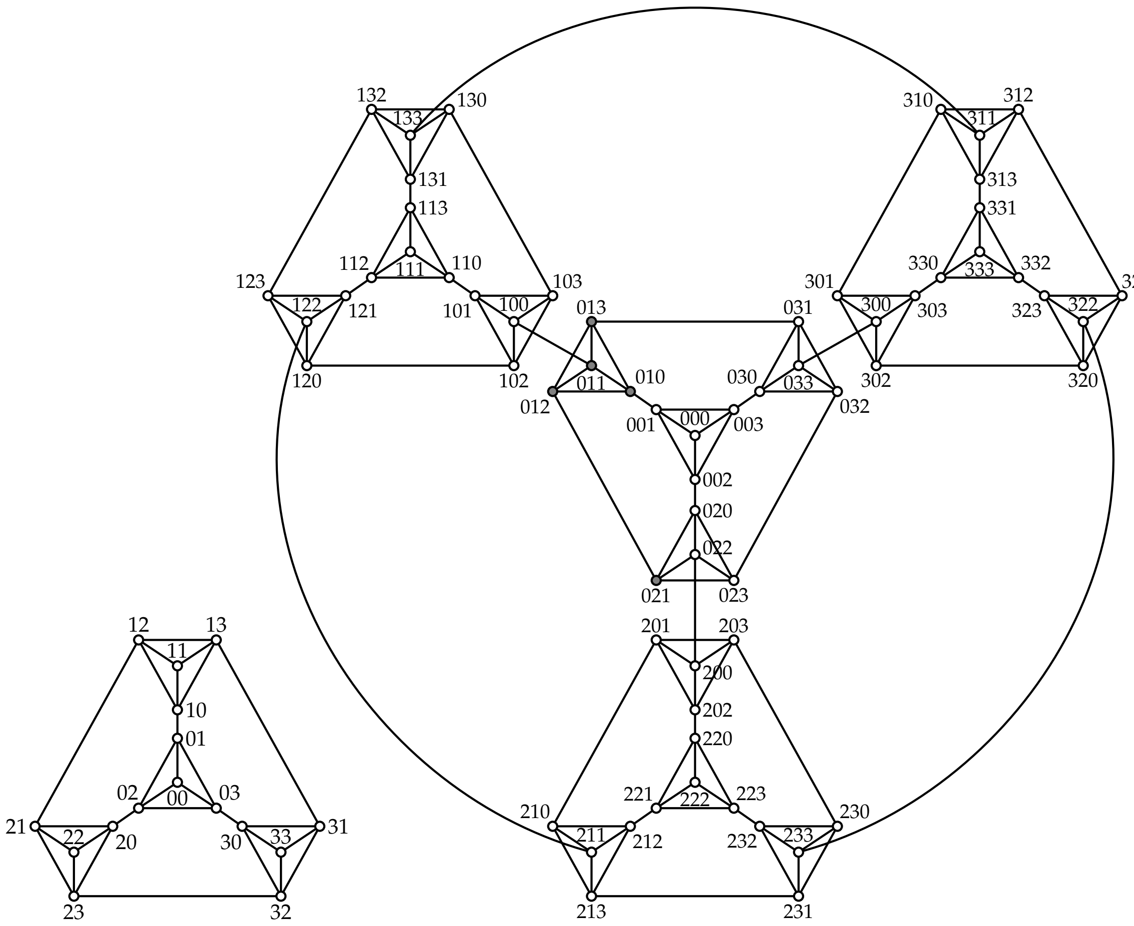


Figure 2.1: A planar drawing of S_4^2 (left) and a drawing of S_4^3 (right) with 12 crossings

This estimation is made using the regularization ${}^+S_p^{n-1}$, but is not always optimal (for example, it also holds for $p = 4$, but the result in Theorem 2.4 gives us a better estimation on the crossing number of S_4^n).

Later, in 2011 Köhler studied crossing numbers of Sierpiński graphs in his diploma thesis [46]. For $n = 2$ he expressed the crossing number of S_p^2 with crossing numbers of complete graphs and the graphs we get from complete graphs by deleting an edge (notation: K_n^-).

Theorem 2.5. [46, Satz 3.11] *If* $p \in \mathbb{N}$, *then*

$$\operatorname{cr}(S_p^2) = p \cdot \operatorname{cr}(K_{p+1}^-) + \operatorname{cr}(K_p)\,.$$

Determining the crossing number of graphs is in general NP-hard, therefore it is extremely satisfactory that we at least have these two results for Sierpiński graphs. Although, it would be interesting to find a better upper bound for arbitrary p.

2.2 Colorings

Different colorings have been studied on Sierpiński graphs so far. Before summarizing these results let us list the basic definitions about colorings of graphs.

A (proper) *k-coloring* of a graph G is a mapping c from the vertex set $V(G)$ to a set of size k (the *colors*), such that adjacent vertices receive different colors. If there is a k-coloring of G, we say that G is *k-colorable*. Then the *chromatic number*, $\chi(G)$, of G is the minimum integer k, such that G is k-colorable.

A (proper) *k-edge-coloring* of a graph G is a mapping c' from the edge set $E(G)$ to a set of size k, such that adjacent edges receive different colors. If there is a k-edge-coloring of G, we say that G is *k-edge-colorable*. Then the *chromatic index*, $\chi'(G)$, of G is the minimum integer k, such that G is k-edge-colorable.

A (proper) *k-total-coloring* of a graph G is a mapping c'' from the set $V(G) \cup E(G)$[1] to a set of size k, such that adjacent vertices or edges and incident vertices and edges receive different colors. If there is a k-total-coloring of G, we say that G is *k-total-colorable*. Then the *total chromatic number*, $\chi''(G)$, of G is the minimum integer k, such that G is k-total-colorable.

It was already observed by Parisse [52, p. 147] that $\chi(S_p^n) = p$ (for $p \geq 2$). Since K_p is a subgraph of S_p^n, it is obvious that $\chi(S_p^n) \geq p$. A coloring of S_p^n with p colors can be defined by

$$\begin{aligned} c : [p]_0^n &\to [p]_0 \,, \\ s_n \ldots s_1 &\mapsto s_1 \,. \end{aligned}$$

Later Klavžar [39] showed $\chi'(S_3^n) = 3$ and even more, these graphs are also uniquely 3-edge-colorable. The proof uses the fact that $\chi(ST_3^n) = 3$ and the 3-colorings of ST_3^n are in 1-1 correspondence with the 3-edge-colorings of S_3^n.

Afterwards Jakovac and Klavžar studied vertex, edge- and total-colorings of Sierpiński graphs [36]. Some of their results were also independently proved by Hinz and Parisse [30]. The next theorem about the chromatic index of Sierpiński graphs is one of them.

Theorem 2.6. [36, Theorem 4.1], [30, Theorem 3] *If $n, p \in \mathbb{N}$ and $n, p \geq 2$, then $\chi'(S_p^n) = p$.*

Since chromatic number and index was now known for all Sierpiński graphs, the only open question was the total chromatic number of Sierpiński graphs. Jakovac and Klavžar [36] proved that it is bounded by $p + 2$ and also showed the exact value when p is odd.

[1] For our purposes we may assume that $V(G) \cap E(G) = \emptyset$.

Proposition 2.7. [36, Proposition 4.3] *If $n, p \in \mathbb{N}$, $n \geq 2$ and $p \geq 3$ is odd, then $\chi''(S_p^n) = p + 1$.*

It seemed a little bit more complicated if p is even. In [36] it was proved that the total chromatic number of S_4^n is 5 and conjectured that the total chromatic number of S_p^n for even $p > 4$ equals $p+2$. Hinz and Parisse [30] disproved the conjecture and found the missing result about total chromatic number of Sierpiński graphs.

Theorem 2.8. [30, Theorem 4] *If $n, p \in \mathbb{N}$ and $n, p \geq 2$, then $\chi''(S_p^n) = p + 1$.*

In this article they also gave explicit vertex-, edge- and total-colorings of Sierpiński graphs. An example is shown in Figure 2.2. In the figure one can find a 5-edge-coloring of S_5^2 and a 5-total coloring of S_4^2.

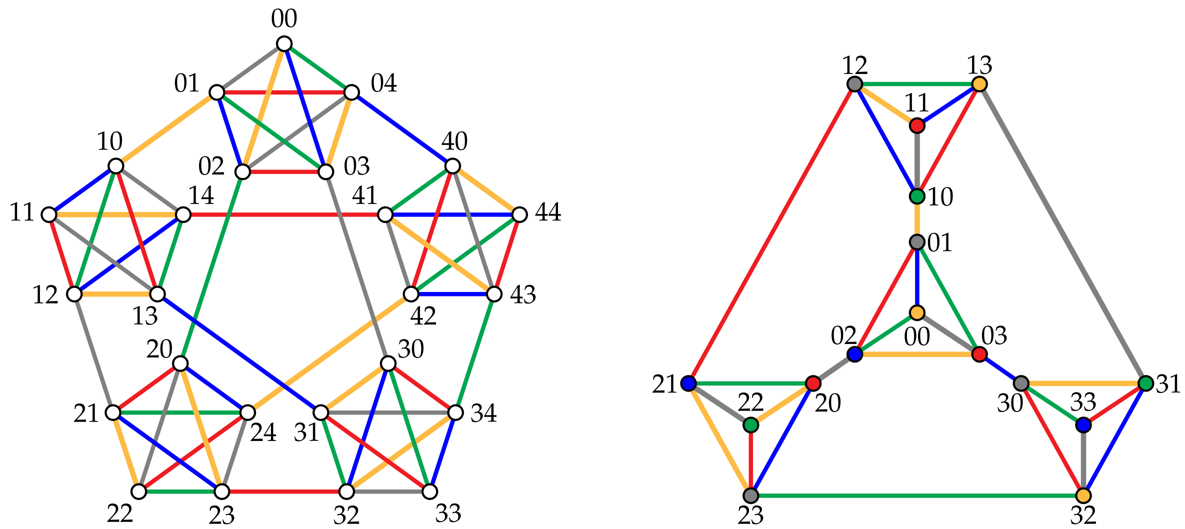


Figure 2.2: A 5-edge-coloring of S_5^2 (left) and a 5-total-coloring of S_4^2 (right)

In the last years different colorings with special properties have been defined, for example b-colorings, distance colorings, $\{P_r\}$-free colorings and linear colorings. A k-coloring of a graph G is a *b-coloring* of G, if there is a vertex in each color class that is adjacent to a vertex in every other color class. The *b-chromatic number*, $\varphi(G)$, of G is the maximum integer k, such that there exists a b-coloring of G with k colors. It is well known, that any proper $\chi(G)$-coloring of G is also a b-coloring. The b-chromatic number of a graph G is bounded above with $\Delta(G) + 1$:

$$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1 \,.$$

Jakovac studied b-colorings of Sierpiński graphs in his Ph.D. thesis [35], where he determined their b-chromatic number. For $n = 1$ we have by the above estimation $\varphi(S_p^1) = \varphi(K_p) = p$. The same holds for $p = 1$, $\varphi(S_1^n) = \varphi(K_1) = 1$, and for other values of n and p the following result holds.

Proposition 2.9. [35, Trditev 5.1] *If $n, p \in \mathbb{N}$ and $n, p \geq 2$, then $\varphi(S_p^n) = p + 1$.*

Suppose $\mathcal{F}$ is a nonempty family of connected bipartite graphs, where each member F of $\mathcal{F}$ has at least 3 vertices. Then a k-coloring of a graph G is *$\mathcal{F}$-free* if G contains no 2-colored subgraphs isomorphic to any graph F of $\mathcal{F}$. The $\mathcal{F}$-free chromatic number, $\chi_{\mathcal{F}}(G)$, of G is the minimum integer k, such that there exists an $\mathcal{F}$-free coloring of G with k colors. If $\mathcal{F} = \{P_3\}$, then an $\mathcal{F}$-free coloring of G is equivalent to a 2-distance coloring of G, and similarly if $\mathcal{F} = \{P_4\}$ we get a star coloring. Fu examined the $\{P_r\}$-free colorings of Sierpiński graphs in [12]. In particular he determined some of their $\{P_r\}$-free chromatic numbers.

Proposition 2.10. [12, Lemma 4.1, Theorem 4.2] *If $n, p \in \mathbb{N}$ and $n, p \geq 2$, then*

$$\chi_{P_3}(S_p^n) = p + 1 = \chi_{P_4}(S_p^n) .$$

As a consequence of this result, Fu showed [12, Corollary 4.4] that for every $n \geq 1$, $p \geq 2$ and for arbitrary $5 \leq r \leq p^n$, the $\{P_r\}$-free chromatic number of Sierpiński graphs is bounded by

$$p \leq \chi_{P_r}(S_p^n) \leq p + 1 .$$

Xue et al. studied path t-colorings [70] and linear t-colorings [71] on Sierpiński graphs. For the definition of these colorings we need the concept of linear forests. A *linear forest* is a graph, whose connected components are paths. Let c be a mapping from the set of vertices of a graph G to a set of size t, whose elements we will call colors. Then $G[c^{-1}(i)]$ denotes the subgraph of G induced by the vertices of color i. The mapping c is called a *path t-coloring* of G if for each i, $G[c^{-1}(i)]$ is a linear forest. The *vertex linear arboricity*, $\mathrm{vla}(G)$, of G is the minimum t such that there exists a path t-coloring of G. The authors of [70] determined the vertex arboricity of Sierpiński graphs.

Proposition 2.11. [70, Theorem 4.1] *If $n, p \in \mathbb{N}$ and $p \geq 3$, then*

$$\mathrm{vla}(S_2^n) = 1 , \quad \textit{and} \quad \mathrm{vla}(S_p^n) = \frac{p + [p \textit{ odd}]}{2} .$$

A *linear t-coloring* of G is a proper t-coloring such that the graph induced by the vertices of any two colors is a linear forest. The *linear chromatic number*, $lc(G)$, of G is the minimum t such that there exists a linear t-coloring of G. Xue et al. determined the linear chromatic number of Sierpiński graphs and it equals their chromatic number.

Proposition 2.12. [71, Theorem 3.4] *If $n, p \in \mathbb{N}$, then*

$$lc(S_p^n) = p .$$

In the meantime also the edge ranking number was studied on Sierpiński graphs. Let c' be a t-edge-coloring of a graph G. We assume that the set of colors is $[t]$. Then we say that c' is an *edge t-ranking* if for any two edges of the same color, every path between them contains an intermediate edge with a larger color. The *edge ranking number*, $\chi'_r(G)$, is the smallest integer t, such that there exists an edge t-ranking of G. Lin et al. [50] proved a relation between the edge ranking number of Sierpiński graphs and the edge ranking number of complete graphs.

Proposition 2.13. [50, Theorem 7] *If* $n, p \in \mathbb{N}$ *and* $n, p \geq 2$, *then*

$$\chi'_r(S_p^n) = n \cdot \chi'_r(K_p)\,.$$

Proposition 2.13 implies the following result.

Corollary 2.14. [50, Corollary 8] *If* $n, p \in \mathbb{N}$ *and* $n, p \geq 2$, *then*

$$\chi'_r(S_p^n) = \frac{n}{3}(p^2 + g(p))\,,$$

where g is the Bodlaender function, defined as $g(1) = -1$ and

$$g(m) = \begin{cases} g\left(\frac{m}{2}\right), & m \text{ even}, \\ g\left(\frac{m+1}{2}\right) + \frac{m-1}{2}, & m \text{ odd}. \end{cases}$$

2.3 Codes, domination and $L(2,1)$-labelings

Several paper about codes and related topics on Sierpiński graphs have been published so far. Some of them also very recently. To summarize their main results, let us start with some background about codes.

Let G be a graph and $t \in \mathbb{N}$. A set of vertices $C \subseteq V(G)$ is a *t-code* in G, if for any two (distinct) vertices u, v of G, $d_G(u,v) \geq 2t+1$. The set C is called a *t-perfect code*, if for any $v \in V(G)$ there is exactly one $c \in C$ such that $d(c,v) \leq t$. In particular, if C is a 1-perfect code of G, then $N_G[C] = V(G)$. The elements of a code are often called *codewords*.

A subset $D \subseteq V(G)$ is *dominating*, if every vertex in $V(G) \setminus D$ has at least one adjacent vertex in D, i.e., $N_G[D] = V(G)$. The *domination number* of a graph G, $\gamma(G)$, is the order of a smallest dominating set in G.

1-perfect codes of a graph are obviously also dominating sets in it. For this reason they are sometimes called efficient dominating sets. Thus, if C is a 1-perfect code of G, $\gamma(G) \leq |C|$. Even more, the following result was independently proven several times (cf. [41, Proposition 2.1] and references therein). We will give a nice short proof.

Proposition 2.15. [25] *If C is a 1-perfect code of a graph G, then $\gamma(G) = |C|$. In particular, all perfect codes of G have the same cardinality.*

Proof. Let $C = \{c_1, \dots, c_\ell\}$ be a 1-perfect code of G. Then $N[C] = V(G)$ and this implies $\gamma(G) \leq \ell$.

Let $D = \{d_1 \dots, d_{\ell'}\}$ be a dominating set of G. Then for an arbitrary $i \in [\ell]$ there is a $j \in [\ell']$, so that $d_j \in N[c_i]$. By taking the minimal such j, we get an injective mapping from $[\ell]$ to $[\ell']$. It is indeed injective, since for arbitrary distinct $c, c' \in C$, $N[c] \cap N[c'] = \emptyset$. Now, by using the pigenhole principle, $\ell \leq \ell'$. $\square$

Although determining whether a graph has a 1-perfect code or not is NP-complete, Klavžar, Milutinović and Petr proved [41] that all Sierpiński graphs possess 1-perfect codes. More precisely, they proved the following theorem.

Theorem 2.16. [41, Theorem 3.6] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then the graph S_p^n has a unique 1-perfect code, if n is even, and there are exactly p 1-perfect codes, if n is odd. Moreover, if n is odd, then each 1-perfect code is determined by the only extreme vertex it contains.*

An example of 1-perfect codes in Sierpiński graphs is given in Figure 2.3 on graphs S_4^2 and S_3^3. The three 1-perfect codes of S_3^3 are shown on the right side of the figure in red, blue and yellow, respectively.

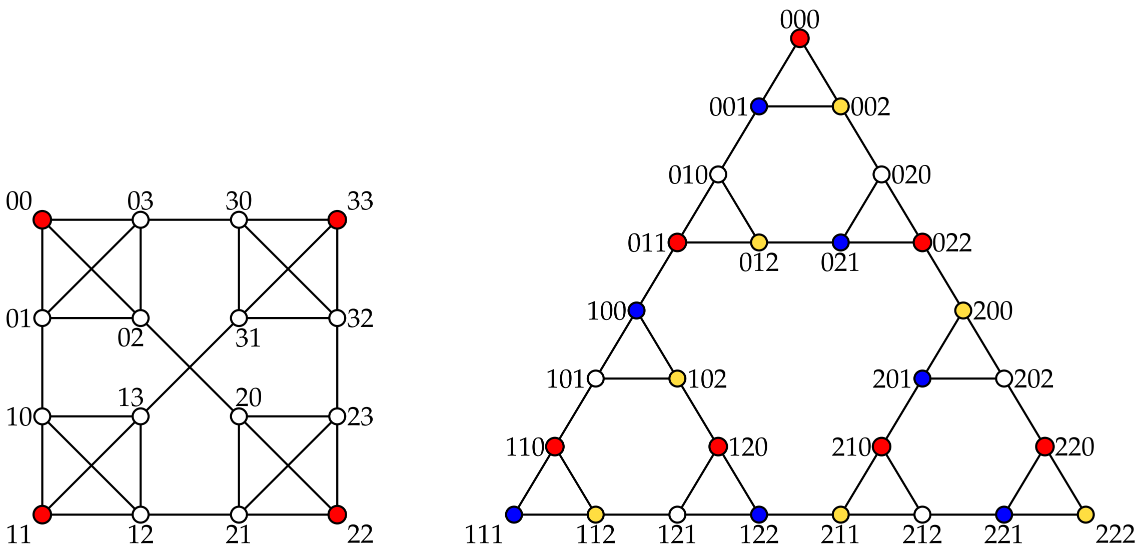


Figure 2.3: 1-perfect codes of S_4^2 (left) and S_3^3 (right)

In [41] the authors also gave an algorithm that decides, for a given 1-perfect code C of S_p^n and a vertex v of S_p^n, whether v is a codeword of C, and if not, the algorithm determines the neighbor vertex of v in C.

With Theorem 2.16 we are able to determine the domination number of S_p^n. All that needs to be done is to count the vertices in (one of) the 1-perfect code(s).

Theorem 2.17. [41, Theorem 3.8] *If* $n \in \mathbb{N}_0$ *and* $p \in \mathbb{N}$, *then*

$$\gamma(S_p^n) = \frac{p^n + p^{[n \text{ even}]}}{p+1}\,.$$

Very recently Dorbec and Klavžar studied generalized power domination on Sierpiński graphs [9]. The problem of generalized power domination is often called k-power domination and generalizes both, the concept of domination and the concept of power domination. We want to determine a subset S of vertices of a graph G, such that starting with $X = N[S]$ and iteratively adding vertices to $X = N[S]$ which have a neighbor v in X and at most k neighbors of v are not yet in X, we get $X = V(G)$. The *k-power domination number,* $\gamma_{P,k}(G)$ of G is the minimum size of such a subset of vertices S. In [9] the authors determined the k-power domination number of Sierpiński graphs.

Theorem 2.18. [9, Theorem 3.1] *Let* $n \in \mathbb{N}_0$ *and* $p, k \in \mathbb{N}$. *Then*

$$\gamma_{P,k}(S_p^n) = \begin{cases} 1\,, & p \in [2] \text{ or } n \in [2]_0 \text{ or } p \leq k+1\,,\\ p-k\,, & n = 2 \text{ and } p \geq k+1\,,\\ (p-k-1)p^{n-2}\,, & \text{otherwise.} \end{cases}$$

The proof of this theorem is not straightforward. It uses for example the fact that Sierpiński graphs are hamiltonian to prove the upper bound in some cases.

Since there may be many k-power dominating sets in a graph G and not all of them have the same efficiency, Dorbec and Klavžar introduced the concept of the *propagation radius*, $\mathrm{rad}_{P,k}(G)$. This is a measure of the efficiency of a k-power dominating set and is defined as 1 + a minimum number of iterations in the process of k-power dominating the graph G, when starting with a k-power dominating set S, taken over all minimum k-power dominating sets of G.

The propagation radius of Sierpiński graphs was almost completely determined:

Theorem 2.19. [9, Theorem 5.3] *Let* $n, p, k \in \mathbb{N}$ *and* $n \geq 3$. *Then*

$$\mathrm{rad}_{P,k}(S_p^n) = \begin{cases} 3\,, & p \geq 2k+3\,,\\ 4 \text{ or } 5\,, & 2k+2 \geq p \geq k+1+\sqrt{k+1}\,,\\ 5\,, & k+1+\sqrt{k+1} > p \geq k+2\,,\\ \mathrm{rad}(S_p^n)\,, & p \leq k+1\,. \end{cases}$$

Another similar concept was studied on Sierpiński graphs in [47]. We call a nonempty set of vertices $S \subseteq V(G)$ of a graph G a *defensive alliance*, if for every vertex $v \in S$, $|N_S[v]| \geq |N_{V(G)\setminus S}(v)|$. A subset S of vertices is called a *strong defensive alliance*, if for every vertex $v \in S$,

$|N_S[v]| > |N_{V(G)\setminus S}(v)|$. Further, a strong defensive alliance of G is *global*, if it forms a dominating set in G. Lin et al. [47] examined the global strong defensive alliance number $\gamma_{\hat{d}}(S_p^n)$ of Sierpiński graphs; this is the minimum cardinality of a global strong defensive alliance.

Theorem 2.20. [47, Theorem 3.9] *If $n \in \mathbb{N}$, $n \geq 2$ and $p \in \mathbb{N}$, $p \geq 3$, then*

$$\gamma_{\hat{d}}(S_p^n) = \frac{p + [p\ odd]}{2} \cdot p^{n-1}\,.$$

The proof of the above theorem is constructive. An example of an optimal global strong defensive alliance can be found in Figure 2.4 for the case S_4^3.

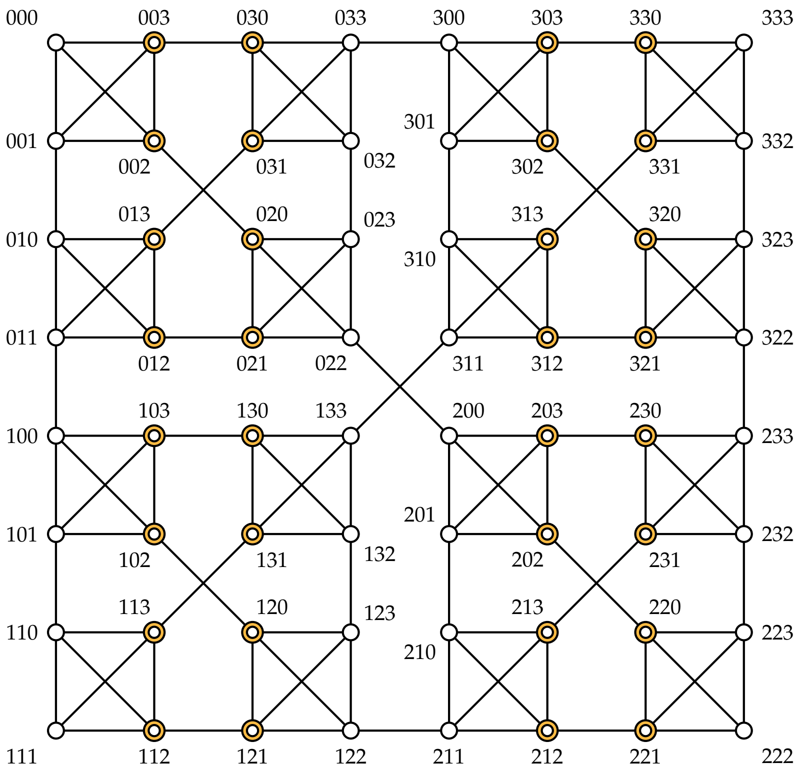


Figure 2.4: An optimal global strong defensive alliance of S_4^3

An $L(2,1)$-*labeling* of a graph G is a labeling of its vertices with labels $\{0, 1, \ldots, \lambda\}$ such that vertices at distance two get different labels and the labels of adjacent vertices differ by at least 2. The concept comes from a more general labeling, namely $L(\ell_1, \ldots, \ell_k)$-labeling. This is a labeling of vertices of G such that the labels of vertices at distance i differ by at least ℓ_i. The maximum label used in an $L(\ell_1, \ldots, \ell_k)$-labeling f is called the *span* of the labeling f and the aim is to minimize the span of a labeling. In the case of Sierpiński graphs we will only deal with $L(2,1)$-labelings. A minimum span of an $L(2,1)$-labeling of a graph G is denoted by $\lambda(G)$

and is called the λ-*number* or $L(2,1)$-*labeling number* of G.

An $L(2,1)$-labeling of a graph G also gives us a partition of its vertex set $V(G)$ into 1-codes. Indeed, let f be an $L(2,1)$-labeling of G with span λ and for each $i \in [\lambda+1]_0$ denote by C_i the set of vertices u with $f(u) = i$. Then the sets $C_0, \ldots, C_\lambda$ form a partition of $V(G)$ and two distinct vertices in C_i are at distance at least three. In 2005 [17], the authors studied codes of Sierpiński graphs in order to obtain an $L(2,1)$-labeling of Sierpiński graphs. They proved a general result connecting codes and the λ-number of a graph.

Proposition 2.21. [17, Proposition 1.1] *If G is a graph and $\{C_0, \ldots, C_k\}$ is a partition of $V(G)$, such that for each $i \in [k+1]_0$, C_i is a code in G, then $\lambda(G) \leq 2k$.*

With this approach they were able to determine the λ-number of Sierpiński graphs.

Theorem 2.22. [17, Theorem 3.2] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then*

$$\lambda(S_p^n) = 2p\,.$$

A special type of $L(2,1)$-labelings are equitable $L(2,1)$-labelings. An $L(2,1)$-labeling is *equitable*, if the orders of its color classes differ by at most one. The *equitable $L(2,1)$-labeling number*, $\lambda_e(G)$, of a graph G is then the smallest integer ℓ, such that there is an equitable $L(2,1)$-labeling of G with span ℓ. Fu and Xie [13] determined the equitable $L(2,1)$-labeling number of Sierpiński graphs and it equals their $L(2,1)$-labeling (or λ-)number.

Theorem 2.23. [13, Theorem 3.3] *If $n, p \in \mathbb{N}$, $n, p \geq 2$, then*

$$\lambda_e(S_p^n) = 2p\,.$$

Another type of codes that were studied on the family of Sierpiński graphs are (a,b)-codes. First let us define the concept of covering codes. We say that a subset $C \subseteq V(G)$ *covers a vertex* $u \in V(G)$, if $u \in C$ or there exists a neighbor of u in C and a code C is a *covering code*, if C covers all the vertices of G.

If G is a graph and $a, b \in \mathbb{N}_0$, then an (a,b)-*code* of G is a set C of vertices with the property that a vertex in C has exactly a neighbors in C and a vertex, which is not in C, has exactly b neighbors in C. The (a,b)-codes are obviously covering codes of a graph (as soon as $a + b \geq 1$). Beaudou et al. [4] determined all possible pairs (a,b), for which there exist (a,b)-codes in a Sierpiński graph S_p^n. The main result of the paper is

Theorem 2.24. [4, Theorem 1.1] *If $n \in \mathbb{N}$, $n \geq 2$ and $p \in \mathbb{N}$, $p \geq 2$, then S_p^n contains an (a,b)-code if and only if $a < p$ and one of the following statements holds.*

(i) $a \geq 1$, $b = a$ *and* p *is even;*
(ii) $a \geq 2$ *is even,* $b = a$ *and* p *is odd;*

(iii) $a = 0$ *and* $b = 1$ *(the case of 1-perfect codes);*

(iv) $a \geq 1$, $b = a + 1$ *and* n *is odd;*

(v) $a \geq 1$, $b = a + 2$, $n = 2$ *and* $p = 2a + 1$.

From the construction of the proof of the above theorem in [4] it also follows that all existing (a, b)-codes in graphs S_p^n are unique up to symmetries.

This concept of codes was further extended. First let us define some properties of codes. Let G be a graph and $C \subseteq V(G)$ a code. Then

- x *covers* or *dominates* a vertex $u \in V(G)$, if $u \in N[x]$;
- C *covers* or *dominates* a vertex $u \in V(G)$, if u is dominated by some vertex $v \in C$ (i.e., $u \in N[C]$);
- C *covers* or *dominates* a set $S \subseteq V(G)$, if every vertex of S is dominated by a vertex of C (i.e., $S \subseteq N[C]$);
- x *separates* vertices u and v of G, if x dominates exactly one of the vertices u and v;
- C *separates* a set $S \subseteq V(G)$, if every pair of vertices u and v of S is separated by at least one vertex of C (i.e., $N[u] \cap C \neq N[v] \cap C$).

With these terms we can define different codes. We say that C is (in G)

- a *total-dominating code* if it totally covers all the vertices of G,
- an *identifying code* if it is a covering code of G that separates all pairs of distinct vertices of G.
- a *locating-dominating code* if it is a covering code of G that separates all pairs of distinct vertices of $V(G) \setminus C$.

In [18] Gravier et al. gave the minimum sizes of identifying codes, locating-dominating codes, and total-dominating codes of Sierpiński graphs.

Theorem 2.25. [18, Theorems 2.1, 3.1, and 4.1] *If* $n \in \mathbb{N}_0$ *and* $p \in \mathbb{N}$, *then*

(i) the minimum cardinality of an identifying code in S_p^n *is*

$$p^{n-1}(p-1)\,,$$

(ii) the minimum cardinality of a locating-dominating code in S_p^n *is*

$$\frac{p^{n-1}(p-1)}{2}\,, \textit{ and}$$

(iii) the minimum cardinality of a total-dominating code in S_p^n *is*

$$p^{n-1} + [n \textit{ odd}]\,.$$

A set $Q \subseteq V(G)$ is a *hub set* of a graph G if, for every pair of vertices $u, v \in V(G) \setminus Q$, there exists a u, v-path such that all intermediate vertices on this path are in Q. The *hub number*,

$h(G)$, of a graph G is the size of a smallest hub set of G. A hub set Q of G is *connected*, if Q is a connected set (i.e., the subgraph of G induced by Q is connected). The *connected hub number*, $h_c(G)$, of a graph G is the size of a smallest connected hub set of G. In a similar way we can also define the *connected domination number*, $\gamma_c(G)$, of a graph G as the size of a smallest connected dominating set. The concept of hub sets was introduced in 2006 by Walsh [65]. He also showed that for a graph G, $\gamma(G) \leq h(G)+1$ and if G is connected, also $h_c(G) \leq \gamma_c(G)$ holds. A bit later in 2008 Grauman et al. [16] combined these properties into the following result.

Theorem 2.26. [16, Theorem 2.1] *If G is a connected graph, then*

$$h(G) \leq h_c(G) \leq \gamma_c(G) \leq h(G)+1\,.$$

Walsh [65] also showed that the problem to determine whether a given graph G has a hub set of (a given) size k is NP-hard. Lin et al. [48] determined the hub number of Sierpiński graphs.

Theorem 2.27. [48, Theorem 9] *If $n, p \in \mathbb{N}$, then*

$$h_c(S_p^n) = h(S_p^n) = 2(p^{n-1}-1)\,.$$

The proof is constructive. An optimal hub set of a Sierpiński graph S_p^n which was used for it is the following:

$$Q_{S_p^n} = \{\underline{s}0\ell^d, \underline{s}\ell 0^d \mid d \in [n-1], \underline{s} \in [p]_0^{n-d-1}, \ell \in [p-1]\}\,.$$

An example of an optimal hub set of S_4^3 is shown in Figure 2.5. Using symmetry we could get p different optimal hub sets by replacing 0 with any $i \in [p]_0$ in the upper set. If $i \in [p]_0$ is fixed, then

$$Q_{S_p^n}^{(i)} = \{\underline{s}i\ell^d, \underline{s}\ell i^d \mid d \in [n-1], \underline{s} \in [p]_0^{n-d-1}, \ell \in [p-1]\}$$

is also an optimal hub set for S_p^n, and $Q_{S_p^n}^{(0)} = Q_{S_p^n}$.

2.4 Other properties

In the final section of this chapter we will just briefly summarize the other properties that have been studied on Sierpiński graphs.

Not many algebraic properties of Sierpiński graphs have been studied so far, although they have some symmetries. While studying crossing numbers on Sierpiński graphs, Klavžar and Mohar [42] determined the group of automorphisms for Sierpiński graphs.

Theorem 2.28. [42, Lemma 2.2] *If $n \in \mathbb{N}$ and $p \in \mathbb{N}$, then the automorphism group of S_p^n is isomorphic to* $\mathrm{Sym}(p)$, *where* $\mathrm{Aut}(S_p^n)$ *acts as* $\mathrm{Sym}(p)$ *on the extreme vertices of S_p^n.*

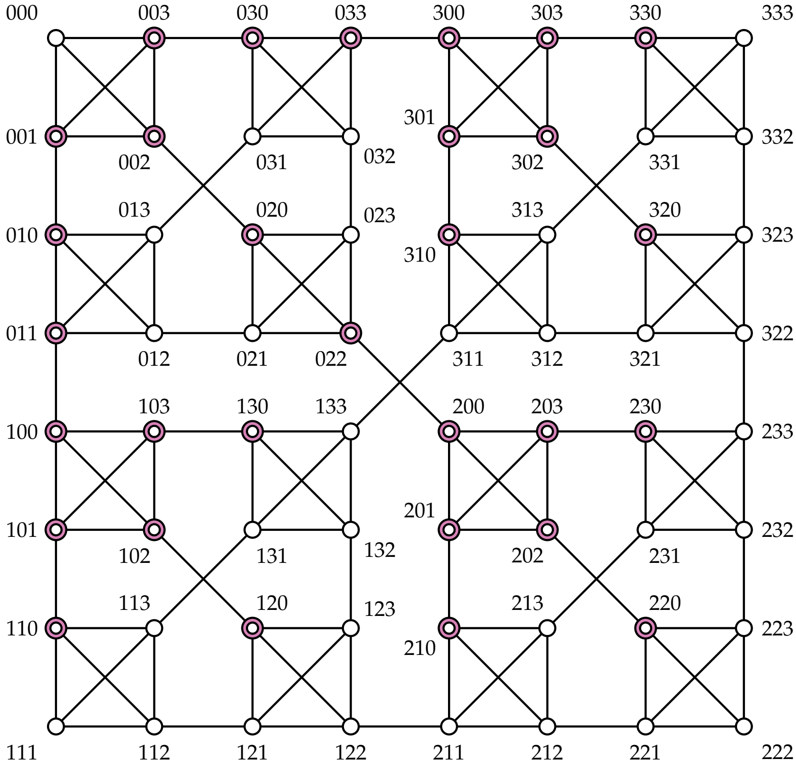


Figure 2.5: An optimal hub sets of S_4^3

In other words, an automorphism of S_p^n is uniquely determined by the permutation of its extreme vertices.

In a very recent book on the Tower of Hanoi problem [27] Hinz et al. proved the following proposition on the clique number of Sierpiński graphs. The proof goes simply by induction.

Proposition 2.29. [27, Theorem 4.3] *If $n, p \in \mathbb{N}$ and $p \geq 3$, then the only maximal cliques (with respect to inclusion) in S_p^n are the p-cliques $\underline{s}S_p^1$ with $\underline{s} \in [p]_0^{n-1}$ and 2-cliques induced by the non-clique edges. In particular, $\omega(S_p^n) = p$.*

Hinz et al. determined the connectivity of Sierpiński graphs in their very recent book [27] on the Tower of Hanoi puzzle and related problems. It equals the connectivity of complete graphs. This is not surprising – Sierpiński graphs are built in a similar manner as complete graphs.

Proposition 2.30. [27, Exercise 4.7] *If $n, p \in \mathbb{N}$, then $\kappa(S_p^n) = p - 1$.*

To see that we need at most $p-1$ vertices, we can simply delete the neighbors of an extreme vertex. To see that deleting $p-2$ does not suffice, one should use induction.

In general, for any (connected) graph G it holds that

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)\,,$$

hence the edge-connectivity is trivial to determine, $\kappa'(S_p^n) = p - 1$.

The next interesting property which was studied on base-3-Sierpiński graphs is the number of spanning trees. This field of graph theory is closely related to the analysis of electrical networks as well as to statistical physics, in particular to the Potts model. For more information on these topics see the fundamental article by Kirchhoff [38][2] and a tutorial review on the statistical properties of the Potts model by Wu [69], respectively. Due to the connection between spanning trees in graphs and physics, many studies on this subject were done. Teufl and Wagner determined the number of spanning trees of base-3-Sierpiński graphs in 2011 [63]. Their results were extensively presented in the book [27, p. 101–104] with an additional proof for the number of spanning trees.

The number of spanning trees in a graph G is denoted by $\tau(G)$ and is also called the *complexity* of G. The classical way to obtain this number is by computing the *Kirchhoff matrix*, $K(G)$, of G according to the Matrix-Tree theorem (see for instance [64, Theorem VI.29]). Let $A(G)$ be the adjacency matrix of G and $D(G)$ the diagonal matrix whose diagonal entries are the degrees of the corresponding vertices. Then

$$K(G) = D(G) - A(G)\,.$$

Next we choose a vertex of G and delete the row and column of K corresponding to it. Denote the matrix obtained by $K^-(G)$. Then

$$\tau(G) = \det(K^-(G))\,.$$

This procedure is unfortunately not very helpful for Sierpiński graphs, since the matrices are very large. Therefore an alternative approach has been used for the proof of the next theorem, see [27, p. 101–104] for more details.

Theorem 2.31. [63, p. 892], [27, Theorem 2.24] *If $n \in \mathbb{N}_0$, then the complexity of S_3^n equals*

$$\tau(S_3^n) = 3^{\frac{1}{4}(3^n-1)+\frac{1}{2}n} \cdot 5^{\frac{1}{4}(3^n-1)-\frac{1}{2}n} = \left(\sqrt{\frac{3}{5}}\right)^n \left(\sqrt[4]{15}\right)^{3^n-1}\,.$$

With a similar approach one can also derive a recurrence relation for matchings in S_3^n. Apart from the asymptotic behavior of the number of matchings in S_3^n determined by Teufl and Wagner [62], its exact value remains unknown.

Donno studied weighted spanning trees on the base-3-Sierpiński graphs with D'Angeli [6,

[2]Although the article is from 1847, it can be found online in the Wiley Online Library.

Section 3] and the Tutte polynomial of the same family of graphs with Iacono [8]. We will not go into details of these results, since they are rather technical.

Chapter 3

Metric properties

Metric properties have been studied quite intensively for Sierpiński graphs so far. One of the main reasons to study them has already been mentioned in Section 1.3.1 and comes from the Tower of Hanoi puzzle. In the first section of this chapter we will summarize some important results known about distances and other metric properties of Sierpiński graphs. Then we will develop some improvements for distances to almost-extreme vertices. We will also determine distances of almost-extreme vertices. To conclude this chapter we will determine the metric dimension of Sierpiński graphs in the final section.

By distance $d_G(u,v)$ between two vertices u and v of a graph G, we mean as usual the length of a shortest u,v-path. A little less known is the term of a distance of a vertex. The *(total) distance* $d_G(u)$ *of a vertex* u in G equals the sum of all the distances to u:

$$d_G(u) = \sum_{v \in V(G)} d_G(u,v)\,.$$

The distance of a vertex, for example, plays an important role in mathematical chemistry, cf. [49], because it is a building block for the extensively investigated Wiener index of a graph. In section 3.2 we also determine the distance of almost-extreme vertices of Sierpiński graphs.

Let us first list properties of distances in Sierpiński graphs S_p^n for $p = 1$ or $n = 0, 1$. As already mentioned, $S_p^0 \cong K_1$ for any $p \in \mathbb{N}$ and $S_1^n \cong K_1$ for any $n \in \mathbb{N}_0$, therefore there is nothing to say about the distances in the cases $n = 0$ or $p = 1$. Since $S_p^1 \cong K_p$ for any $p \in \mathbb{N}$, it is also well known that the distance between arbitrary (distinct) vertices of S_p^1 equals 1 for any $p \geq 2$. Thus we will mainly focus on $n, p \geq 2$ in the rest of the chapter.

3.1 Known results

When Klavžar and Milutinović introduced the family of Sierpiński graphs in 1997 [40], they also presented the following key lemma about the distance in a Sierpiński graph between an arbitrary vertex and an extreme vertex of the graph.

Lemma 3.1. [40, Lemma 4] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for any $j \in [p]_0$ and any vertex $s = s_n \dots s_1$ of S_p^n,*

$$d(s, j^n) = \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}\,.$$

Moreover, there is exactly one shortest path between s and j^n. In particular, for any distinct $i, j \in [p]_0$, $d(i^n, j^n) = 2^n - 1$.

From Lemma 3.1 some important results about distances in Sierpiński graphs can be derived. Let us first list some corollaries that follow immediately from the lemma and were first observed by Parisse in 2009 [52]. It is straightforward to sum the distances between an arbitrary fixed vertex and all the extreme vertices.

Corollary 3.2. [52, Proposition 2.5] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for any vertex s of S_p^n,*

$$\sum_{i=0}^{p-1} d(s, i^n) = (p-1)(2^n - 1)\,.$$

It was also established that the distance between arbitrary vertices does not depend on a common prefix.

Corollary 3.3. [52, Corollary 2.2(i)] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for arbitrary vertices js and jt of S_p^{n+1},*

$$d_{S_p^{n+1}}(js, jt) = d_{S_p^n}(s, t)\,.$$

Finally, the diameter of the family of Sierpiński graphs can be derived from the lemma above. For a fixed $n \in \mathbb{N}_0$, the diameter of a Sierpiński graph S_p^n is equal to the distance between two arbitrary extreme vertices and is p-independent.

Proposition 3.4. [52, Corollary 2.2(ii)] *If $n \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $p \geq 2$, then the diameter of the Sierpiński graph S_p^n equals*

$$\operatorname{diam}(S_p^n) = 2^n - 1\,. \tag{3.1}$$

Now that we know that the shortest paths to extreme vertices are unique we can use this fact together with the recursive structure of Sierpiński graphs to obtain all possible candidates for a shortest path between two arbitrary vertices of a Sierpiński graph. There are exactly $p - 1$ such paths, let us define them explicitly:

Definition 3.5. *Let $n, p \in \mathbb{N}$, and let $i, j \in [p]_0$ be distinct. Further let $s = \underline{s}i\overline{s}$ and $t = \underline{s}j\overline{t}$ be vertices of S_p^n, where $\overline{s}, \overline{t} \in [p]_0^{\delta-1}$ and $\underline{s} \in [p]_0^{n-\delta}$ for a $\delta \in [n]$. Then define*

$$\begin{aligned} & d_i(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_j(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_{S_p^{\delta-1}}(\overline{s}, j^{\delta-1}) + 1 + d_{S_p^{\delta-1}}(\overline{t}, i^{\delta-1}) \,, \\ \forall \ell \in [p]_0 \setminus \{i,j\} : \quad & d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_{S_p^{\delta-1}}(\overline{s}, \ell^{\delta-1}) + 1 + 2^{\delta-1} + d_{S_p^{\delta-1}}(\overline{t}, \ell^{\delta-1}) \,. \end{aligned}$$

The distances $d_i(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ and $d_j(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ are called the direct distances *between s and t.*

We will usually write just one of the direct distances, since they are the same. The s,t-path corresponding to the direct distance will be called the *direct s,t-path*.

First observe that the vertices defined in the above definition both belong to the subgraph $\underline{s}S_p^\delta$. For these two vertices distances $d_\ell(s,t)$, $\ell \in [p]_0 \setminus \{i,j\}$, correspond to the path through the subgraph $\underline{s}\ell S_p^{\delta-1}$. It is easy to see that a shortest path between these vertices is one of the paths corresponding to the distances d_ℓ for $\ell \in [p]_0$. Other possibilities would be to go through more than just one subgraph isomorphic to $S_p^{\delta-1}$, but then this path would already be longer than the diameter of the subgraph $\underline{s}S_p^\delta$. Note also that the shortest path between an arbitrary vertex s and an extreme vertex j^n of S_p^n is the direct s, j^n-path.

In Figure 3.1 we present the graph S_4^4 with emphasized paths that correspond to distances $d_\ell(0231, 2301)$, $\ell \in [4]_0$. The direct path, i.e., the path corresponding to the direct distance $d_0(0231, 2301) = d_2(0231, 2301)$, is drawn in red, the path for $d_1(0231, 2301)$ is green, and the path for $d_3(0231, 2301)$ is blue. Obviously the shortest path for these two vertices is the direct $0231, 2301$-path and $d_{S_4^4}(0231, 2301) = 9$.

Theorem 3.6. [40, Theorem 5] *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. If $s = \underline{s}i\overline{s}$ and $t = \underline{s}j\overline{t}$ are vertices of S_p^n, where $i, j \in [p]_0$, are distinct, $\delta \in [n]$, $\overline{s}$, $\overline{t} \in [p]_0^{\delta-1}$, and $\underline{s} \in [p]_0^{n-\delta}$, then*

$$d_{S_p^n}(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{ d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p]_0 \right\} \tag{3.2}$$

The minimum (3.2) can be obtained by at most two of the distances d_ℓ, $\ell \in [p]_0 \setminus \{i\}$, i.e., there are at most two shortest paths between any two vertices (cf. [40, Theorem 6] or alternative very recent proof [26, Corollary 1.1]). Moreover, if there are two shortest paths between two vertices, one of them is the direct path. In [40, Corollary 7] the authors also showed that the distance between arbitrary vertices of a Sierpiński graph S_p^n can be computed in $O(n)$ time.

Now we have all the tools we need to prove that the cycle $C_{ij\ell}^{(n)}$ is an isometric subgraph of S_p^n.

Proposition 3.7. *If $p \in \mathbb{N}$, $p \geq 3$ and $n \in \mathbb{N}$, then for any pairwise distinct $i, j, \ell \in [p]_0$ the cycle $C_{ij\ell}^{(n)}$ is an isometric cycle in S_p^n.*

Proof. Note first that any path $kP_{gh}^{(n-1)}$ is isometric in S_p^n for any $g, h, k \in [p]_0$, $g \neq h$, because it is the shortest path between kg^{n-1} and kh^{n-1}. To show that $C_{ij\ell}^{(n)}$ is isometric in S_p^n, assume the

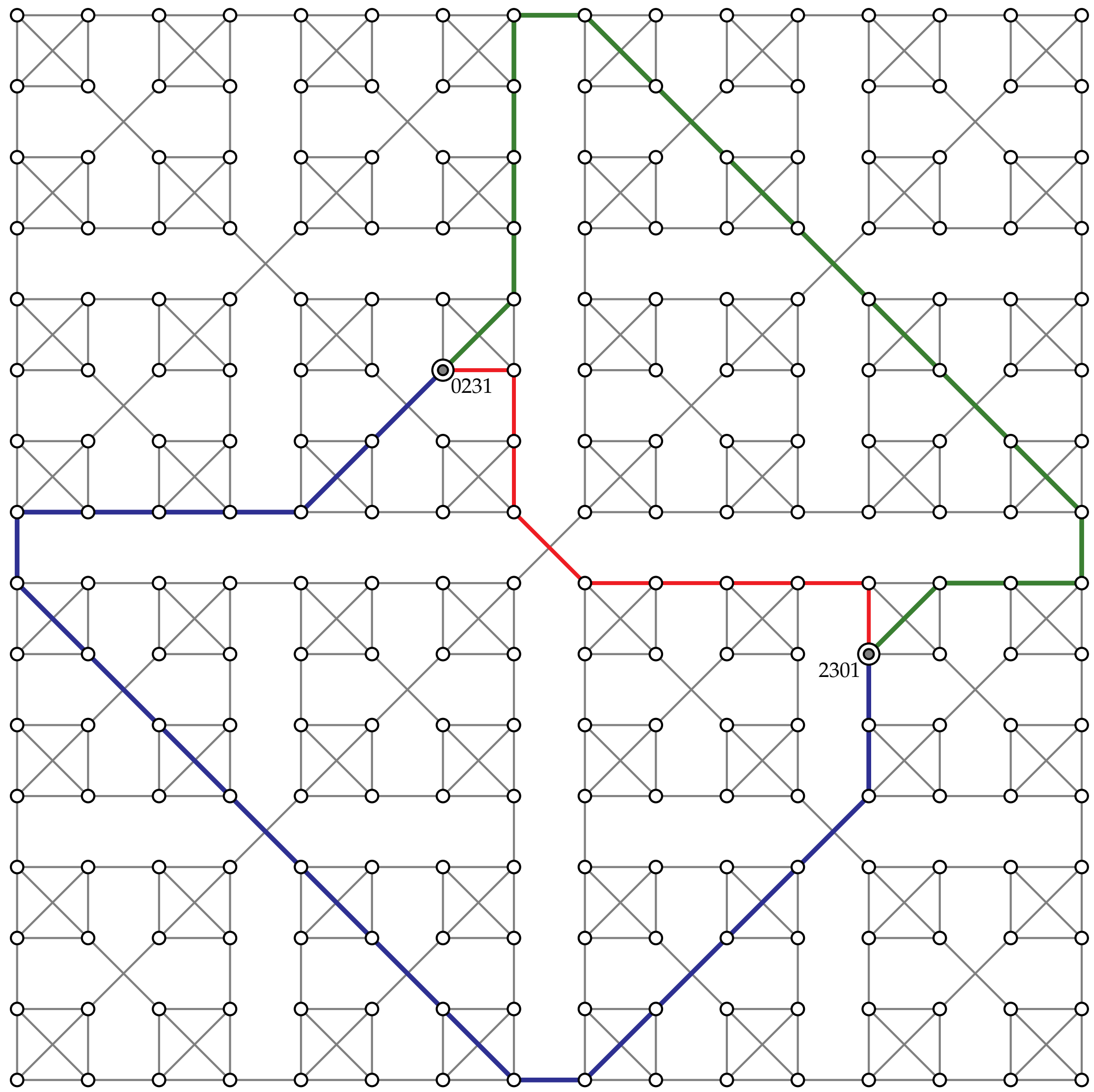


Figure 3.1: Distances d_ℓ, $\ell \in [4]_0$ for vertices $0231, 2301$ of S_4^4

contrary, i.e., we assume that $i\overline{s}$ and $j\overline{t}$ of $C_{ij\ell}^{(n)}$ are such that

$$d_{C_{ij\ell}^{(n)}}(i\overline{s}, j\overline{t}) > d_{S_p^n}(i\overline{s}, j\overline{t})\,.$$

So the shortest $i\overline{s}, j\overline{t}$-path is the path corresponding to $d_k(i\overline{s}, j\overline{t})$, for some $k \in [p]_0 \setminus \{i, j, \ell\}$.

Note that $i\bar{s} \in iP_{j\ell}^{(n-1)}$ and $j\bar{t} \in jP_{i\ell}^{(n-1)}$ and therefore $\bar{s} \in \{j,\ell\}^{n-1}$, $\bar{t} \in \{i,\ell\}^{n-1}$. So we have

$$\begin{aligned} d_k(i\bar{s}, j\bar{t}) &= d_{S_p^{n-1}}(\bar{s}, k^{n-1}) + 1 + 2^{n-1} + d_{S_p^{n-1}}(\bar{t}, k^{n-1}) \\ &= 3 \cdot 2^{n-1} - 1 > \frac{|C_{ij\ell}^{(n)}|}{2} \geq d_i(i\bar{s}, j\bar{t}) \,, \end{aligned}$$

a contradiction. □

For $\ell \in \mathbb{N}$ let us denote the number of non-zero binary digits of ℓ by $q(\ell)$. Then we can state the following result about the number of vertices at distance ℓ from some fixed extreme vertex. This result is a consequence of Lemma 3.1 and Proposition 3.4.

Corollary 3.8. [52, Corollary 2.4] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for an arbitrary extreme vertex i^n of S_p^n and $\ell \in [2^n]_0$,*

$$|\{s \in [p]_0^n \mid d(s, i^n) = \ell\}| = (p-1)^{q(\ell)} \,,$$

and

$$\sum_{\ell=0}^{2^n-1} (p-1)^{q(\ell)} = p^n \,.$$

In addition to all the results listed above, Parisse [52] also presented some outcomes related to the eccentricity in the Sierpiński graphs. First, let us recall some theory about these terms. The *eccentricity, $\epsilon_G(v)$, of a vertex* $v \in V(G)$ is the maximum distance in graph G between v and any other vertex $u \in V(G)$,

$$\epsilon_G(v) = \max\{d_G(u,v) \mid u \in V(G)\} \,.$$

The diameter of a graph can therefore also be interpreted as the maximum eccentricity in a graph. The minimum eccentricity in a graph G is the *radius of a graph,* $\mathrm{rad}(G)$. A vertex with $\epsilon_G(v) = \mathrm{rad}(G)$ is called a *central vertex* of G and the set of central vertices $C(G) = \{v \in V(G) \mid \epsilon_G(v) = \mathrm{rad}(G)\}$ is the *center of a graph G.* The *average eccentricity* of a graph G is the arithmetic mean of all eccentricities, that is

$$\bar{\epsilon}(G) = \frac{1}{|G|} \sum_{v \in V(G)} \epsilon_G(v) \,.$$

Proposition 3.9. [52, Lemma 2.3] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for an arbitrary vertex s of S_p^n,*

$$\epsilon_{S_p^n}(s) = \max\{d(s, i^n) \mid i \in [p]_0\} \,.$$

With the eccentricity of vertices we can determine both, the radius and the center of the Sierpiński graphs:

Theorem 3.10. [52, Theorem 3.1] *Let $n, p \in \mathbb{N}$. The radius of S_p^n is*

$$\operatorname{rad}\left(S_p^n\right) = \lfloor 2^{n-p+1}\left(2^{p-1}-1\right)\rfloor = \begin{cases} 2^n - 1\,, & n < p\,, \\ 2^{n-p+1}\left(2^{p-1}-1\right)\,, & n \geq p\,. \end{cases}$$

For $n \geq p$ let

$$C_p^n = \left\{ z \in [p]_0^n \mid z = z_p \ldots z_2 z_1^{n-p+1}\,, \{z_p, \ldots, z_1\} = [p]_0 \right\}\,.$$

The center of S_p^n is then

$$C\left(S_p^n\right) = \begin{cases} [p]_0^n\,, & n < p\,, \\ C_p^n\,, & n \geq p\,. \end{cases}$$

The center has

$$\left|C\left(S_p^n\right)\right| = \begin{cases} p^n\,, & n < p\,, \\ p!\,, & n \geq p \end{cases}$$

vertices and the graph induced by the center has

$$\left|E\left(C\left(S_p^n\right)\right)\right| = \begin{cases} \frac{p}{2}(p^n - 1)\,, & n < p\,, \\ \frac{p!}{2}\,, & n \geq p \end{cases}$$

edges. In particular, for $n \geq p > 1$, the center of S_p^n induces a 1-regular graph with $\frac{p!}{2}$ disconnected edges, i.e., $C(S_p^n)$ induces a subgraph of S_p^n isomorphic to $\frac{p!}{2}K_2$.

A bit later Hinz and Parisse [31] determined the average eccentricity of Sierpiński graphs.

Theorem 3.11. [31, Corollary 3.5] *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then the average eccentricity of the graph S_p^n equals*

$$\overline{\epsilon}(S_p^n) = \left(1 - \binom{2p}{p-1}^{-1}\right) 2^n - \frac{p-1}{p} - \sum_{k=0}^{p-2} (-1)^{p-k} \frac{p-1-k}{2p-k} \binom{p}{k} \left(\frac{k}{p}\right)^n\,.$$

As one can see from the proof of Theorem 3.11 (see [31]), some results are not easy to prove, although the structure of the Sierpiński graphs is quite easy to explain. To illustrate this even further, let us present another fascinating formula - the average distance in Sierpiński graphs, given by Wiesenberger in his graduation thesis.

Theorem 3.12. [68, Satz 3.1.11] *For $p \in \mathbb{N}$ let*

$$\begin{aligned} \alpha_p &= p^4 - 12p^3 + 56p^2 - 104p + 68\,, \\ \lambda_{p,\pm} &= \frac{1}{2}p^2 - p + 1 \pm \frac{1}{2}\sqrt{\alpha_p}\,, \\ \gamma_{p,\pm} &= (p^2 + 3p - 2) \mp (p^4 + p^3 - 30p^2 + 58p - 36)\sqrt{\alpha_p}\,. \end{aligned}$$

Then for all $n \in \mathbb{N}_0$

$$\begin{aligned}\overline{d}(S_p^n) = &\frac{(p-1)(2p^4+6p^3-17p^2+26p-16)}{p(2p-1)(p^3+4p^2-4p+8)}2^n \\ &-\frac{p-2}{p}+\frac{p^2+3p-6}{(2p-1)(p^2-7p+8)}p^{-n} \\ &-\frac{p(p-1)\gamma_{p,+}}{2(p^2-7p+8)(p^3+4p^2-4p+8)}\left(\frac{\lambda_{p,+}}{p^2}\right)^n \\ &-\frac{p(p-1)\gamma_{p,-}}{2(p^2-7p+8)(p^3+4p^2-4p+8)}\left(\frac{\lambda_{p,-}}{p^2}\right)^n .\end{aligned}$$

The formula in Theorem 3.12 is fascinating, because for particular values of p we actually get perfect squares for α_p. For example $\alpha_2 = 4$ and $\alpha_4 = 36$. In such cases the formula from the theorem for the average distance is simplified at least a little bit. For example, in the mentioned cases (for $p = 2, 4$) we have:

$$\begin{aligned}\overline{d}(S_2^n) &= \frac{1}{3}\left(2^n - 2^{-n}\right) \\ \overline{d}(S_4^n) &= \frac{89}{140}2^n - \frac{1}{2} + \frac{1}{4}2^{-n} - \frac{11}{14}4^{-n} + \frac{2}{5}8^{-n} .\end{aligned}$$

By now we have seen that there are at most two shortest paths between arbitrary two vertices of S_p^n. Therefore it seems reasonable to examine whether there is one shortest path (and which one) or there are two shortest paths between given two vertices of S_p^n. For the puzzle of the Switching Tower of Hanoi this corresponds to the decision whether the largest disc moves once or twice or both strategies are optimal. Already in 2006 Romik [57] has developed an automaton for the classical case, i.e., $p = 3$, which for given two states returns the answer to the question on shortest paths. See [27, Figure 2.27] for a nice drawing of Romik's automaton. However, the answer to the same question with $p > 3$ remained unsolved until very recently when Hinz and Holz auf der Heide [26] generalized the previous automaton. The general automaton is depicted in Figure 3.2 and will be explained in the following example.

Example 3.13. *The input for the automaton are two (arbitrary) vertices is, jt of S_p^{n+1} for $n \in \mathbb{N}$. (For $S_p^0 \cong K_1$ and $S_p^1 \cong K_p$ everything about shortest paths is already known.) Without loss of generality we may assume that $i \neq j$, since the distance and the shortest paths between is and jt do not depend on a common prefix.*

Vertices are entered into the automaton as pairs (s_d, t_d) one by one with $d = n$ downto 1. The very first pair fixes all the values i and j in the automaton. For example, in the case of $0s$ and $1t$ we replace any i with 0 and j with 1. Note that all dots in the automaton are arbitrary entries. Further on, at the starting state 0, $k \in [p]_0 \setminus \{i, j\}$. After starting with the pair $(g, h) = (s_n, t_n)$ in state 0, $k \in \{g, h\} \setminus \{i, j\}$, $\ell \in [p]_0 \setminus \{g, h, j\}$, and $m \in [p]_0 \setminus \{g, h, i\}$. Depending on the values s_n and t_n we move either to state 1, any of the states A and B or we end the procedure in state D. Note that the states

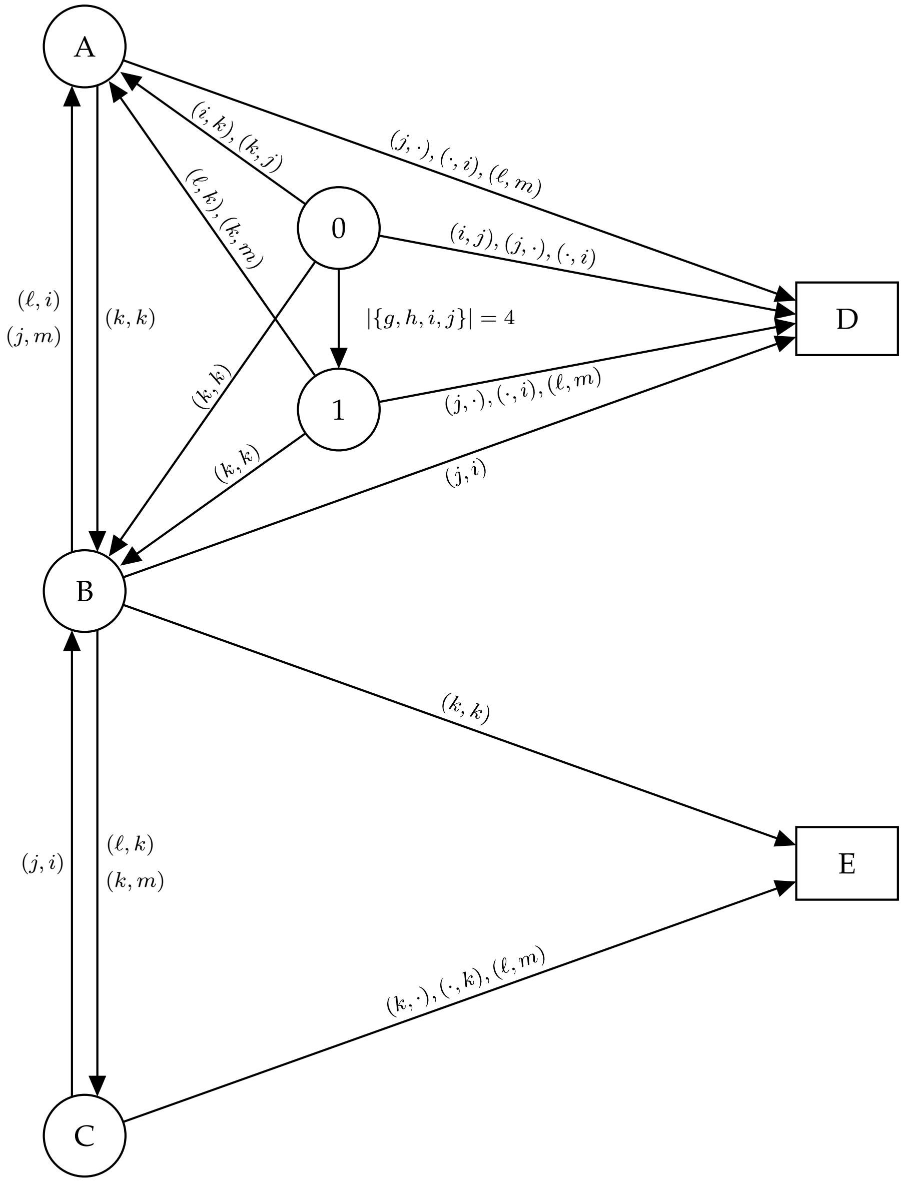


Figure 3.2: P2 decision automaton for S_p^n

D *and* E *are* absorbing, *meaning that if we come to one of these states, we already know the answer to the decision whether to move the largest disc once or twice (or there are two shortest paths), although we have possibly entered less than* n *pairs of* s *and* t*. After leaving the states* 0 *and* 1 *(if entered), we either finish in* D *or the value of* k *is fixed and* $\ell \in [p]_0 \setminus \{j,k\}$*, and* $m \in [p]_0 \setminus \{i,k\}$*. The value* k *gives us the second candidate for a shortest path between* is *and* jt*, namely the path corresponding to the distance* d_k*. How to interpret the states* 1, A, B, C, D, *and* E *when these are the endstate of the automaton is* *explained in Table 3.1.*

1, A, D	the largest disc moves once, i.e., the unique shortest path is the direct path
B	both strategies are optimal, i.e., there are two shortest paths, the direct path and the path corresponding to d_k
C, E	the largest disc moves twice, i.e., the unique shortest path is the path corresponding to d_k

Table 3.1: Meanings of the states 1, A, B, C, D, and E

To get a better perception of the automaton, let us run it in S_4^4 for $is = 02^3$ and $it = 1320$. We insert $i = 0$, $j = 1$, $g = 2$, and $h = 3$ to the state 0. This way we move to state 1 with the next pair $(2, 2)$, so we get $k = 2$ and move to B. After inserting the last pair $(2, 0)$ we stay at B and can thus conclude that both, the direct path and the path through the subgraph $2S_4^3$, are shortest $02^3, 1320$-paths (cf. Proposition 3.25).

Let now $is = 02^3$ and $it = 13^3$. Then we insert all the pairs and end in state 1. If this happens, then no k is fixed for another candidate for a shortest path and obviously the direct path is the shortest path.

3.2 Almost-extreme vertices

Beside the initial cases we mentioned at the beginning of this chapter, it is also easy to determine the distance between arbitrary vertices of S_2^n, for any $n \in \mathbb{N}$. Recall that $S_2^n \cong P_{2^n}$ so shortest paths are unique and the distance between arbitrary vertices $0s$ and $1t$ of S_p^{n+1} can be computed using Lemma 3.1. We are also able to determine an explicit formula for it

$$d_{S_2^{n+1}}(0s, 1t) = 1 + \sum_{d=1}^{n}(1 - s_d + t_d)2^{d-1} = 2^n + \sum_{d=1}^{n}(t_d - s_d)2^{d-1}\,.$$

Although the structure of Sierpiński graphs is easily understandable and Theorem 3.6 provides us with an approach to determine the distance between two arbitrary vertices of S_p^n, the distance is in general equal to the minimum of p (not necessarily different) values. We want to find an easier or a more effective way to compute these distances. An explicit formula for computing distances to extreme vertices already exists, and the neighbors of extreme vertices are quite similar to them. The similarity between these two types of vertices was the key starting point for finding the explicit formula for distances to outer almost-extreme vertices. The results presented in the sequel are taken mainly from the article [45].

From Corollary 3.3 we know that the distance between arbitrary vertices does not depend on a common prefix. Therefore we will consider only distances between an outer almost-extreme vertex of a subgraph jS_p^n and an arbitrary vertex of a subgraph iS_p^n, for $i \neq j$.

Proposition 3.14. *If $n, p \in \mathbb{N}$ and $j^n k$ is an outer almost-extreme vertex of S_p^{n+1}, then for $i \in [p]_0 \setminus \{j\}$ the distance between an arbitrary vertex is of S_p^{n+1} and $j^n k$ equals*

$$d_{S_p^{n+1}}(is, j^n k) = d(s, j^n) + 2^n - [i = k] .$$

Proof. Let $n \in \mathbb{N}$. Using Theorem 3.6 we have

$$\begin{aligned} d_i(is, j^n k) = d_j(is, j^n k) &= d(s, j^n) + 1 + d(j^{n-1}k, i^n) \\ &= d(s, j^n) + 2^n - [i = k] \end{aligned}$$

Our goal is to show that the minimum (3.2) for the almost-extreme vertex $j^n k$ is achieved at $d_i(is, j^n k)$. For an arbitrary $\ell \in [p]_0 \setminus \{j\}$, we have

$$\begin{aligned} d_\ell(is, j^n k) &= d(s, \ell^n) + 1 + 2^n + d(j^{n-1}k, \ell^n) \\ &\overset{j \neq \ell}{=} d(s, \ell^n) + 2^{n+1} - [k = \ell] \\ &\geq d(s, j^n) + 2^n - [i = k] \qquad (3.3) \\ &= d_i(is, j^n k) . \end{aligned}$$

(Note that by the definition of almost-extreme vertices $j \neq k$, therefore (3.3) holds for $\ell = k$ as well.) $\square$

Corollary 3.15. *If $n, p \in \mathbb{N}$ and $i, j \in [p]_0$ are distinct, then there are two shortest paths between an arbitrary vertex is of S_p^{n+1} and an outer almost-extreme vertex $j^n k$ of S_p^{n+1} if and only if $s = k^n$.*

Proof. Equality in (3.3) holds if and only if $i \neq k = \ell$, $d(s, j^n) = 2^n - 1$, and $d(s, \ell^n) = 0$. This is only in the case if $is = ik^n$, $i \neq k$. $\square$

This result was further improved by Xue et al. [72]. They determined all the vertices with two shortest paths to an outer almost-extreme vertex, not just those that are not in the same subgraph isomorphic to S_p^n as the almost-extreme vertex of S_p^{n+1} under consideration. This can also be obtained by Corollary 3.15 by applying it recursively.

Proposition 3.16. [72, Theorem 3.3], [26, Proposition 2.3] *If $n, p \in \mathbb{N}$ and $j^n k$ is an outer almost-extreme vertex of S_p^{n+1}, then there are two shortest paths between an arbitrary vertex s of S_p^{n+1} and $j^n k$ if and only if $s = j^{n-m} i k^m$ with $m \in [n]$ and $i \in [p]_0 \setminus \{j, k\}$.*

Figure 3.3 shows the graph S_5^3 with emphasized vertices (red) for which there are two shortest paths to the almost-extreme vertex 002 (gray). Xue et al. also determined the distance between an outer almost-extreme vertex and a vertex with two shortest paths to it.

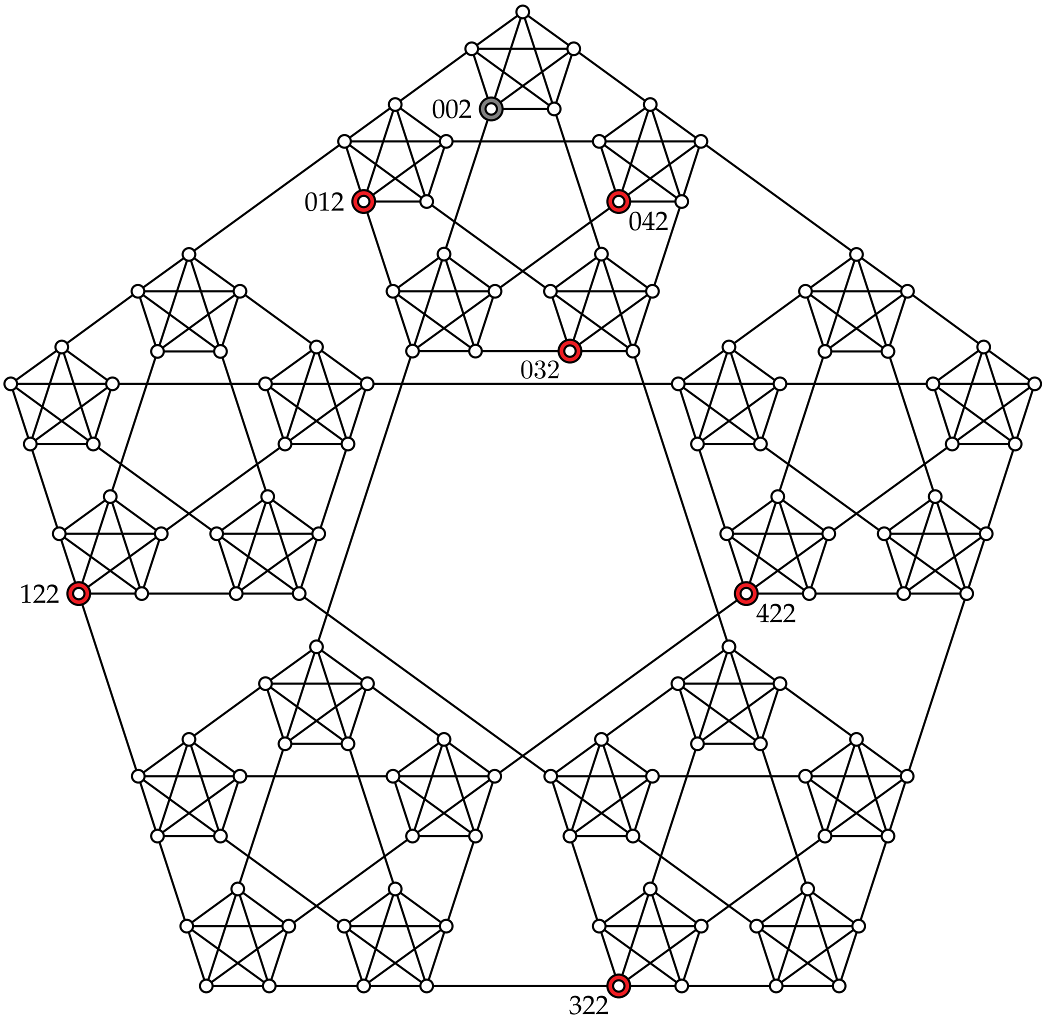


Figure 3.3: Vertices with two shortest paths to 002 in the Sierpiński graph S_5^3

Proposition 3.17. [72, Corollary 3.4] *If $n, p \in \mathbb{N}$ and $j^n k$ is an outer almost-extreme vertex of S_p^{n+1}, then the distance between $j^n k$ and the vertex $j^{n-m} i k^m$ with $m \in [n]$, $i \in [p]_0 \setminus \{j, k\}$ of S_p^{n+1} can be expressed as*

$$d_{S_p^{n+1}}(j^n k, j^{n-m} i k^m) = 2^{m+1} - 1\,.$$

Remark 3.18. *Although we defined almost-extreme vertices for $n \in \mathbb{N}$, Proposition 3.14 holds also for $n = 0$. In that case we have $S_p^1 \cong K_p$, where every vertex is extreme and the distance is*

$$d_{S_p^1}(i, k) = [i \neq k] = 1 - [i = k]\,,$$

as stated in the proposition.

With Proposition 3.14 we can determine the distance of an outer almost-extreme vertex of a Sierpiński graph. To do so, we require the distance of extreme vertices. Since Sierpiński graphs possess certain symmetry properties (see Theorem 2.28) and automorphisms are distance preserving, it is obvious that all the extreme vertices have the same distance.

Lemma 3.19. [52, p. 7], [68, Satz 3.1.10], [45, Lemma 8] *If $n, p \in \mathbb{N}$, then for any $i \in [p]_0$,*

$$d_{S_p^n}(i^n) = p^{n-1}(p-1)\,(2^n - 1)\ .$$

Proof. Let $d \in [n]$ and $i \in [p]_0$. Then there are $p^{n-1}(p-1)$ vertices $s = s_n \dots s_1$ with $s_d \neq i$ and hence Lemma 3.1 implies

$$\begin{aligned}
\sum_{s\in[p]^n} d(s, i^n) &= \sum_{s\in[p]^n} \sum_{d=1}^{n} [s_d \neq i] \cdot 2^{d-1} \\
&= \sum_{d=1}^{n} \left(\sum_{s\in[p]^n} [s_d \neq i] \right) \cdot 2^{d-1} \\
&= p^{n-1}(p-1) \sum_{d=1}^{n} 2^{d-1} = p^{n-1}(p-1)\,(2^n - 1)\ ,
\end{aligned}$$

which completes the proof. □

Now we are ready to prove the distance of the outer almost-extreme vertices. By the symmetry of Sierpiński graphs it is again obvious that all the outer almost-extreme vertices have the same distance.

Theorem 3.20. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for any distinct $j, k \in [p]_0$,*

$$d_{S_p^{n+1}}(j^n k) = \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right) p^{n+1} + \frac{p}{p-1}\ .$$

Proof. We proceed by induction on $n \in \mathbb{N}_0$. For $n = 0$, we have

$$d_{S_p^1}(k) = p - 1 = \frac{p-1}{p} 2p - p - \frac{1}{p-1} + \frac{p}{p-1}\ .$$

Let now $n \in \mathbb{N}_0$, then

$$d_{S_p^{n+1}}(j^n k) = \sum_{i\in[p]_0} \sum_{s\in[p]_0^n} d_{S_p^{n+1}}(is, j^n k)\ .$$

By Corollary 3.3 and Proposition 3.14 we have

$$d_{S_p^{n+1}}(is, j^n k) = \begin{cases} d_{S_p^n}(s, j^{n-1}k)\ , & i = j\ , \\ d_{S_p^n}(s, j^n) + 2^n - 1\ , & i = k\ , \\ d_{S_p^n}(s, j^n) + 2^n\ , & i \in [p]_0 \setminus \{j, k\}\ . \end{cases} \tag{3.4}$$

Therefore we split the sum above into three sums:

$$\begin{aligned}
d_{S_p^{n+1}}(j^n k) &\overset{(3.4)}{=} \sum_{s\in[p]_0^n} d(s, j^{n-1}k) + \sum_{s\in[p]_0^n} \big(d_{S_p^n}(s, j^n) + 2^n - 1\big) + (p-2) \sum_{s\in[p]_0^n} \big(d_{S_p^n}(s, j^n) + 2^n\big) \\
&\overset{3.19}{=} d_{S_p^n}(j^{n-1}k) + p^{n-1}(p-1)^2(2^n-1) + p^n(2^n-1) + (p-2)(2p)^n \\
&= d_{S_p^n}(j^{n-1}k) + \frac{(2p-1)(p-1)}{p}(2p)^n - \left(1 + \frac{(p-1)^2}{p}\right) p^n .
\end{aligned}$$

Using induction hypothesis we get the desired result. □

Remark 3.21. *The expression of Theorem 3.20 can be further transformed as follows:*

$$\begin{aligned}
d_{S_p^{n+1}}(j^n k) &= \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right) p^{n+1} + \frac{p}{p-1} \\
&= p^n(p-1)2^{n+1} - p^n(p-1) + p^n(p-1) - p^{n+1} - \frac{p^n}{p-1} + \frac{p}{p-1} \\
&= p^n(p-1)(2^{n+1}-1) - p \cdot \frac{p^n - 1}{p-1} \\
&= d_{S_p^{n+1}}(j^{n+1}) - \sum_{\ell=1}^{n} p^\ell .
\end{aligned}$$

This is an alternative way to calculate $d_{S_p^{n+1}}(j^n k)$. It can be interpreted as $d_{S_p^{n+1}}(j^{n+1})$ minus the additional step to all the vertices reachable directly from $j^n k$. There are $p + p^2 + p^3 + \cdots + p^n$ such vertices.

Another type of vertices in Sierpiński graphs that are similar to extreme vertices are inner almost-extreme vertices. In the rest of this section we will develop analogue results as we have just proved for outer almost-extreme vertices. As before, we consider the distance between an inner almost-extreme vertex of a subgraph jS_p^n and an arbitrary vertex of a subgraph iS_p^n, where $i \neq j$. In order to express a formula for this distance we need the concept of direct and special vertices.

Definition 3.22. *Let $n, p \in \mathbb{N}$ and let jk^n be an inner almost-extreme vertex of S_p^{n+1}. A vertex s of S_p^{n+1} is* direct *with respect to jk^n, if one of the following statements hold:*

(i) $s \in kS_p^n$,
(ii) there exists a $\delta \in [n+1]$ such that $s = \underline{s} j \overline{s}$ with $\underline{s} \in ([p]_0 \setminus \{j,k\})^{n+1-\delta}$ and $\overline{s} \in [p]_0^{\delta-1}$, or
(iii) $s \in \big([p]_0 \setminus \{j,k\}\big)^{n+1}$.

In other words, if s is direct with respect to jk^n then $s_d = k$ holds only if $d = n+1$ or there is a $\delta \in [n+1] \setminus [d]$ with $s_\delta = j$. Obviously, in S_p^{n+1} there are

$$\frac{1}{2}\big((p+2)p^n + (p-2)^{n+1}\big)$$

direct vertices with respect to jk^n. The choice for their name becomes apparent because of the following proposition. Recall that the direct path between two vertices is and jt of S_p^{n+1} is the path corresponding to the direct distance $d_i(is, jt)$ (cf. Definition 3.5).

Proposition 3.23. *If $n, p \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{n+1}, then the direct path between an arbitrary vertex is and jk^n is the (only) shortest path if and only if the vertex is is direct with respect to jk^n.*

Proof. For $i = j$ this is trivial, since $d_{S_p^{n+1}}(js, jk^n) = d_{S_p^n}(s, j^k)$. The corresponding shortest path is direct for these two vertices and unique by Lemma 3.1. On the other hand, the vertex js is also direct with respect to jk^n, since coordinate j appears before any k.

If $i = k$, then the length of the direct path is $d_k(ks, jk^n) = d_{S_p^n}(s, j^n) + 1$, which is strictly smaller than the length of any of the paths $d_\ell(ks, jk^n) = d_{S_p^n}(s, \ell^n) + 2^{n+1}$, $\ell \in [p]_0 \setminus \{j, k\}$.

So let now $j \neq i \neq k$. To prove the assertion, we have to see that for any $\ell \in [p]_0 \setminus \{i, j\}$

$$d_i(is, jk^n) < d_\ell(is, jk^n)\,.$$

Let first $\ell \neq k$. Then

$$\begin{aligned} d_\ell(is, jk^n) &= d_{S_p^n}(s, \ell^n) + 1 + 2^n + d_{S_p^n}(k^n, \ell^n) \\ &= d_{S_p^n}(s, \ell^n) + 2^{n+1} > 2^{n+1} - 1 = \mathrm{diam}(S_p^{n+1})\,, \end{aligned}$$

so $d_\ell(is, jk^n)$ is not a shortest is, jk^n-path. For $\ell = k$,

$$d_k(is, jk^n) = d_{S_p^n}(s, k^n) + 1 + 2^n + d_{S_p^n}(k^n, k^n) = d_{S_p^n}(s, k^n) + 2^n + 1\,,$$

while on the other hand

$$d_i(is, jk^n) = d_{S_p^n}(s, j^n) + 2^n\,.$$

Thus let us consider

$$d_{S_p^n}(s, k^n) + 1 - d_{S_p^n}(s, j^n) = 1 + \sum_{d=1}^{n} ([s_d \neq k] - [s_d \neq j]) 2^{d-1}\,. \tag{3.5}$$

Note that $\sigma_d := ([s_d \neq k] - [s_d \neq j]) = ([s_d = j] - [s_d = k]) \in \{-1, 0, 1\}$, in particular

$$\sigma_d = \begin{cases} -1\,, & s_d = k\,, \\ 0\,, & s_d \in [p]_0 \setminus \{j, k\}\,, \\ 1\,, & s_d = j\,. \end{cases}$$

Now the expression in (3.5) is greater than 0 if $\sigma_d = 0$ for all $d \in [n]$ or if for the first time $\sigma_d \neq 0$, it is positive (i.e., $\sigma_d = 1$). But this is equivalent to is being direct with respect to jk^n. $\square$

Definition 3.24. *Let $n, p \in \mathbb{N}$ and let jk^n be an inner almost-extreme vertex of S_p^{n+1}. A vertex s of S_p^{n+1} is* special *with respect to jk^n, if there exists a $\delta \in [n]$, such that $s = \underline{s}kj^{\delta-1}$ with $\underline{s} \in ([p]_0 \setminus \{j,k\})^{n+1-\delta}$.*

By the above definition there are

$$\frac{p-2}{p-3}\big((p-2)^n - 1\big)$$

special vertices with respect to jk^n in S_p^{n+1}. Again, the name for the special vertices was chosen because of the next result.

Proposition 3.25. *If $n, p \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{n+1}, then there are two shortest paths between an arbitrary vertex s of S_p^{n+1} and jk^n if and only if the vertex s is special with respect to jk^n.*

Proof. Let $s = i\overline{s}$ and consider first the case $i = k$. Then we already know by Proposition 3.23 that there is only one shortest path from $k\overline{s}$ to jk^n. Similarly, if $i = j$, then jk^n is an extreme vertex in jS_p^n and by Lemma 3.1 shortest paths to extreme vertices are unique.

Therefore let $i \in [p]_0 \setminus \{j,k\}$. To prove the proposition, we have to show that $d_i(i\overline{s}, jk^n) = d_k(i\overline{s}, jk^n)$ is equivalent to $i\overline{s}$ being special with respect to jk^n. So let us determine when

$$d_k(i\overline{s}, jk^n) - d_i(i\overline{s}, jk^n) = 1 + \sum_{d=1}^{n}([\overline{s}_d \neq k] - [\overline{s}_d \neq j])2^{d-1} = 0\,. \tag{3.6}$$

Note again $\sigma_d := ([\overline{s}_d \neq k] - [\overline{s}_d \neq j]) = ([\overline{s}_d = j] - [\overline{s}_d = k])$. Recall from the proof of the previous proposition that

$$\sigma_d = \begin{cases} -1\,, & \overline{s}_d = k\,, \\ 0\,, & \overline{s}_d \in [p]_0 \setminus \{j,k\}\,, \\ 1\,, & \overline{s}_d = j\,. \end{cases}$$

If $\sigma_d \neq 0$ for some $d \in [n]$, let δ be the largest such index. Then (3.6) holds if and only if $\sigma_\delta = -1$ and for all $d \in [\delta - 1]$, $\sigma_d = 1$. But if $\sigma_d = 0$ for all $d \in [n]$, then (3.6) cannot hold. In other words, (3.6) holds if and only if the vertex is is of the form $i\underline{s}kj^{\delta-1}$ for some $\delta \in [n]$ and $i\underline{s} \in \big([p]_0 \setminus \{j,k\}\big)^{n+1-\delta}$. This is equivalent to the vertex $i\overline{s}$ being special with respect to jk^n. □

Now we can state the following proposition, an analogue to Proposition 3.14 but for inner almost-extreme vertices.

Proposition 3.26. *If $n, p \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{n+1}, then for any $i \in [p]_0$ with $i \neq j$, the distance between an arbitrary vertex is of S_p^{n+1} and jk^n can be expressed as follows*

$$d_{S_p^{n+1}}(is, jk^n) = \begin{cases} d(s, j^n) + 2^n - [i = k](2^n - 1), & \textit{if } is \textit{ is direct for } jk^n\,, \\ d(s, k^n) + 2^n + 1, & otherwise\,. \end{cases}$$

Proof. For $i \neq j$ we have

$$d_i(is, jk^n) = d_{S_p^n}(s, j^n) + 1 + d_{S_p^n}(i^n, k^n) = d(s, j^n) + 2^n - [i = k](2^n - 1)$$

and for $\ell \in [p] \setminus \{i, j\}$,

$$d_\ell(is, jk^n) = d(s, \ell^n) + 1 + 2^n + d(\ell^n, k^n)\,. \tag{3.7}$$

The expression in (3.7) is strictly larger than $d_i(is, jk^n)$, if $\ell \neq k$. So we may assume that $i \neq k$ and the distances $d_k(is, jk^n)$ and $d_i(is, jk^n)$ are the only two possible lengths of a shortest path between is and jk^n. Now the assertion follows by Propositions 3.23 and 3.25. □

Note that in the above proposition any special vertex with respect to jk^n could also be in the first line of the formula, since there are two shortest paths for these vertices and each shortest path corresponds to one line of the equation.

An example of direct and special vertices is illustrated in Figure 3.4 on the graph S_6^3 for the almost-extreme vertex 144. All vertices circled green are direct for 144 and thus belong to the first line of the formula in Proposition 3.25, for all the others we use the second line. Orange vertices are special for 144, so for these vertices both lines of the equation in Proposition 3.25 hold.

Based on Corollary 3.3, Lemma 3.19, and Proposition 3.26, the distance of the inner almost-extreme vertices reads as follows.

Theorem 3.27. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for any distinct $j, k \in [p]_0$,*

$$d_{S_p^{n+1}}(jk^n) = \frac{p^2 - 2}{p(p+2)}(2p)^{n+1} - \frac{p-2}{2p}p^{n+1} - \frac{p}{2(p+2)}(p-2)^{n+1}\,.$$

Proof. Let us calculate

$$\begin{aligned} d_{S_p^{n+1}}(jk^n) &= \sum_{is \in [p]_0^{n+1}} d_{S_p^{n+1}}(is, jk^n) \\ &= \sum_{s \in [p]_0^n} d_{S_p^n}(s, k^n) + \sum_{s \in [p]_0^n} \big(d_{S_p^n}(s, j^n) + 1\big) + (p-2) \sum_{s \in [p]_0^n} d_{S_p^{n+1}}(is, jk^n)\,. \end{aligned} \tag{3.8}$$

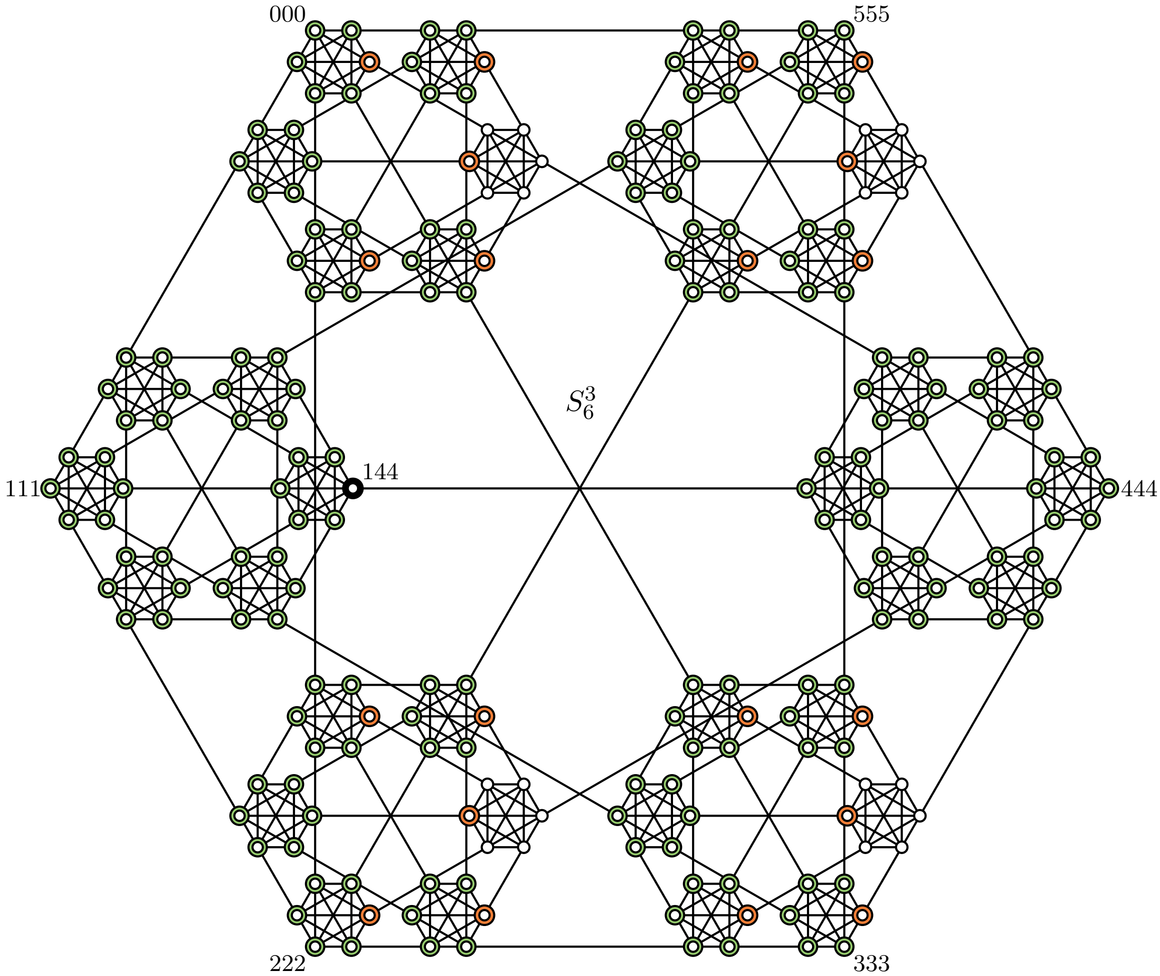


Figure 3.4: Direct and special vertices with respect to 144 in S_6^3

For fixed $i \in [p]_0 \setminus \{j,k\}$ define

$$\rho(is) := d_k(is, jk^n) - d_j(is, jk^n) = 1 + \sum_{d=1}^{n} \sigma_d \cdot 2^{d-1}\,,$$

where $\sigma_d = ([s_d = j] - [s_d = k])$. Now, if the vertex is is direct with respect to jk^n, then we have two options. Either $s \in \big([p]_0 \setminus \{j,k\}\big)^n$, in this case $\rho(is) = 1$; or $s = \underline{s}j\overline{s}$, $\underline{s} \in \big([p]_0 \setminus \{j,k\}\big)^{n-\delta}$, $\overline{s} \in [p]_0^{\delta-1}$, for $\delta \in [n]$, and then $\rho(is) = 1 + 2^{\delta-1} + \sum\limits_{d=1}^{\delta-1} \sigma_d \cdot 2^{d-1}$. For $i \in [p]_0 \setminus \{j,k\}$ define

$$\mathcal{P}_i := \sum_{s \in [p]_0^n} d_{S_p^{n+1}}(is, jk^n) = \sum_{s \in [p]_0^n} d_k(is, jk^n) - \sum_{is \text{ direct}} \rho(is)\,.$$

Then

$$\begin{aligned}
\mathcal{P}_i &= \sum_{s\in[p]_0^n} \left(d_{S_p^n}(s,k^n)+1+2^n\right)\\
&\quad - \left((p-2)^n + \sum_{\delta=1}^{n}(p-2)^{n-\delta}\cdot \sum_{\overline{s}\in[p]_0^{\delta-1}}\left(1+2^{\delta-1}+\sum_{d=1}^{\delta-1}\sigma_d\cdot 2^{d-1}\right)\right)\\
&= d_{S_p^n}(k^n)+(1+2^n)p^n-(p-2)^n-\sum_{\delta=1}^{n}(p-2)^{n-\delta}p^{\delta-1}(1+2^{\delta-1})\\
&= d_{S_p^n}(k^n)+(2p)^n+p^n-(p-2)^n-\frac{(p-2)(p^{n+1}+2p^n(2^n+1)-(p+4)(p-2)^n)}{2(p+2)}\,.
\end{aligned}$$

Note that $\mathcal{P}_i$ is i-independent, thus inserting the outcome into (3.8) we get

$$d_{S_p^{n+1}}(jk^n) = d_{S_p^n}(k^n) + d_{S_p^n}(j^n) + p^n + (p-2)\cdot\mathcal{P}_i\,,$$

which gives us the desired result. □

As we have already seen in Section 1.2, for $n = 2$, both kinds of almost-extreme vertices coincide and their total distances must be equal. Indeed, for $n = 2$, Theorems 3.20 and 3.27 both give the value $d_{S_p^2}(jk) = p(3p-4)$. Similarly as before with outer almost-extreme vertices, the expression of the distance of an inner almost-extreme vertex can be rewritten as follows:

$$d_{S_p^{n+1}}(jk^n) = \frac{1}{2}p^n(p-2)(2^{n+1}-1) + \frac{p}{2}\sum_{\ell=0}^{n}(2p)^{n-\ell}(p-2)^{\ell}\,.$$

In this case, however, we have no interpretation for this formula such as in Remark 3.21.

Xue et al. also studied the distances and shortest paths for the inner almost-extreme vertices. Their result about vertices with two shortest paths to an inner almost-extreme vertex of S_p^{n+1} is equivalent to Proposition 3.25 (cf. [72, Theorem 3.3]). Like for the outer almost-extreme vertices, they determined a similar result about distances between special vertices and inner almost-extreme vertices.

Proposition 3.28. [72, Corollary 3.2] *If $n,p\in\mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{n+1}, then the distance between jk^n and the vertex $s=\underline{s}kj^{\delta-1}$ with $\delta\in[n]$, $\underline{s}\in([p]_0\setminus\{j,k\})^{n+1-\delta}$ of S_p^{n+1} can be expressed as*

$$d_{S_p^{n+1}}(jk^n, \underline{s}kj^{\delta-1}) = 2^{n+1}-2^{\delta-1}\,.$$

Let us conclude with a listing of the distances of extreme and almost-extreme vertices for the classical case, i.e., when $p=3$. In this case S_3^n is isomorphic to the Hanoi graph H_3^n with extreme vertices mapped onto perfect ones and almost-extreme vertices being transformed into vertices of the same form. By Lemma 3.19 and Theorems 3.20 and 3.27 we get:

Corollary 3.29. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for $i, j, k \in [p]$, $j \neq k$,*

$$\begin{aligned} d_{S_3^n}(i^n) &= \frac{2}{3}3^n(2^n-1) = d_{H_3^n}(i^n)\,, \\ d_{S_3^{n+1}}(j^nk) &= \frac{2}{3}\cdot 6^{n+1} - \frac{7}{6}\cdot 3^{n+1} + \frac{3}{2} = d_{H_3^{n+1}}(j^nk)\,, \\ d_{S_3^{n+1}}(jk^n) &= \frac{7}{15}\cdot 6^{n+1} - \frac{1}{6}\cdot 3^{n+1} - \frac{3}{10} = d_{H_3^{n+1}}(jk^n)\,. \end{aligned}$$

3.3 Metric dimension

The concept of metric dimension of a graph was independently introduced by Harary and Melter in 1974 [22] and by Slater in 1975 [59]. A few years ago Bailey and Cameron published a semi-survey paper [2], which is a great source on historical developments, connections of this dimension to other invariants and a long list of references on this topic. Another survey source for the metric dimension is [14]. In this final section of the chapter on metric properties of Sierpiński graphs we will determine the metric dimension of S_p^n.

Definition 3.30. *Let G be a graph and $k \in \mathbb{N}$. A subset $R = \{u_1, \ldots, u_k\} \subseteq V(G)$ is a* resolving set *(for G) if for any two distinct vertices $x, y \in V(G)$*

$$\big(d(x,u_1), \ldots, d(x,u_k)\big) \neq \big(d(y,u_1), \ldots, d(y,u_k)\big)\,.$$

The metric dimension *of G, $\mu(G)$, is the minimal size of a resolving set.*

In other words, the set $R \subseteq V(G)$ is resolving if each vertex of G is uniquely determined by the distances to the vertices of R. This way any two distinct vertices $x, y \in V(G)$ are resolved by some vertex of R, which means that there exists a vertex $u_i \in R$ such that $d(x,u_i) \neq d(y,u_i)$.

Returning to Sierpiński graphs, since $S_p^0 \cong S_1^n \cong K_1$ it is obvious that $\mu(S_p^n) = 0$ for $n = 0$ or $p = 1$. To determine the metric dimension of other Sierpiński graphs, let us construct resolving sets in these graphs. Let $n, p \in \mathbb{N}$ and $\ell \leq p$. Then denote the set of (the first) ℓ extreme vertices of S_p^n by

$$R_\ell^n := \{i^n \mid i \in [\ell]_0\}\,.$$

By Theorem 2.28 on symmetry of Sierpiński graphs the set R_ℓ^n could be replaced by any set of ℓ extreme vertices of S_p^n.

It is easy to see that the set R_p^n forms a resolving set for the graph S_p^n. Assume the opposite, i.e., that for some distinct vertices s, t of S_p^n

$$\big(d(s,0^n), \ldots, d(s,(p-1)^n)\big) = \big(d(t,0^n), \ldots, d(t,(p-1)^n)\big)\,.$$

Then $d(s, s_d^n) = d(t, s_d^n)$, for every $d \in [n]$, and thus Lemma 3.1 implies $s_d = t_d$ for every $d \in [n]$, which means $s = t$, a contradiction. The set remains resolving even if we remove a vertex:

Lemma 3.31. *If $n, p \in \mathbb{N}$, then R^n_{p-1} is a resolving set of S^n_p.*

Proof. For $p = 1$, $R^n_{p-1} = \{0^n\} = V(S^n_1)$, so R^n_0 definitely forms a resolving set for S^n_1. Let now $p \geq 2$ and let s and t be vertices of S^n_p with $d(s, i^n) = d(t, i^n)$ for all $i \in [p-1]_0$. Then by Corollary 3.2, $d(s, (p-1)^n) = d(t, (p-1)^n)$ holds as well. But then by Lemma 3.1 $s = t$, a contradiction. □

To obtain the metric dimension of Sierpiński graphs we also require the following immediate consequence of Proposition 3.14:

Corollary 3.32. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then for any pairwise distinct $i, j, k \in [p]_0$ and $s \in [p]^n_0$,*

$$d_{S^{n+1}_p}(is, j^n k) = d_{S^{n+1}_p}(is, j^{n+1})\,.$$

By combining these results we are able to determine the metric dimension of S^n_p. It it equal to the metric dimension of a complete graph K_p. This is not surprising, since K_p is the main building block for the construction of S^n_p.

Theorem 3.33. *If $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, then*

$$\mu(S^{n+1}_p) = p - 1\,.$$

Moreover, if R is a minimum resolving set, then $|R \cap V(jS^n_p)| \leq 1$ holds for any $j \in [p]_0$.

Proof. Let $R \subset V(S^{n+1}_p)$ and assume that $R \cap jS^n_p = \emptyset = R \cap kS^n_p$ for distinct $j, k \in [p]_0$. Corollary 3.32 implies that for each $r \in R$ we have $d(r, j^n k) = d(r, j^{n+1})$. This means R can not be a resolving set for S^{n+1}_p and each resolving set must contain at least one element of at least $p-1$ subgraphs isomorphic to S^n_p. So each resolving set must contain at least $p-1$ elements. Since by Lemma 3.31 any $p-1$ extreme vertices form a resolving set, we deduce that $\mu(S^{n+1}_p) = p-1$ and, with recourse to the pigeonhole principle, that no jS^n_p can contain more than one element of a minimal resolving set. □

The first assertion of Theorem 3.33 has been found independently and at the same time by Aline Parreau [54, Théorème 3.6].

Chapter 4

Embeddings

Before we start with particular embeddings of Sierpiński graphs, we will explain some basic theory about embeddings of graphs. The theory presented in the sequel is mainly adapted from the books [33] and [21]. Some of the embeddings considered later will be into Cartesian product graphs. The *Cartesian product* of graphs G and H, $G \,\square\, H$, is a graph defined with

$$V(G \,\square\, H) = V(G) \times V(H)\,,$$
$$E(G \,\square\, H) = \{\{(g,h),(g',h')\} \mid g = g', \{h,h'\} \in E(H) \text{ or } \{g,g'\} \in E(G), h = h'\}\,.$$

Special representatives of Cartesian product graphs are *Hamming graphs*. They are defined as Cartesian products of complete graphs. An equivalent definition of Hamming graphs is the following. Let $r_\ell \geq 2$, $\ell \in [n]$, be given integers. Then a Hamming graph G is the graph whose vertex set is $[r_1] \times [r_2] \times \cdots \times [r_n]$, and two vertices are adjacent if the corresponding n-tuples differ in precisely one coordinate. With the notation of Cartesian products this means $G = K_{r_1} \,\square\, K_{r_2} \,\square\, \cdots \,\square\, K_{r_n}$. The number of factors in a Hamming graph is the *dimension of the Hamming graph*. If all the factors in a Hamming graph are of order p, we denote it by K_p^n.

In our case an *embedding* of a graph G into a graph H will be an injective homomorphism, i.e., an injective mapping $f : V(G) \to V(H)$, such that if $\{u,v\}$ is an edge in G, $\{f(u), f(v)\}$ is also an edge in H. An *image* $f(G)$ of G under the embedding f will be a graph with $V(f(G)) = f(V(G))$ and $E(f(G)) = \{\{f(u), f(v)\} \mid \{u,v\} \in E(G)\}$. Note that not every edge of H with endvertices in $f(V(G))$ is necessarily in $f(G)$. In particular we will consider isometric and induced embeddings. As usual, isometric means distance preserving, formally:

Definition 4.1. *Let G and H be graphs. An embedding $f : V(G) \to V(H)$ is* isometric *if for every pair of vertices $u, v \in V(G)$*

$$d_H(f(u), f(v)) = d_G(u,v)\,.$$

A weaker condition is if the embedding is induced:

Definition 4.2. *Let G and H be graphs. An embedding $f : V(G) \to V(H)$ is* induced *if $f(G)$ is an induced subgraph of H.*

Note that every isometric embedding is also induced, since every isometric subgraph of a graph is also an induced subgraph. But an induced embedding is not necessarily isometric, see for example Figure 4.1, where an induced embedding of P_3 into C_5 is shown but P_3 is not an isometric subgraph of C_5.

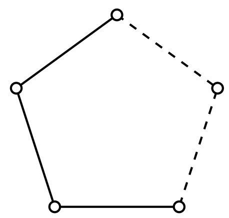


Figure 4.1: Embedding of P_3 into C_5

While dealing with embeddings, we will often use quotient graphs. For a quotient graph we require a partition of either the vertex set or the edge set of a graph. The definition is similar in both cases, but for our purposes we will define it with a partition of the edge set.

Definition 4.3. *Let G be a graph and let $\mathcal{F} = \{F_1, \ldots, F_r\}$ be a partition of its edge set. Then for $i \in [r]$ the* quotient graph *G/F_i is the graph whose vertices are the connected components of $G \setminus F_i$, where two components C_i and C_j are connected (in G/F_i) if there is an edge in G that connects a vertex of C_i with a vertex of C_j.*

When embedding into Cartesian product graphs, we are usually interested in having no unused factors or vertices. Therefore we define an irredundant embedding.

Definition 4.4. *Let G be a graph and $H = \mathop{\square}\limits_{i=1}^{k} H_i$ be a Cartesian product graph. An embedding $f : V(G) \to V(H)$ is* irredundant *if*

(i) $|V(H_i)| \geq 2$ for every $i \in [k]$, and

(ii) every vertex $h \in \bigcup\limits_{i=1}^{k} V(H_i)$ occurs as a coordinate in the image of some vertex $g \in G$.

If f is an irredundant embedding, we say that the image of G under f is an irredundant subgraph *in $H = \mathop{\square}\limits_{i=1}^{k} H_i$.*

In the rest of the chapter we will first discuss embeddings of Sierpiński graphs into Hanoi graphs [28], then we will give the canonical metric representation [44] of Sierpiński graphs and finally we will study their Hamming dimension [44]. Recall from Section 1.2 that for $n = 0$

a Sierpiński graph $S_p^0 \cong K_1$ for any $p \in \mathbb{N}$. When it comes to embeddings we are mainly interested in getting more information about the structure of a graph. Since there is not much information to obtain about the structure of a one-vertex graph, we will exclude the case $n = 0$ from this chapter entirely.

4.1 Embeddings into Hanoi graphs

Hanoi and Sierpiński graphs are defined on the same vertex set. From Section 1.3.1 we already know that $H_3^n \cong S_3^n$, therefore we were wondering whether we can generalize this relation to any $p \in \mathbb{N}$. We will see that although the Hanoi graph H_p^n has significantly more edges than the Sierpiński graph S_p^n, as soon as $p > 3$, we are not always able to embed S_p^n into H_p^n as a spanning subgraph. The reason for this is that the edge of S_p^n between any two subgraphs isomorphic to S_p^{n-1} is unique, and any two such edges are non-adjacent.

Theorem 4.5. *If $n, p \in \mathbb{N}$, then S_p^n can be embedded into H_p^n if and only if p is odd or $n = 1$.*

Proof. The case $n = 1$ is clear, because $S_p^1 \cong K_p \cong H_p^1$. The same applies to $p = 1$ since $S_1^n \cong K_1 \cong H_1^n$. Moreover, for $n \geq 2$, we have $\|S_2^n\| = 2^n - 1 > 2^{n-1} = \|H_2^n\|$, so that S_2^n cannot be embedded into H_2^n. (In fact, H_2^n is a spanning subgraph of S_2^n.)

In the rest of the proof we will first show that it is not possible to embed S_p^n into H_p^n for any even p and afterwards we will describe an embedding of S_p^n into H_p^n for any odd p.

So let first $p \geq 4$ be even and $n = 2$. Assume that there is an embedding $\alpha : S_p^2 \to H_p^2$. By Lemma 1.12, the p-cliques of S_p^2 are mapped onto the p-cliques of H_p^2. The remaining edges of S_p^2, these are exactly the edges $e_{ij}^{(2)}$, $i, j \in [p]_0$, $i \neq j$, have to be mapped by α to edges in H_p^2 corresponding to moves of disc 2. There are $\binom{p}{2}$ edges $e_{ij}^{(n)}$ of S_p^2 and they are pairwise non-adjacent. On the other hand, edges in H_p^2 corresponding to moves of disc 2 induce p cliques of order $p-1$. Among the edges of these cliques, we can select at most $p\left\lfloor\frac{p-1}{2}\right\rfloor$ independent ones. Since p is even, $p\left\lfloor\frac{p-1}{2}\right\rfloor < p\,\frac{p-1}{2} = \binom{p}{2}$. We conclude that S_p^2 cannot be embedded into H_p^2.

We will now reduce the more general case for even p and $n \geq 3$ to the case just dealt with. Let α' be an embedding of S_p^n into H_p^n. The key idea is to consider the image $\alpha'(0^{n-2}S_p^2)$. Since non-extreme vertices of S_p^n are of degree p, they cannot be mapped by α' to perfect vertices. Hence, the p extreme vertices of S_p^n are mapped to p perfect vertices of H_p^n so that $\alpha'(0^n) = j^n$ for some j. Using Lemma 1.12 again, $\alpha'(0^{n-1}S_p^1) = j^{n-1}H_p^1$. Moreover, the subgraph $0^{n-2}S_p^2$ of S_p^n contains $p-1$ p-cliques that are at distance 1 from the clique $0^{n-1}S_p^1$. All the other cliques of S_p^n are at distance more than 1 from $0^{n-1}S_p^1$. Similarly, the subgraph $j^{n-2}H_p^2$ of H_p^n contains p p-cliques that are pairwise at distance 1. Every other p-clique of H_p^n is at distance at least two from $j^{n-1}H_p^1$. (Indeed, suppose another clique which is not in $j^{n-2}H_p^2$, say $j^{n-3}iH_p^2$, $i \neq j$, would be connected to a vertex $j^{n-1}k$ of $j^{n-1}H_p^1$. Then the vertex of $j^{n-3}iH_p^2$ would have the form $j^{n-3}ijk$, but then we get a contradiction by Definition 1.11.) Therefore, $\alpha'(0^{n-2}S_p^2) = j^{n-2}H_p^2$. Hence α' would embed $0^{n-2}S_p^2 \cong S_p^2$ into $j^{n-2}H_p^2 \cong H_p^2$, a possibility which we already excluded.

Suppose next that $p \geq 3$ is odd. We will show by induction on n that there is an embedding of S_p^n into H_p^n. The case $n = 1$ was already considered at the beginning of the proof and is trivial. By the degree condition, any such embedding must map extreme vertices of S_p^n onto perfect vertices of H_p^n. For $n \geq 1$ let ι_n be an embedding from S_p^n into H_p^n. Since an arbitrary permutation of the perfect states of H_p^n extends to an automorphism of H_p^n (cf. [53]), we may without loss of generality assume that $\iota_n(k^n) = k^n$ for all $k \in [p]_0$. We construct the mapping $\iota_{n+1} : V(S_p^{n+1}) \to V(H_p^{n+1})$ in the following way. For $k \in [p]_0$ let π_k be the permutation on $[p]_0$ defined by

$$\forall\, i \in [p]_0 : \ \pi_k(i) = \frac{1}{2}\left(k(p+1) - i(p-1)\right) \bmod p\,.$$

It has precisely one fixed point, namely k. Next, let π_k^n denote the bijection on $[p]_0^n$ with $\pi_k^n(s_n \ldots s_1) = \pi_k(s_n) \ldots \pi_k(s_1)$. Define

$$\forall\, k \in [p]_0\ \forall\, s \in [p]_0^n : \ \iota_{n+1}(ks) = k\pi_k^n\left(\iota_n(s)\right)\,.$$

This obviously constitutes a bijection with

$$\iota_{n+1}(k^{n+1}) = k\pi_k^n\left(\iota_n(k^n)\right) = k\pi_k^n(k^n) = k^{n+1}\,.$$

This construction is illustrated in Figure 4.2 for the case of S_5^2 and H_5^2.

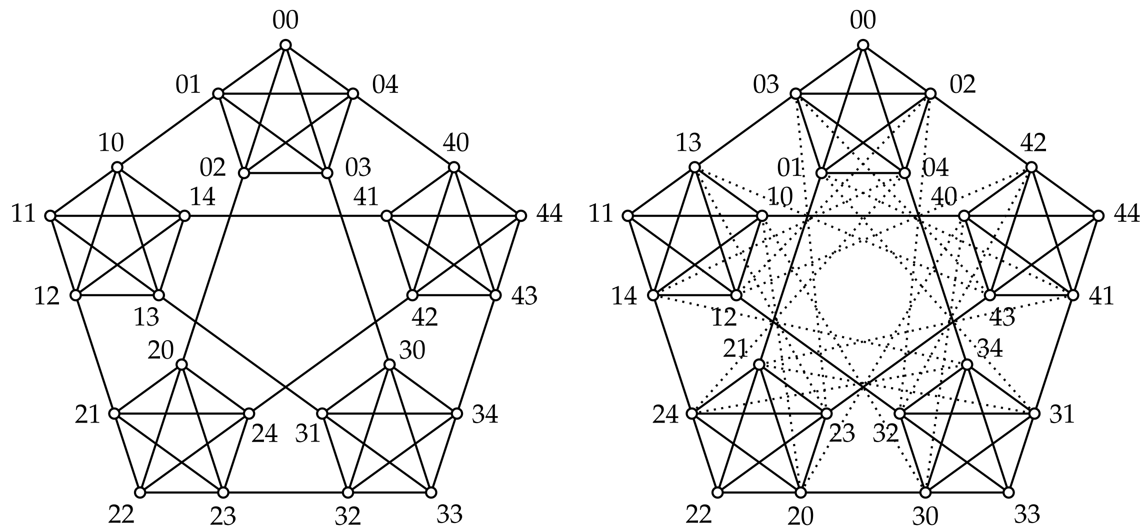


Figure 4.2: The embedding ι_2 from S_5^2 into H_5^2

It remains to show that $\{\iota_{n+1}(ij^n), \iota_{n+1}(ji^n)\} \in E(H_p^{n+1})$ for distinct $i, j \in [p]_0$. We have $\iota_{n+1}(ij^n) = i\pi_i^n(\iota_n(j^n)) = i\pi_i(j)^n$ and similarly $\iota_{n+1}(ji^n) = j\pi_j(i)^n$. Moreover,

$$i \neq \pi_i(j) = \frac{1}{2}(ip + i - jp + j) \bmod p = \frac{1}{2}(jp + j - ip + i) \bmod p = \pi_j(i) \neq j\,,$$

and so the two vertices are adjacent in H_p^n. □

As observed in [33, Section 2.2], Hanoi graphs H_p^n are spanning subgraphs of K_p^n. Therefore, we get

Corollary 4.6. *If $p \in \mathbb{N}$ is odd, then for any $n \in \mathbb{N}_0$, S_p^n is a spanning subgraph of the Hamming graph K_p^n.*

Although Corollary 4.6 holds only for odd values of p, we believe it could be generalized to arbitrary p. We will explain more details about the possibilities of this extension in the final chapter, where we discuss some open problems.

4.2 Canonical metric representation

The classical theory due to Graham and Winkler [15] asserts that there is precisely one isometric embedding of a graph into Cartesian product graphs that is irredundant and has the largest number of factors. It is called the *canonical metric representation*. Let us start with a brief overview of the theory required to describe the embedding. For more details see [21, Chapters 11 and 13] and [33, Chapter 14].

Definition 4.7. *Let G be a graph and let $e = uv$ and $f = xy$ be edges of G. The edges e and f are in relation Θ (in G) if and only if*

$$d(u,x) + d(v,y) \neq d(u,y) + d(v,x)\,.$$

Relation Θ is reflexive and symmetric, but not transitive in general. In order to get an equivalence relation we build the transitive closure Θ^* of the relation Θ. Its equivalence classes form a partition of the edge set of G. We will denote it by $\mathcal{E} = \{E_1, \ldots, E_\rho\}$. Other properties of relations Θ and Θ^*, which we will be using, are gathered in the subsequent lemma.

Lemma 4.8. [21, 33] *Let G be a graph. Then:*

(i) No two distinct edges on a shortest path in G are in relation Θ.
(ii) If P is a walk connecting the endpoints of an edge e in G, then P contains an edge $f \neq e$ with $e\Theta f$.
(iii) Two adjacent edges of G are in relation Θ if and only if they belong to a common triangle.

(iv) If C is an isometric cycle in G, then two edges e and f on cycle C are in relation Θ if and only if they are antipodal edges[1] *of C.*

(v) No two edges from different 2-connected components of G are in relation Θ.

We will also need the following modification of Lemma 4.8(v).

Lemma 4.9. *If H is an isometric subgraph of a graph G, and e and f are edges from different 2-connected components of H, then e is not in relation Θ with f in G.*

Proof. Let $e = uv$ and $f = xy$ be edges from different 2-connected components of H. By Lemma 4.8(v), e and f are not in relation Θ in H, that is,

$$d_H(u,x) + d_H(v,y) = d_H(u,y) + d_H(v,x)\,.$$

Since H is an isometric subgraph of G, it follows that

$$d_G(u,x) + d_G(v,y) = d_G(u,y) + d_G(v,x)\,,$$

hence e and f are not in relation Θ in G. □

Note that we cannot conclude in Lemma 4.9 that e and f are not in relation Θ^* in G. For instance, consider P_3 as a subgraph of $K_{2,3}$ shown in Figure 4.3. It is easy to see that all the edges of $K_{2,3}$ form a single Θ^*-class (Lemma 4.8(iv)). P_3 is an isometric subgraph of $K_{2,3}$ yet its edges are in relation Θ^*. Similarly, we also cannot assume that the properties in Lemma 4.8 hold for Θ^*.

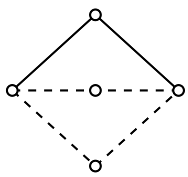


Figure 4.3: P_3 as an isometric subgraph of $K_{2,3}$

By now, we have familiarized ourselves well with the relation Θ^*. Next we would like to derive an embedding from it. We call the embedding *canonical*, due to its definition. To define the canonical embedding with respect to a partition $\mathcal{F} = \{F_1, \ldots, F_r\}$ of the set $E(G)$, we also require the concept of the *natural projections*. These are projections obtained from $\mathcal{F} =$

[1]If a cycle has odd length, then every edge has two antipodal edges; for example, in a triangle any two edges are antipodal to the third one.

$\{F_1, \ldots, F_r\}$ in the following way

$$f_i : V(G) \to V(G/F_i)\,,$$

where G/F_i is the quotient graph with respect to F_i and a vertex v is mapped to the connected component of $G - F_i$ that contains v.

Definition 4.10. *Let G be a graph and let $\mathcal{F} = \{F_1, \ldots, F_r\}$ be a partition of its edge set. Further, let $f_1, \ldots, f_r$ be the natural projections derived from $\mathcal{F}$. The* canonical embedding *of G (with respect to $\mathcal{F}$) is the mapping*

$$f : V(G) \to V(G/F_1) \,\Box\, \cdots \,\Box\, V(G/F_r)\,,$$

with

$$f(v) = (f_1(v), \ldots, f_r(v))\,.$$

In the case when a partition of the edge set of a graph G consists of the Θ^*-classes of G, the canonical embedding is called *canonical metric representation* of G. We denote the embedding by α and the natural projections by α_i:

$$\begin{aligned} &\alpha : V(G) \to V(G/E_1) \,\Box\, \cdots \,\Box\, V(G/E_\rho)\,, \\ &\alpha(v) = (\alpha_1(v), \ldots, \alpha_\rho(v))\,. \end{aligned}$$

We say that the canonical metric representation is *trivial* if G contains only one Θ^*-class. It is also isometric, see for instance [15, Theorem 1].

It follows immediately from Lemma 4.8 that S_p^1 has a trivial canonical metric representation for any $p \in \mathbb{N}$, as S_1^n does for any $n \in \mathbb{N}$. Since $S_2^n = P_{2^n}$, every edge of S_2^n represents its own Θ^*-class. So for any $i \in [2^n - 1]$, $S_2^n/F_i = K_2$. The canonical metric representation of S_2^n is therefore an isometric embedding into the hypercube Q_{2^n-1}.

The next observation is crucial to determine most of the Θ^*-classes of the rest of the Sierpiński graphs.

Lemma 4.11. *If $n, p \in \mathbb{N}$ and $n \geq 2$, $p \geq 3$, then for any pairwise distinct $i, j, \ell \in [p]_0$,*

$$e_{ij}^{(n)} \;\Theta\; \ell e_{ij}^{(n-1)}\,.$$

Proof. The edge $e_{ij}^{(n)}$ is the antipodal edge of the edge $\ell e_{ij}^{(n-1)}$ in $C_{ij\ell}^{(n)}$. By Proposition 3.7 the cycle $C_{ij\ell}^{(n)}$ is isometric in S_p^n, so the assertion follows by Lemma 4.8(iv). □

Keep in mind that Lemma 4.11 holds for all $p \geq 3$ and is thus the main reason why most of the Sierpiński graphs have a trivial canonical metric representation.

Proposition 4.12. *If $p \in \mathbb{N}$, $p \geq 4$, then for any $n \in \mathbb{N}$ the canonical isometric representation of S_p^n is trivial.*

Proof. For a fixed $p \geq 4$ we proceed by induction on $n \in \mathbb{N}$. S_p^1 is isomorphic to K_p, hence the assertion clearly holds in this case. Let $n > 1$. Then for any $i \in [p]_0$, the subgraph iS_p^{n-1} contains a single Θ^*-class by the induction hypothesis applied to iS_p^{n-1} (which is isomorphic to S_p^{n-1}). Let $j \in [p]_0 \setminus \{i\}$ be fixed. Then by Lemma 4.11 $e_{ij}^{(n)}$ is in relation Θ with the edge $\ell e_{ij}^{(n-1)}$, for any $\ell \in [p]_0 \setminus \{i, j\}$. Thus $e_{ij}^{(n)}$ is in the same Θ^*-class as ℓS_p^{n-1}. Finally, symmetry of Sierpiński graphs (Theorem 2.28) asserts that the canonical isometric representation of S_p^n is trivial. □

Thus our only hope for a non-trivial canonical metric representation remains the case S_3^n. For some initial base-3-Sierpiński graphs it is easy to determine Θ^*-classes, as it can be seen in Figure 4.4.

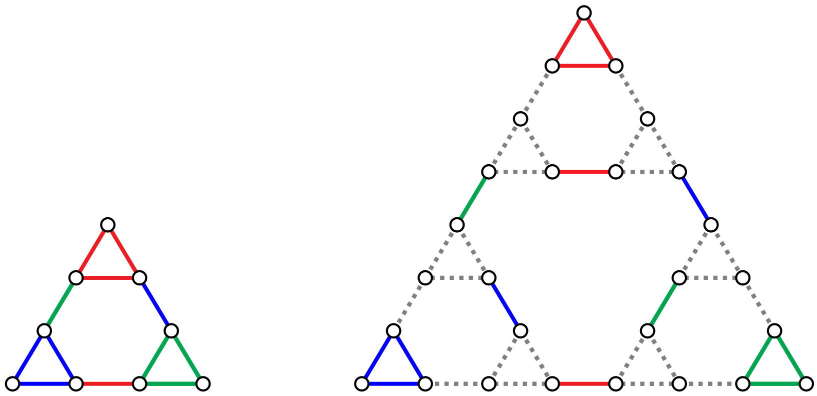


Figure 4.4: Θ^*-classes of S_3^2 (left) and S_3^3 (right)

Similarly we can determine the structure of Θ^*-classes of S_3^n for larger values of n. First note that for any $n \in \mathbb{N}$ there is only one cycle $C_{ij\ell}^{(n)}$ in S_3^n, namely $C_{012}^{(n)}$, and recall that $T = [3]_0$. By Lemma 4.8 all edges in a triangle $\underline{s}S_3^1$, $\underline{s} \in T^{n-1}$, of S_3^n are in one Θ^*-class. Using Lemma 4.11, we can conclude that for $\{i, j, \ell\} = T$, the edges of $i^{n-1}S_3^1$ and all the edges $i^m e_{j\ell}^{(n-m)}$, $m \in [n-1]_0$, are in one Θ^*-class. Our goal is to show, that such a Θ^*- class does not contain any other edge. But first let us prove an observation on Θ^*- classes of S_3^n.

Proposition 4.13. *If $n \in \mathbb{N}$, then for any Θ^*-class F of S_3^n and any distinct $i, j \in T$, $|P_{ij}^{(n)} \cap F| \geq 1$.*

Proof. We proceed by induction on n. The statement is clearly true for $n = 1$, since $S_3^1 = K_3$ and it has only one Θ^*-class. Let $n > 1$ and let F be an arbitrary Θ^*-class of S_3^n. If $|F \cap iS_3^{n-1}| \geq 1$, then by the induction hypothesis (applied to iS_3^{n-1}), F intersects shortest paths $iP_{ij}^{(n-1)}$, $iP_{ik}^{(n-1)}$, and $iP_{jk}^{(n-1)}$ for $\{i, j, k\} = T$. Let e be in $iP_{j,k}^{(n-1)} \cap F$. If the antipodal edge of e on $C_{012}^{(n)}$ is $e_{jk}^{(n)}$, we are done since $e_{jk}^{(n)}$ is on $P_{j,k}^{(n)}$. Otherwise, the antipodal edge of e on $C_{012}^{(n)}$ is either on $jP_{ik}^{(n-1)}$ or $kP_{ij}^{(n-1)}$. In this case we use induction and symmetry of the Sierpiński graphs (Theorem 2.28) until we reach one of the paths $P_{ij}^{(n)}$, $P_{ik}^{(n)}$, and $P_{jk}^{(n)}$. $\square$

In other words, every Θ^*-class is present on any of the paths $P_{ij}^{(n)}$. To describe Θ^*-classes of S_3^n explicitly, let $T = \{i, j, \ell\}$ and set

$$\begin{aligned} F_i^n &:= \left\{\{i^n, i^{n-1}j\}, \{i^n, i^{n-1}\ell\}\right\} \cup \left\{i^{n-m}e_{j\ell}^{(m)} \mid m \in [n]\right\}, \\ \widetilde{F^n} &:= E(S_3^n) \setminus (F_0^n \cup F_1^n \cup F_2^n). \end{aligned}$$

In Figure 4.4, Θ^*-classes F_0^2 and F_0^3 are drawn in red, F_1^2 and F_1^3 in blue, and F_2^2 and F_2^3 in green. Note that $\widetilde{F^2} = \emptyset$ and $\widetilde{F^3}$ is drawn with dotted gray lines. An example of a quotient graph $S_3^n/\widetilde{F^n}$ is shown in Figure 4.5 for $n = 4$.

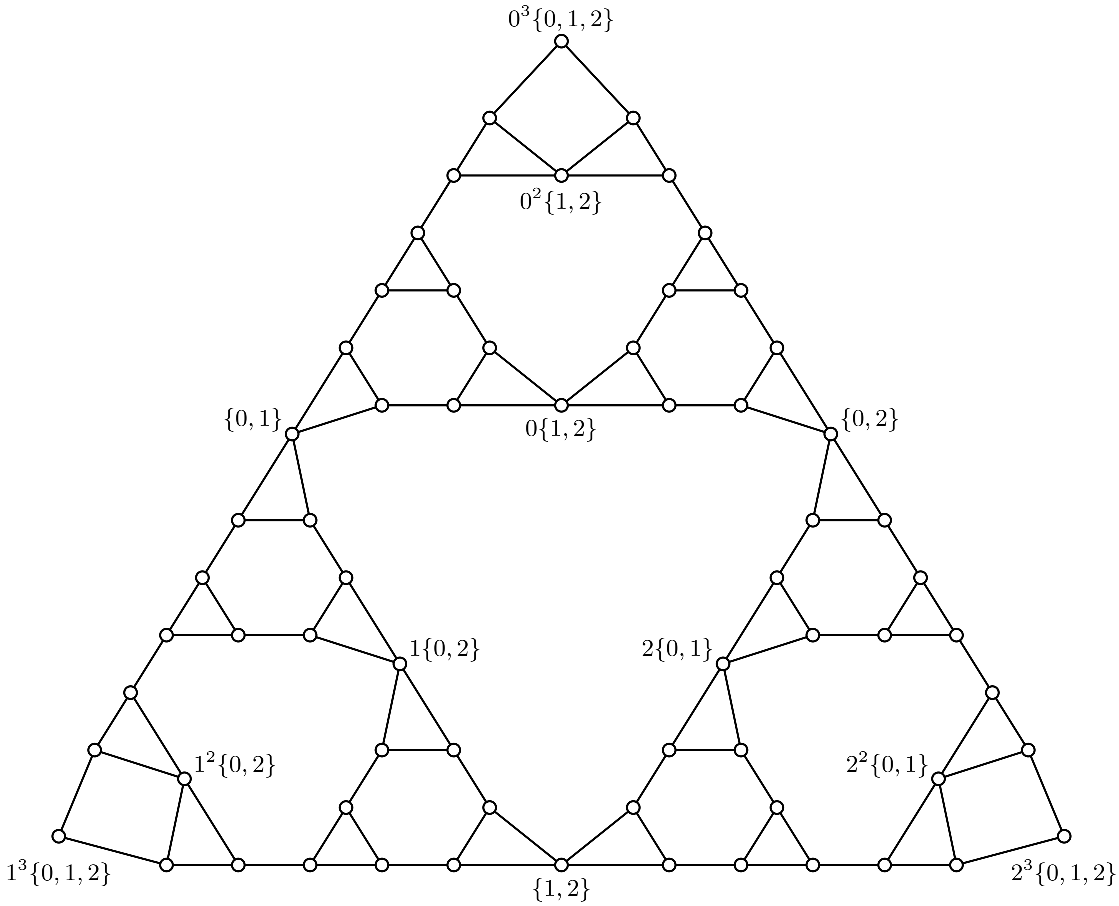


Figure 4.5: The quotient graph $S_3^4/\widetilde{F^4}$

Now we are ready to prove that these sets are the only Θ^*-classes of S_3^n.

Theorem 4.14. *If $n \in \mathbb{N}$ and $n \geq 2$, then the Θ^*-classes of S_3^n are F_0^n, F_1^n, F_2^n, and $\widetilde{F^n}$.*

Proof. It is straightforward to check the result for $n = 2$, where $\widetilde{F_3} = \emptyset$. In this case we have three Θ^*-classes, which are also shown in Figure 4.4.

Let $i \in T$ and consider F_i^n. Recall that iS_3^{n-1} is an isometric subgraph of S_3^n, therefore by Lemma 4.9 and by induction hypothesis it follows that $\{i^n, i^{n-1}j\}, \{i^n, i^{n-1}\ell\} \in F_i^n$, as well as $i^{n-m}e_{j\ell}^{(m)} \in F_i^n$ for $m \in [n-1]$ and $\{i, j, \ell\} = T$. Moreover, Lemma 4.11 asserts $e_{j\ell}^{(n)} \Theta i e_{j\ell}^{(n-1)}$. Hence the edges of F_i^n belong to a common Θ^*-class.

It remains to show that no two edges from different sets F_0^n, F_1^n, F_2^n, and $\widetilde{F^n}$ are in relation Θ and that in $\widetilde{F^n}$ any two edges are in relation Θ^*.

For the first assertion, by symmetry (Theorem 2.28) it suffices to prove that no edge of F_0^n is in relation Θ with any other edge. There are three connected components of $S_3^n \setminus F_0^n$. One is the extreme vertex 0^n and the other two symmetrical components we will denote by G_1 and G_2, where $1^n \in G_1$ and $2^n \in G_2$. An example for the connected components of the graph $S_3^n \setminus F_0^n$ is drawn in Figure 4.6 for $n = 3$. Using symmetry again, it suffices to prove that no edge of F_0^n is in relation Θ with an edge of G_1.

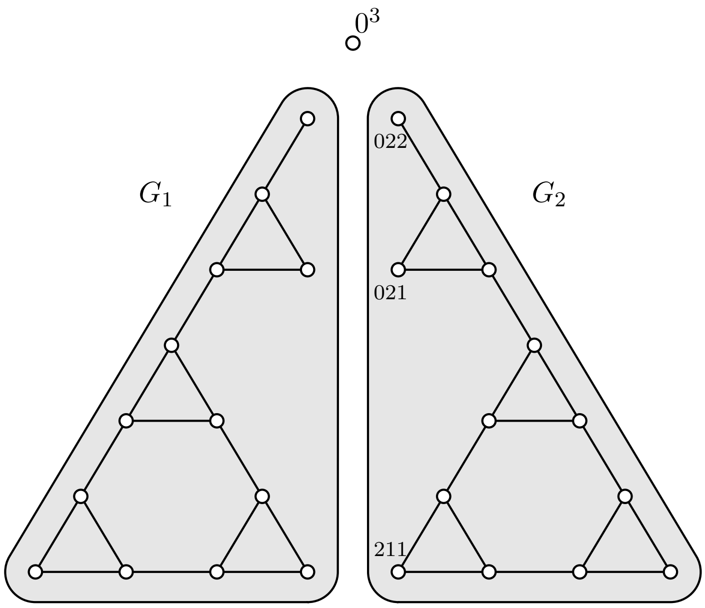


Figure 4.6: The graph $S_3^3 \setminus F_0^3$ with subgraphs G_1 and G_2

Note first that G_1 is isometric in S_3^n. Moreover, the graph induced by $V(G_1)$ and vertices 0^n and $0^{n-1}2$ is also isometric in S_3^n. Then Lemma 4.9 implies that edges $\{0^n, 0^{n-1}1\}$, $\{0^n, 0^{n-1}2\}$, and $\{0^{n-1}1, 0^{n-1}2\}$ are not in relation Θ with any edge in G_1. Let $m \in [n-1]_0$ and consider the subgraph of S_3^n induced by $V(G_1)$ and 0^m21^{n-m-1} (see Figure 4.6 for $n = 3$). We infer again that this subgraph is isometric, hence by applying Lemma 4.9 we conclude that $\{0^{n-1}1, 0^{n-1}2\}$ is in relation Θ with no edge of G_1. This completes the proof that no two edges from different sets F_0^n, F_1^n, F_2^n, and $\widetilde{F^n}$ are in relation Θ.

It remains to prove that any two edges of $\widetilde{F^n}$ are in relation Θ^*. If $n = 3$, it is straightforward to check that $\{001, 010\}\Theta\{211, 210\}\Theta\{011, 012\}$. By symmetry and transitivity the result follows. Let $n \geq 4$. Then because $C_{012}^{(n)}$ is isometric in S_3^n (Proposition 3.7),

$$\{01^{n-1}, 01^{n-2}2\}\Theta\{210^{n-2}, 210^{n-3}1\}$$

as well as

$$\{01^{n-2}2, 01^{n-3}21\}\Theta\{210^{n-3}1, 210^{n-4}10\}\,.$$

Now we apply induction, symmetry, and transitivity of Θ^* to conclude that $\widetilde{F^n}$ is indeed a Θ^*-class. $\square$

For any $i \in T$ we get $S_3^n/F_i^n \cong K_3$, while $S_3^n/\widetilde{F^n}$ is obtained from S_3^n by contracting each edge in $F_0^n \cup F_1^n \cup F_2^n$. See Figure 4.5 for $S_3^4/\widetilde{F^4}$. The vertices are labeled in a similar manner as in Sierpiński triangle graphs ST_p^n. For example, by contracting the edges of the triangle $0^3S_3^1$ we get the vertex $0^3\{0, 1, 2\}$ and by contracting the edge $\{012^2, 021^2\}$ we get the vertex $0\{1, 2\}$.

Note that $|F_i^n| = n + 2$, and thus

$$\left|\widetilde{F^n}\right| = \frac{p}{2}(p^n - 1) - 3n - 6\,.$$

The three Θ^*-classes F_i^n of S_3^n give us small quotient graphs, but the fourth quotient graph has roughly the same number of vertices as S_3^n. Although we found an explicit canonical metric representation of Sierpiński graphs, it does not help us, for example, to determine the Wiener index of a graph. The latter can be computed quite easily with the canonical metric representation of a graph, if the corresponding quotient graphs are (much) smaller than the original graph, cf. [33, Chapter 14]. We will therefore study induced embeddings of Sierpiński graphs into Hamming graphs in the following section.

4.3 Hamming dimension

Since isometric embeddings of Sierpiński graphs from the previous section did not provide us with much new information about the structure of Sierpiński graphs, we now introduce the Hamming dimension of a graph and later study it on Sierpiński graphs.

Definition 4.15. *Let G be a graph. The* Hamming dimension, $\mathrm{Hdim}(G)$, *of G is the largest dimen-*

sion of a Hamming graph into which G embeds as an irredundant induced subgraph. If G is not an induced subgraph of any Hamming graph we set $\mathrm{Hdim}(G) = \infty$.

Clearly, $\mathrm{Hdim}(G) = 1$ if and only if G is a complete graph. To picture the Hamming dimension better, let us list further examples for some other known families of graphs. For a path on n vertices, $\mathrm{Hdim}(P_n) = n - 1$. Another nice example are star graphs where $\mathrm{Hdim}(K_{1,n}) = n$. But there are also graphs for which there is no irredundant embedding into Hamming graphs. Such graphs are for example the wheels W_n and "almost complete graphs" K_n^-, so $\mathrm{Hdim}(W_n) = \mathrm{Hdim}(K_n^-) = \infty$.

To determine the Hamming dimension of a graph, the theory of induced embeddings into Hamming graphs, which was developed in [43], is very useful. Let G be a graph and let $\mathcal{F} = \{F_1, F_2, \ldots, F_\rho\}$ be a partition of $E(G)$. Such a partition naturally yields the corresponding labeling (of the edge set) $\mathcal{L} : E(G) \to \{1, 2, \ldots, \rho\}$ by setting $\mathcal{L}(e) = i$ for $e \in F_i$. We say that a labeling is *nontrivial* if $\rho > 1$. Further, we introduce two conditions for a labeling:

Condition A. *An edge labeling of a graph G fulfills Condition A, if for any triangle of G, its edges have the same label.*

Condition B. *An edge labeling of a graph G fulfills Condition B, if for any vertices u and v of G at distance at least two, there exist different labels i and j which both appear on any induced u, v-path. (An induced path in our case is an induced subgraph X of G isomorphic to a path graph.)*

Conditions A and B are helpful tools for studying Hamming dimension because of the following result of Klavžar and Peterin [43] (expressed in the terms of the Hamming dimension):

Theorem 4.16. [43, Theorem 3.3] *If G is a connected graph, then* $\mathrm{Hdim}(G) < \infty$ *if and only if there exists a labeling of edges of G that fulfills Conditions A and B.*

The proof of Theorem 4.16 provides us with an approach on getting an induced embedding of a graph G into a Hamming graph when we have a labeling of G that satisfies Conditions A and B. We form a partition $\mathcal{F} = \{F_1, \ldots, F_\rho\}$ of the set $E(G)$, where F_i is the set of edges with label $i \in [\rho]$. For each partition set F_i we form the quotient graph G/F_i, and denote by $\psi_i : V(G) \to V(G/F_i)$ the natural projection (i.e., ψ_i maps $u \in V(G)$ to the component of $G \setminus F_i$ to which it belongs). Then

$$\psi = (\psi_1, \ldots, \psi_p) : V(G) \to V\left(G/F_1 \,\Box \cdots \Box\, G/F_\rho\right) \tag{4.1}$$

is an induced embedding of G. Moreover, by adding edges to each factor G/F_i to make it complete, the embedding ψ is still induced. So ψ is actually an induced embedding of G into a Hamming graph. In addition, $\psi(G)$ is an irredundant subgraph of $G/F_1 \,\Box \cdots \Box\, G/F_\rho$ and thus also an irredundant subgraph of a ρ-dimensional Hamming graph.

The following additional properties of a labeling that fulfills Condition B will be helpful.

Lemma 4.17. [43, Lemmas 3.1 and 3.2] *If G is a graph with a labeling of its edges that fulfills Condition B, then*

(i) in an induced cycle of length at least 3, every label must appear at least twice, and

(ii) if every induced path between two vertices contains two distinct labels i and j, then every path between these two vertices contains these two labels.

In addition, it is easy to see that if a maximal part of an induced cycle C is labeled alternatively with labels i and j, then i and j must also exist on the other part of C. In particular, if we have the sequence iji on C, then i appears at least once more on C.

Every S_p^n can be embedded in a Hamming graph with two factors with the following labeling.

Definition 4.18. *Let $n, p \in \mathbb{N}$, $p \geq 3$ and let $a, b \in \mathbb{N}_0$ be distinct. To obtain the* $(a|b)$-labeling *of S_p^n we label its every clique edge with label a and its every non-clique edge with b.*

Clearly, an $(a|b)$-labeling fulfills Condition A, since all the edges of a complete subgraph are labeled with a. Moreover, by the construction of Sierpiński graphs, no two non-clique edges are adjacent, thus Condition B holds as well. This tells us that

$$2 \leq \mathrm{Hdim}(S_p^n) < \infty \,, \tag{4.2}$$

therefore it makes sense to study the Hamming dimension of Sierpiński graphs.

4.3.1 Embeddings into products of Sierpiński triangle graphs

By defining a special labeling of Sierpiński graphs, the *Sierpiński triangle labeling*, we can embed these graphs into Cartesian products of Sierpiński triangle graphs. The labeling is defined as follows:

Definition 4.19. *Let $n, p \in \mathbb{N}$ and $p \geq 3$. The* Sierpiński triangle labeling *of S_p^n is defined inductively. We label the edges of S_p^1 with label 1. Assuming S_p^{n-1} has already been labeled, we label every subgraph iS_p^{n-1}, $i \in [p]_0$, of S_p^n in the same manner as S_p^{n-1}. Finally, we label the remaining edges $e_{ij}^{(n)}$ with label n.*

By the above definition it is obvious that the Sierpiński triangle labeling of S_p^n uses n labels. An example is presented in Figure 4.7 on S_3^4. By applying the Sierpiński triangle labeling to the graph S_p^2 we get the $(1|2)$-labeling (defined in Definition 4.18 for $a = 1$ and $b = 2$). This can also be seen in Figure 4.7 by looking at any subgraph isomorphic to S_3^2, for example $0^2S_3^2$.

Lemma 4.20. *If $n, p \in \mathbb{N}$ and $p \geq 3$, then the Sierpiński triangle labeling of S_p^n fulfills Conditions A and B.*

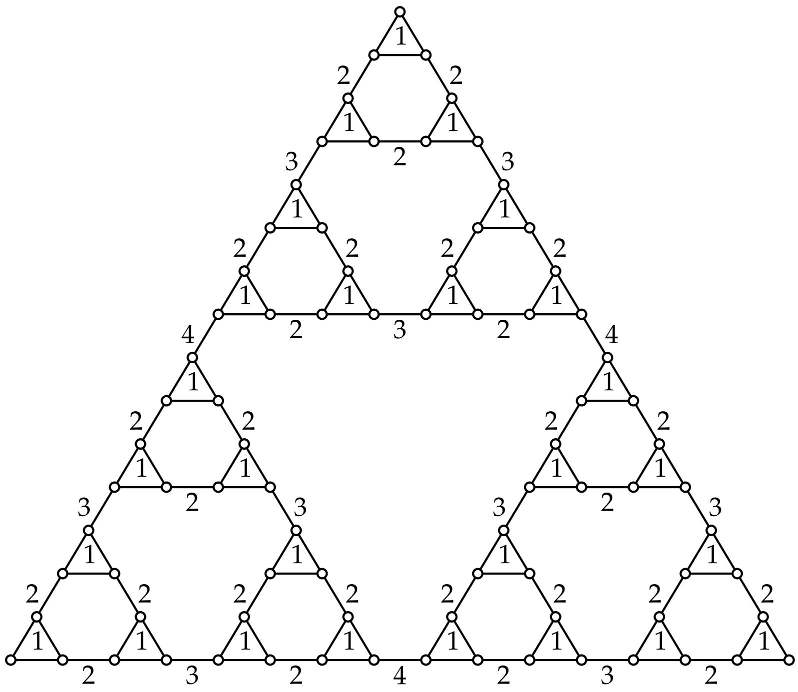


Figure 4.7: The Sierpiński triangle labeling of S_3^4

Proof. Let $p \geq 3$ be a fixed integer. By Proposition 2.29 the only maximal cliques (with respect to inclusion) in S_p^n are K_2 and $K_p \cong S_p^1$, so the triangles occur only in subgraphs sS_p^1, $s \in [p]_0^{n-1}$ of S_p^n. These subgraphs are by Definition 4.19 labeled with the same label, therefore the Sierpiński triangle labeling fulfills Condition A.

To show that the labeling also fulfills Condition B, we take two non-adjacent vertices u, v of S_p^n and a shortest u, v-path P. Note first that on any path in S_p^n of length at least two there is an edge with label 1, and even more, every other label on an induced path in S_p^n of length at least two is also 1. Denote the largest label on P by ℓ. Then $\ell > 1$, otherwise u and v would be adjacent. By the construction of Sierpiński triangle labeling, vertices u and v lie in a common subgraph $\overline{s}S_p^\ell$, $\overline{s} \in [p]_0^{n-\ell}$ of S_p^n, but in different subgraphs of $\overline{s}S_p^\ell$ that are isomorphic to $S_p^{\ell-1}$. Therefore any induced u, v-path must also contain label ℓ, so the labeling fulfills Condition B. $\square$

Combining the theory of induced embeddings discussed before with the Sierpiński triangle labeling, we can embed a Sierpiński graph S_p^n into a product of Sierpiński triangle graphs.

Theorem 4.21. *If $n, p \in \mathbb{N}$ and $p \geq 3$, then there exists an induced embedding*

$$S_p^n \to ST_p^{n-1} \,\Box\, ST_p^{n-2} \,\Box\, \cdots \,\Box\, ST_p^0 \,.$$

Proof. Let $p \geq 3$ be a fixed integer. Since by Lemma 4.20 the Sierpiński triangle labeling of S_p^n fulfills Conditions A and B, it remains to show that this labeling leads to the above-stated embedding. Let F_i, $i \in [n]_0$, be the set of edges of S_p^n labeled with $n-i$ in the Sierpiński triangle labeling of S_p^n. We are going to prove that for any $n \geq 1$ and for any $i \in [n]_0$, $S_p^n/F_i \cong ST_p^i$.

We proceed by induction on n. For $n = 1$, $S_p^1 \cong K_p$ and all of its edges are labeled with 1. Hence $S_p^1/F_0 \cong K_p \cong ST_p^0$. Suppose that the assertion of the theorem holds for $n \geq 1$, and consider S_p^{n+1}. Since $F_0 = \{e_{ij}^{(n+1)} \mid i,j \in [p]_0, i \neq j\}$ we infer that $S_p^{n+1}/F_0 \cong K_p \cong ST_p^0$. Next let $i \geq 1$. Then each edge of F_i lies in some subgraph jS_p^n, $j \in [p]_0$. Let jF_i be the restriction of F_i to jS_p^n, and note that by Definition 4.19 jF_i coincides with F_{i-1} in S_p^n. Hence, by the induction hypothesis, it follows that $jS_p^n/jF_i \cong ST_p^{i-1}$. But then $S_p^{n+1}/F_i \cong ST_p^i$ by the way the Sierpiński triangle graphs are constructed (see for instance Definition 1.5). $\square$

This gives us another lower bound on Hdim(S_p^n), namely Hdim$(S_p^n) \geq n$, which is much better than (4.2). For $p = 3$ we will improve it even further in the next subsection.

4.3.2 A lower bound on Hdim(S_3^n)

In this subsection we derive a better lower bound for the Hamming dimension of base-3-Sierpiński graphs. To do this, we introduce a labeling with a very large number of labels. Because of the way we define the labeling, we also think it has a maximal possible number of labels among all labelings of S_3^n that fulfill Conditions A and B, yet we are still not able to find an appropriate proof for this.

In a way similar to the construction of the Sierpiński triangle labeling, we build a labeling of S_p^n with a large number of labels. It is done inductively, so we take a labeling of S_p^{n-1} with as many different labels as possible and label each subgraph iS_p^{n-1}, $i \in T$ with the same pattern as S_3^{n-1}, but so that for any distinct $i,j \in T$, iS_3^{n-1} and jS_3^{n-1} use different labels. No matter how we label the edges $e_{ij}^{(n)}$, by Lemma 4.17 (for $n \geq 3$) this labeling does not fulfill Condition B, because on the cycle $C_{ij\ell}^{(n)}$ some labels appear only once. Therefore we need to merge these labels. More precisely, we have to merge labels that appear only once on the path $0P_{12}^{(n-1)}$, only once on $1P_{02}^{(n-1)}$, and only once on $2P_{01}^{(n-1)}$ with the exception of $0e_{12}^{(n-1)}$, $1e_{02}^{(n-1)}$, and $2e_{01}^{(n-1)}$, respectively. For a proper description of merging we require the notation of *oriented (sub)paths*. If we say that a path $P_{ij}^{(n)}$ in S_3^n is *oriented*, we mean it has a beginning, in this case the extreme vertex i^n, and an ending, j^n.

Definition 4.22. *Let $n \in \mathbb{N}$ and $n \geq 2$. The* merging labeling *of S_3^n is defined inductively. For $n = 2$ and $\{i,j,k\} = T$ we label every edge of the triangle iS_3^1 and the edge $e_{jk}^{(2)}$ with label i. Assuming S_3^{n-1} has already been labeled with the merging labeling, we label every subgraph iS_3^{n-1} with the same pattern as S_3^{n-1}, but so that for any distinct $i,j \in T$, iS_3^{n-1} and jS_3^{n-1} use different labels. In addition, we label the edges $e_{01}^{(n)}$, $e_{12}^{(n)}$, and $e_{02}^{(n)}$ with the same labels as $2e_{01}^{(n-1)}$, $0e_{12}^{(n-1)}$, and $1e_{02}^{(n-1)}$, respectively.*

Consider the pairs of oriented subpaths of $C_{012}^{(n)}$:

$$[01P_{12}^{(n-2)}, 21P_{10}^{(n-2)}];$$
$$[02P_{12}^{(n-2)}, 12P_{02}^{(n-2)}];\ \textit{and}$$
$$[20P_{01}^{(n-2)}, 10P_{02}^{(n-2)}].$$

Now traverse $01P_{12}^{(n-2)}$ and $21P_{10}^{(n-2)}$ in parallel. As soon as a label ℓ_0 appears on $01P_{12}^{(n-2)}$ that appears only once on $0P_{12}^{(n-1)}$, merge it with the corresponding label ℓ_2 of $21P_{10}^{(n-2)}$. (Note that ℓ_2 also appears only once on $2P_{10}^{(n-1)}$ by construction and symmetry.) More precisely, merging means we replace every such label ℓ_2 in S_3^n with ℓ_0. Do the same procedure for the remaining two pairs of paths.

It is easy to see that the merging labeling of S_3^2 coincides with its Θ^*- classes. Indeed, they both induce the same partition of the edge set of S_3^2. Another example of a merging labeling is shown in Figure 4.8 for S_3^3. Here labels 3 and 5 are merged into 3, labels 6 and 8 into 6, and labels 2 and 9 into 2. In the right copy of S_3^3 we replace label 7 with label 5, since it was not used in the middle copy of S_3^3. Doing so it becomes obvious that the merging labeling of S_3^3 uses 6 labels. Note that the label in a triangle refers to all three edges of the triangle.

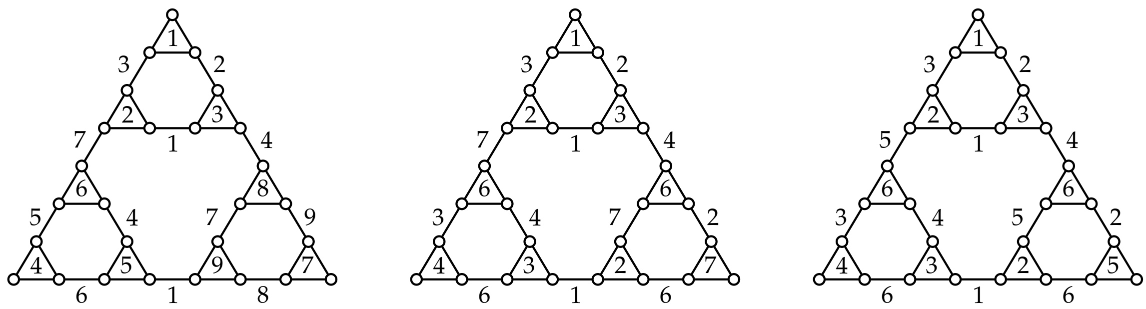


Figure 4.8: A pre-merging (left) and merging labelings of S_3^3 (middle, right)

Proposition 4.23. *If $n \in \mathbb{N}$ and $n \geq 2$, then a merging labeling of S_3^n fulfills Conditions A and B.*

Proof. Edges that form a triangle are labeled with the same label, hence Condition A is fulfilled. Condition B is fulfilled on S_3^2, cf. $0S_3^2$ in Figure 4.8. Let now $n > 2$ and let u, v be vertices of S_3^n with $d(u, v) \geq 2$. Let d be the smallest index such that both u and v are in sS_3^{n-d}, $s \in T^d$. Then $d < n - 1$, since $d(u, v) \geq 2$. Let $u \in siS_3^{n-d-1}$, $v \in sjS_3^{n-d-1}$, and let $\{i, j, k\} = T$.

Let P be a shortest u, v-path. Suppose first that P contains the edges $se_{ik}^{(n-d)}$ and $se_{jk}^{(n-d)}$. Then the labels of these two edges are on any induced u, v-path because of the way the merging labeling is constructed. In the other case, P contains a unique edge of the form $e = se_{h_1h_2}^{(n-d)}$, namely the edge $se_{ij}^{(n-d)}$. By the same argument its label appears on every induced u, v-path.

Since $d(u, v) \geq 2$, the edge e has at least one adjacent edge on P, say f. We may assume without loss of generality that $f \in sjS_3^{n-d-1}$. Then the label of f appears also on the triangle of skS_3^{n-d-1} that is incident with the edge $se_{ik}^{(n-d)}$. Again by construction, the label of f appears on any induced u, v-path. So we found two labels that appear on any induced u, v-path, and the proof is hereby complete. $\square$

Obviously a merging labeling of S_p^n uses many more labels than both of the labelings we have defined before (see Definitions 4.18 and 4.19). This induces smaller factors of the Hamming graphs into which we embed. For example, consider the graph S_3^3 and the Sierpiński triangle labeling (cf. subgraph $0S_3^3$ in Figure 4.7) and its merging labeling from Figure 4.8. The first one gives us an induced embedding into $K_{15} \,\square\, K_6 \,\square\, K_3$, while the merging labeling yields an induced embedding into K_3^6.

Before we continue, we present a more elaborated merging labeling of S_3^5 in Figure 4.9. We will refer to this labeling in the subsequent arguments. Note that in $0^2S_3^3$ we use labels 1 to 6, which is a labeling obtained from the right labeling from Figure 4.8 by replacing label 7 with label 5. Note also that the labeling of the upper subgraph $0S_3^4$ coincides with the merging labeling of S_3^4.

Lemma 4.24. *If $n \in \mathbb{N}$, $n \geq 2$ and S_3^n is labeled with a merging labeling, then every label of a non-clique edge in $P_{ij}^{(n)} \setminus \{e_{ij}^{(n)}\}$, where $i, j \in T$ are distinct, appears exactly twice on $P_{ij}^{(n)} \setminus \{e_{ij}^{(n)}\}$.*

Proof. There is nothing to be proved for $n = 2$. By Theorem 2.28 we may restrict ourselves to $P_{12}^{(n)}$. Note that the labels of the edges $1e_{12}^{(n-1)}$ and $2e_{12}^{(n-1)}$ are merged in S_3^n and have thus the same label. Hence every label of a non-clique edge of $P_{12}^{(n)}$ other than the label of $e_{12}^{(n)}$ appears at least twice on $P_{12}^{(n)}$ by induction.

It remains to prove that no non-clique edge appears more than twice. This clearly holds for $n = 3, 4$, cf. Figures 4.8 and 4.9. Let now $n \geq 5$. Note first that the assertion holds for the label of $1e_{12}^{(n-1)}$ and $2e_{12}^{(n-1)}$. Indeed, their labels were unique on $1P_{12}^{(n-1)}$ and $2P_{12}^{(n-1)}$, respectively, and were subsequently merged in the last step of the construction of the merging labeling. The label of the edges $1^2e_{12}^{(n-2)}$ and $12e_{12}^{(n-2)}$ (which is the same) appears only once on $1P_{02}^{(n-1)}$ and is also merged in the last step of merging in S_3^n. But this label appears on $12P_{02}^{(n-2)}$ and is merged with a label from $02P_{12}^{(n-1)}$. In other words, this label does not appear in $2S_3^{n-1}$ and consequently not on $2P_{12}^{(n-1)}$. By symmetry, the assertion also holds for the label of $21e_{12}^{(n-2)}$ and $2^2e_{12}^{(n-2)}$.

Next we show that the label ℓ of the non-clique edges $1^3e_{12}^{(n-3)}$ and $1^22e_{12}^{(n-3)}$ appears twice on $1P_{02}^{(n-1)}$ and is not merged in the last step of merging in S_3^n. Clearly ℓ appears once on $1^22P_{02}^{(n-3)}$ (on the edge incident with 1^220^{n-3}) and was in $1S_3^{n-1}$ merged with the label of the edge in $102P_{12}^{(n-3)}$ incident with 1021^{n-3}. This label is present in $10S_3^{n-2}$ also on the edges $10^2e_{02}^{(n-3)}$ and $102e_{02}^{(n-3)}$, which are both in $1P_{02}^{(n-1)}$.

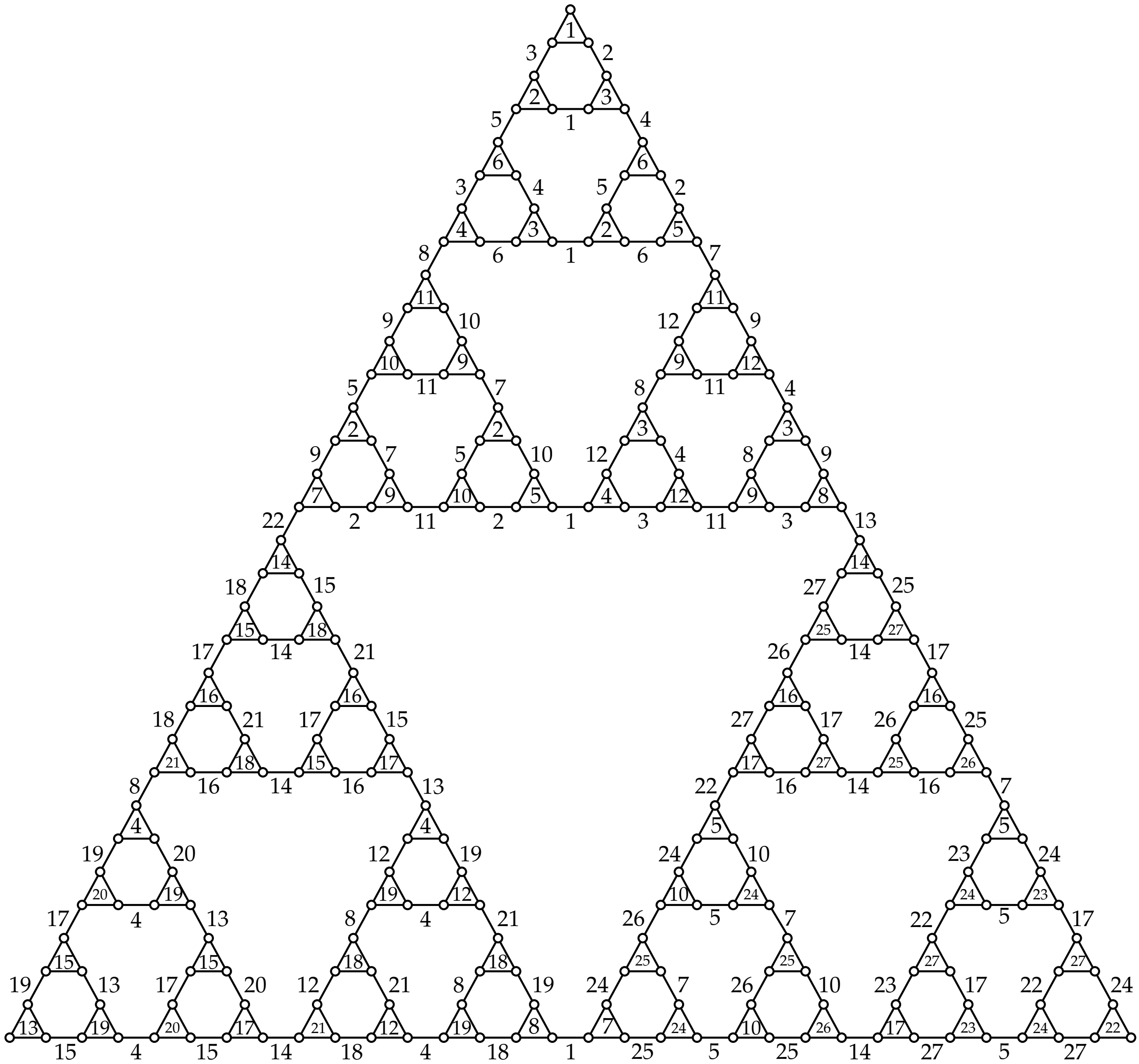


Figure 4.9: A merging labeling of S_3^5

Similarly, the label ℓ' of the edges $121e_{12}^{(n-3)}$ and $12^2e_{12}^{(n-3)}$ appears twice on $1P_{02}^{(n-1)}$ and is not merged in S_3^n. Clearly ℓ' appears once on $1P_{02}^{(n-1)}$, since it is in the triangle $12^20^{n-4}S_3^1$ (in $12^2S_3^{n-3}$). But ℓ' is also in the triangle $1210^{n-4}S_3^1$ (in $121S_3^{n-3}$). Hence it was merged in $1S_3^{n-1}$ with the label of the triangle $1012^{n-4}S_3^1$ (in $101S_3^{n-3}$). But this was again merged in $10S_3^{n-2}$ with the label of the triangle $1002^{n-4}S_3^1$, which lies on $1P_{02}^{(n-1)}$. So ℓ' appears twice on $1P_{02}^{(n-1)}$ and is thus not merged in the last step of merging in S_3^n.

For the labels of $P_{12}^{(n)}$ in $2S_3^{n-1}$ we proceed analogously, since they are symmetric to the edges in the previous two paragraphs. Finally, for all the other non-clique edges of $P_{12}^{(n)}$ the assertion follows by induction, so the proof is complete. $\square$

Since merging labeling fulfills Conditions A and B, we are now able to determine some exact values of the Hamming dimension of base-3-Sierpiński graphs.

Proposition 4.25. $\mathrm{Hdim}(S_3^2) = 3$, $\mathrm{Hdim}(S_3^3) = 6$.

Proof. Merging labeling uses 3 and 6 labels for S_3^2 and S_3^3, respectively. Thus

$$\mathrm{Hdim}(S_3^2) \geq 3\,, \quad \mathrm{Hdim}(S_3^3) \geq 6\,.$$

The cycles $C_{012}^{(2)}$ and $C_{012}^{(3)}$ are isometric by Proposition 3.7 and therefore also induced in S_3^2 and S_3^3, respectively. By Lemma 4.17(i), every label on an induced cycle must appear at least twice and since the length of $C_{012}^{(2)}$ is 6 and the length of $C_{012}^{(3)}$ is 12,

$$\mathrm{Hdim}(S_3^2) \leq 3\,, \quad \mathrm{Hdim}(S_3^3) \leq 6\,,$$

so the proposition is proved. $\square$

To determine a better lower bound on $\mathrm{Hdim}(S_3^n)$ we calculate the number of labels of a merging labeling of S_3^n. Let b_n be the number of labels different from 1 that appear on $P_{12}^{(n)}$ exactly once. In other words, b_n is the number of labels of $0S_3^n$ that will be merged with some other label in S_3^{n+1}. (By construction of the merging labeling label 1 will not be merged, since it appears on edges $0e_{12}^{(n-1)}$ and $e_{12}^{(n)}$.) Hence

$$b_n = 2b_{n-1} - 2c_n\,, \tag{4.3}$$

where c_n represents the number of labels that appear twice on $P_{12}^{(n)}$ for the first time. To determine c_n, Lemma 4.24 implies that we only need to find clique edges whose labels appear twice on $P_{12}^{(n)}$ for the first time and, moreover, one edge must be in $1S_3^{n-1}$ and the second one in $2S_3^{n-1}$. By the way merging is defined this can only happen if the first edge is in $1^2 2S_3^{n-3}$ and its label appears on both $1^2 2P_{12}^{(n-2)}$ and $1^2 P_{02}^{(n-2)}$ exactly once. The label of such an edge is then merged with the label of some edge in $102S_3^{n-3}$ that again appears on $10P_{12}^{(n-2)}$ and $10P_{02}^{(n-2)}$ exactly once. The edge in $10P_{02}^{(n-2)}$ is then on $C_{012}^{(n)}$ and its label is merged with the label of an edge in $201S_3^{n-3}$ that appears on $20P_{12}^{(n-2)}$ and $20P_{12}^{(n-2)}$ exactly once by symmetry. Finally, this was merged with a label in $2^2 1S_3^{n-3}$ that again appears only once on $2^2 P_{12}^{(n-2)}$ and $2^2 P_{12}^{(n-2)}$. Looking at Figure 4.9 we infer that $c_4 = 1$ (label 9) and $c_5 = 1$ (label 17).

To determine c_n completely we need to observe clique edges on $1^2 2P_{12}^{(n-3)}$. For this sake we define even and odd clique edges of $P_{12}^{(n)}$. Let $T_1, T_2, \ldots, T_{2^{n-1}}$ be consecutive triangles with edges in $P_{12}^{(n)}$, for example $T_1 = 1^{n-1}S_3^1$ and $T_{2^{n-1}} = 2^{n-1}S_3^1$. (In Figure 4.9, triangle T_1 is labeled with 13, and T_{16} with 22.) Then we say that a clique edge $e \in T_i \cap P_{12}^{(n)}$ is *even* if i is even; otherwise e is *odd*. Note that the label of an odd clique edge from $1^2 2 P_{12}^{(n-3)}$ appears twice on $1^2 P_{02}^{(n-2)}$. Hence it appears twice on $1C_{012}^{(n-1)}$ and is not merged at this step. For this reason we only need to consider even clique edges from $1^2 2 P_{12}^{(n-3)}$. We will show by induction that $c_n = n-4$ for $n \geq 5$. For $n=5$ there is only one such label, namely label 17 (cf. Figure 4.9). For $n > 5$ every even clique edge of $1^2 2^2 P_{12}^{(n-4)}$ in S_3^n has this property as well as the even clique edge of $T_{3 \cdot 2^{n-5}}$. Hence $c_n = n-4$ for $n \geq 5$.

By inserting the obtained outcome for c_n into (4.3) we get

$$b_n = 2b_{n-1} - 2n + 8 \quad \text{for } n \geq 6\,, \quad \text{and} \quad b_5 = 10\,,$$

which yields

$$b_n = 2^{n-3} + 2n - 4,\ n \geq 5\,.$$

Actually, this formula holds also for $n = 4$.

Let finally a_n, for $n \geq 4$, be the number of labels in a merging labeling of S_3^n. Then

$$\begin{aligned} a_n &= 3a_{n-1} - \frac{3}{2} b_{n-1} \\ &= 3a_{n-1} - \frac{3}{2}(2^{n-4} + 2n - 6)\,, \quad a_4 = 12\,, \end{aligned}$$

since we merge six parts into three in pairs.

Clearly $\mathrm{Hdim}(S_3^n) \geq a_n$, so the solution of the recurrence gives us:

Theorem 4.26. *If $n \in \mathbb{N}$ and $n \geq 4$, then*

$$\mathrm{Hdim}(S_3^n) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2} n - \frac{9}{4}\,.$$

4.3.3 An upper bound on Hdim(S_p^n)

Finally, let us prove an upper bound on the Hamming dimension of S_p^n (for $p \geq 3$). We first establish an exact value for $n = 2$.

Proposition 4.27. *If $p \in \mathbb{N}$ and $p \geq 4$,* $\mathrm{Hdim}(S_p^2) = 2$.

Proof. Let $p \in \mathbb{N}$, $p \geq 4$. We claim that the $(1|2)$-labeling of S_p^2 yields the unique induced embedding of S_p^2 into a Hamming graph and this would imply $\mathrm{Hdim}(S_p^2) = 2$.

Since S_p^2 is not a complete graph we need at least two labels. By Condition A, all edges of iS_p^1, $i \in [p]_0$, must receive the same label. By Condition B, every edge $e_{ij}^{(2)}$, for $j \neq i$, must have a different label from the labels of iS_p^1 and jS_p^1. If all subgraphs isomorphic to S_p^1 have the same label, then all the non-clique edges of any cycle $C_{ijk}^{(2)}$ must have the same label, for otherwise one label appears only once on $C_{ijk}^{(2)}$. Since i, j, and k are arbitrary (but pairwise distinct), we obtain the $(1|2)$-labeling.

Suppose next that two subgraphs isomorphic to S_p^1 are labeled with 1 and that among the others there is at least one labeled with 2. We may choose the notation so that $0S_p^1$ and $1S_p^1$ have label 1 and $2S_p^1$ label 2. Then by Condition B the edges $e_{01}^{(2)}$, $e_{02}^{(2)}$, and $e_{12}^{(2)}$ cannot have label 1. Moreover, $e_{02}^{(2)}$ and $e_{12}^{(2)}$ cannot have label 2 for the same reason. But then $e_{01}^{(2)}$ must have label 2, for otherwise we have a contradiction with Condition B in $C_{012}^{(2)}$. Now consider vertices 02 and 12 to find the final contradiction with Condition B.

Assume finally that all the iS_p^1, $i \in [p]_0$, have different labels, say iS_p^1 has label i. To satisfy Condition B, the edge $e_{01}^{(2)}$ of $C_{012}^{(2)}$ must have label 2, $e_{02}^{(2)}$ label 1, and $e_{12}^{(2)}$ label 0. By the same argument applied to $C_{013}^{(2)}$, the edge $e_{01}^{(2)}$ must have label 3, a final contradiction. □

We are able to derive an upper bound on $\mathrm{Hdim}(S_p^n)$ simply by using the recursive construction of Sierpiński graphs. It is obvious that

$$\mathrm{Hdim}(S_p^n) \leq p \cdot \mathrm{Hdim}(S_p^{n-1}), \quad n \geq 3 .$$

With the initial conditions from Propositions 4.25 and 4.27, we get

$$\mathrm{Hdim}(S_p^n) \leq 2 \cdot p^{n-2}, \qquad \text{and} \qquad \mathrm{Hdim}(S_3^n) \leq 3^{n-1} .$$

But with a bit more work we can further improve this upper bound:

Theorem 4.28.

$$(i) \quad \mathrm{Hdim}(S_3^n) \leq 5 \cdot 3^{n-3} + 1 \quad (n \geq 3) .$$

$$(ii) \quad \mathrm{Hdim}(S_p^n) \leq \frac{2}{p-1} p^{n-2} + \frac{2p-4}{p-1} \quad (p \geq 4 \textit{ and } n \geq 2) .$$

Proof. Labels that appear in more than one subgraph iS_p^{n-1}, $i \in [p]_0$, of S_p^n will be called *common labels*.

For a fixed p and $n \geq 3$ let us examine a labeling of S_p^n that fulfills Conditions A and B and uses $\mathrm{Hdim}(S_p^n)$ labels. We know that such a labeling exists, for instance, the $(1|2)$-labeling fulfills the Conditions. Further on, iS_p^{n-1}, $i \in [p]_0$, is isomorphic to S_p^{n-1}, so the fixed labeling has at most $\mathrm{Hdim}(S_p^{n-1})$ different labels in each such subgraph. In addition, by Condition B, there must be at least two labels in each iS_p^{n-1} that appear also in $S_p^n \setminus iS_p^{n-1}$. To see this, consider two inner almost-extreme vertices ij^{n-1} and ik^{n-1} for pairwise distinct $i, j, k \in [p]_0$ and the two induced ij^{n-1}, ik^{n-1}-paths in $C_{ijk}^{(n)}$. Then two labels of $iP_{jk}^{(n-1)}$ must also appear on the other induced ij^{n-1}, ik^{n-1}-path (in $C_{ijk}^{(n)}$). Consequently we get

$$\mathrm{Hdim}(S_p^n) \leq p(\mathrm{Hdim}(S_p^{n-1}) - 2) + \alpha_n ,$$

where α_n denotes the maximum number of common labels in the labeling under consideration.

Setting

$$a_n = p(a_{n-1} - 2) + \alpha_n \,,$$

we have $\mathrm{Hdim}(S_p^n) \leq a_n$ for the same initial conditions. By Propositions 4.25 and 4.27, the initial conditions are $\mathrm{Hdim}(S_3^3) = 6$ and $\mathrm{Hdim}(S_p^2) = 2$, for $p \geq 4$.

We now derive α_n. Consider iS_p^{n-1} and $C_{ijk}^{(n)}$. As before, take the inner almost-extreme vertices ij^{n-1} and ik^{n-1} for pairwise distinct $i, j, k \in [p]_0$. Applying Condition B to the cycle C_{ijk} shows that we need (at least) two labels of iS_p^{n-1} on the other part of $C_{ijk}^{(n)}$. Hence for every $i \in [p]_0$ there are at most $a_{n-1} - 2$ labels that appear only in iS_p^{n-1}. First we assume that the maximum number of labels is attained when we have $a_{n-1} - 2$ different labels in every iS_p^{n-1}. Moreover, these two labels cannot be on $e_{ij}^{(n)}$ or $e_{ik}^{(n)}$, for otherwise we can include these two edges and consider the vertices ji^{n-1} and ki^{n-1}. Thus we have 6 positions on $C_{ijk}^{(n)}$ for new labels in iS_p^{n-1} (new in the sense that the labels did not appear in iS_p^{n-1} before merging), jS_p^{n-1}, and kS_p^{n-1}, and additional 3 edges $e_{ij}^{(n)}$, $e_{ik}^{(n)}$ and $e_{jk}^{(n)}$ – altogether 9 positions. By the above argument, each position in iS_p^{n-1}, jS_p^{n-1}, and kS_p^{n-1} can contain more than one edge but all such edges can be viewed just as one. But then in $C_{ijk}^{(n)}$ we may have at most $4 = \lfloor \frac{9}{2} \rfloor$ common labels.

Suppose now that we can use 5 common labels. First we consider a longer path P_{ijk} between ik^{n-1} and jk^{n-1} in C_{ijk} for arbitrary pairwise distinct $i, j, k \in [p]_0$. If every C_{ijk} contains at most two common labels, then P_{ijk} clearly contains both labels. But then $P_{ij\ell} = P_{ijk}$ for every $\ell \notin \{i, j, k\}$ and every $C_{ij\ell}$ contains these two labels. This is a contradiction since we have used 5 common labels. Next suppose that every C_{ijk} contains at most 3 common labels. If P_{ikj} contains only two of these labels, then both P_{ijk} and P_{jki} contain all three of them. Again $P_{ij\ell} = P_{ijk}$ for every $\ell \notin \{i, j, k\}$ and every $C_{ij\ell}$ contains these three labels – a contradiction. Next suppose that C_{ijk} contains 4 common labels. If P_{ijk} contains only three common labels, we have only four positions in $C_{ijk} - P_{ijk}$ and one label, say 4, is present only on $C_{ijk} \setminus P_{ijk}$. By the above, both $e_{ik}^{(n)}$ and $e_{jk}^{(n)}$ must have label 4. The label of $e_{ij}^{(n)}$, say 3, must be in pS_p^{n-1} together with a common label 2. Label 2 must also be in one of iS_p^{n-1} or jS_p^{n-1}. We may assume that it is in iS_p^{n-1} (together with label 1). Hence P_{ikj} contains 4 common labels. If label 5 exists in ℓS_p^{n-1}, $\ell \notin \{i, j, k\}$, then $C_{ik\ell}$ contains 5 common labels which is not possible.

Let $e_{h\ell}^{(n)}$ have label 5. If $h \in \{i, k\}$ (or by symmetry $\ell \in \{i, k\}$) then $C_{ik\ell}$ (or C_{ikh}) contains 5 common labels again. If finally $h, \ell \notin \{i, j, k\}$, either $e_{hi}^{(n)}$ or $e_{\ell i}^{(n)}$ has label 5, which is not possible. Thus $\alpha_n \leq 4$, hence

$$a_n \leq p(a_{n-1} - 2) + 4 \,, \quad a_3 = 4 \,.$$

Solving the recurrence yields the result. □

A direct consequence of the above theorem gives us another exact value on Hamming dimension.

Corollary 4.29. *If $p \in \mathbb{N}$ and $p \geq 4$, then* $\mathrm{Hdim}(S_p^3) = 4$.

Proof. By Theorem 4.28, $\mathrm{Hdim}(S_p^3) \leq 4$. A 4-labeling of S_p^3 that satisfies Conditions A and B can be constructed as follows. Use the $(1|2)$-, $(2|3)$-, $(3|4)$-, and $(4|1)$-labelings on $0S_p^2$, $1S_p^2$, $2S_p^2$, and $3S_p^2$, respectively. Label the edges $e_{01}^{(3)}$, $e_{12}^{(3)}$, $e_{23}^{(3)}$, and $e_{03}^{(3)}$ with 4, 1, 2, and 3, respectively. Next, we may choose labels 2 or 4 for the edge $e_{02}^{(3)}$ and labels 1 or 3 for the edge $e_{13}^{(3)}$. Finally, for every $i \in [p]_0 \setminus [4]_0$ use the $(1|3)$-labeling on iS_p^2, label edges $e_{i0}^{(3)}$ and $e_{i1}^{(3)}$ with 4, edges $e_{i2}^{(3)}$ and $e_{i3}^{(3)}$ with 2, and all the other edges $e_{ij}^{(3)}$, $j \in [p]_0 \setminus [4]_0$, $i \neq j$, with 2. For this labeling, Condition A clearly holds. Moreover, a direct check of labels on cycles $C_{ijk}^{(3)}$ shows that Condition B is fulfilled as well. $\square$

Note that in Theorem 4.28 equality holds for S_p^2 and S_p^3, $p \geq 4$. The upper bound (ii) is also exact for S_4^4. Indeed, the bound is 12 and two different appropriate labelings of S_4^4 are shown in Figure 4.10.

We have already proven that $\mathrm{Hdim}(S_p^3) = 4$ for $p \geq 4$. Actually, we are able to find an induced embedding of S_p^3, $p \geq 4$, into the 2-, 3-, or 4-dimensional Hamming graphs.

Proposition 4.30. *If $p \in \mathbb{N}$ and $p \geq 4$, then there exists an induced embedding of S_p^3 into a Hamming graph with τ factors if and only if $2 \leq \tau \leq 4$.*

This is clear because the $(1|2)$-labeling and the Sierpiński triangle labeling of S_p^3 give induced embeddings into a Hamming graph with 2 and 3 factors, respectively. While the $(1|2)$-labeling of S_p^3 is unique, the 4-labeling from the proof of Theorem 4.28 is not. Namely, if we change the labeling $(2|3)$ of $1S_p^2$ into $(3|2)$ ($e_{13}^{(3)}$ must have label 1 in this case), we obtain a labeling that still satisfies Conditions A and B, but gives a different embedding.

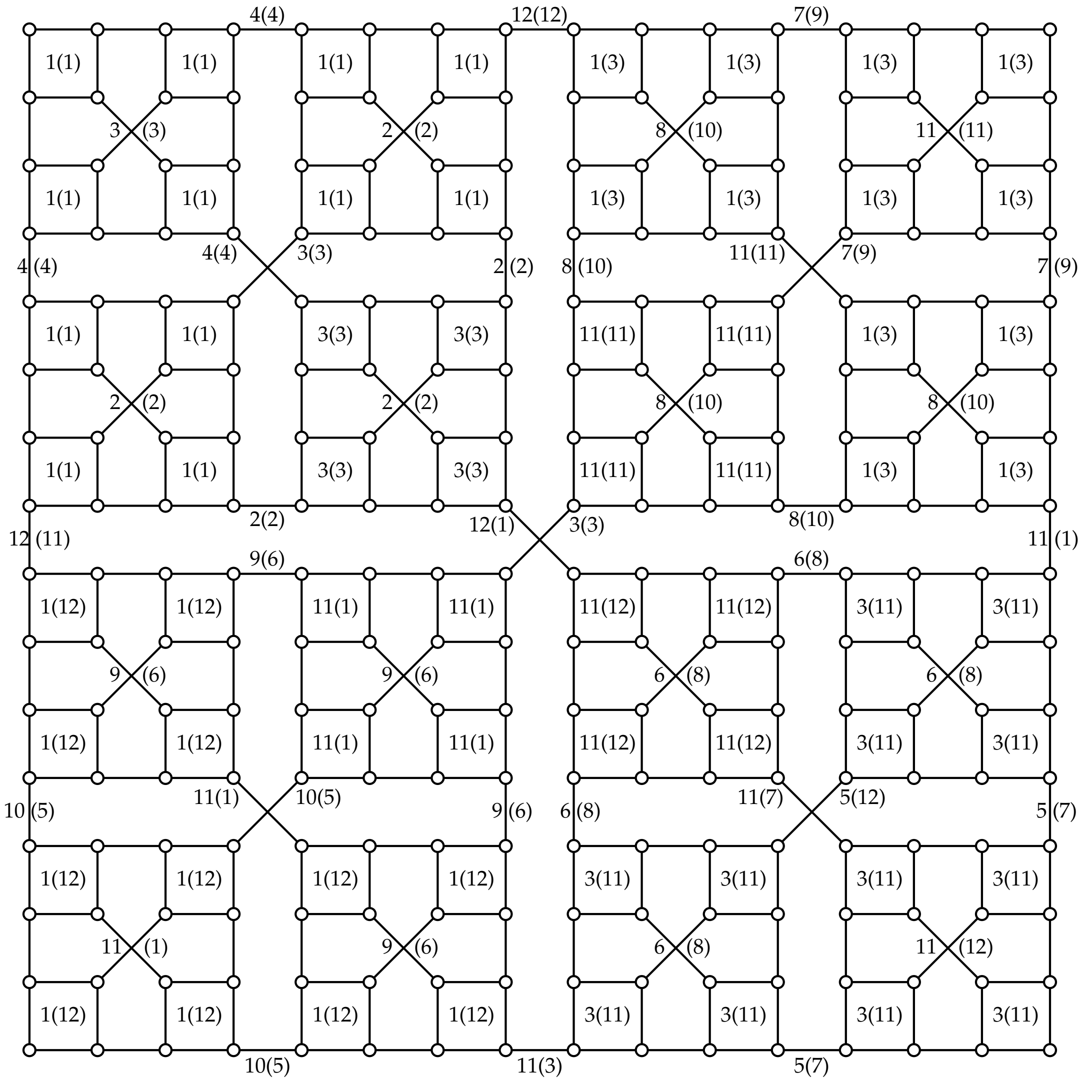


Figure 4.10: Two labelings of S_4^4

Chapter 5

Future research topics

During our research we came across some problems that remain to be solved. The most interesting we will present in this chapter.

In Section 3.3 we determined the metric dimension of Sierpiński graphs. Later on we studied some other dimensions related to metric properties. Among them was the Wiener dimension of a graph, introduced in [1]. Suppose that $\{d_G(u) \mid u \in V(G)\} = \{\delta_1, \ldots, \delta_k\}$. Then the *Wiener dimension*, $\dim_W(G)$, of G is k. In other words, the Wiener dimension of G is the number of different (total) distances of vertices of G. For some initial cases of Sierpiński graphs one may easily derive their Wiener dimensions with the help of a computer. The obtained values are presented in the table below.

$p \setminus n$	2	3	4	5	6	7	8	9	10
2	2	4	8	16	32	64	128	256	512
3	2	4	13	40	120	356	1084	3268	9832
4	2	5	15	50	187	715	2793	?	?
5	2	5	15	52	201	854	?	?	?
6	2	5	15	52	203	?	?	?	?
7	2	5	15	52	203	?	?	?	?

These results suggest the following proposition.

Proposition 5.1. *If $n, p \in \mathbb{N}$ and $p \geq 2$, then*

$$\dim_W(S_p^2) = 2 \quad \textit{and} \quad \dim_W(S_2^n) = 2^{n-1}\,.$$

Theorem 2.28 applied to $n = 2$ implies that the vertices of S_p^2 form only two orbits: one consists of all extreme vertices and the other one of all almost-extreme vertices. This gives us an upper bound $\dim_W(S_p^2) \leq 2$ for $p \geq 2$. It is also not hard to see that $d(i^2) \neq d(ij)$ for $i \neq j$.

Indeed,

$$d_{S_p^2}(ij) = (p-1) + (2p-1) + (p-2)\cdot(2p + d_{S_p^1}(i)) = p(3p-4)\,,$$

and

$$d_{S_p^2}(i^2) = (p-1) + (p-1)\cdot(2p + d_{S_p^1}(i)) = p(3p-3) < p(3p-4) = d_{S_p^2}(ij)\,.$$

So the first assertion of Proposition 5.1 holds. The second assertion follows directly from the fact that $S_2^n \cong P_{2^n}$.

However, the problem to determine the Wiener dimension of a general Sierpiński graph S_p^n still remains open:

Problem 5.2. *Let $n, p \in \mathbb{N}$ and $n, p \geq 3$. Determine the Wiener dimension of the Sierpiński graph S_p^n.*

In Section 4.1 we have considered embeddings of Sierpiński graphs into Hanoi graphs. We concluded the section with Corollary 4.6, which says that for all odd values of p, Sierpiński graphs S_p^n are spanning subgraphs of the Cartesian product of complete graphs K_p^n. Since this was a direct consequence of Theorem 4.5, we did not consider the cases when p is even. However, this result can probably be proven for any value of p. Consider for example the case $p \geq 2$ and $n = 2$. Then the embedding is given by

$$\iota_2 : \; S_p^2 \to K_p^2\,, \quad \iota_2(ij) = i(i+j) \in [p]_0^2\,.$$

According to this we state the following conjecture:

Conjecture 5.3. *If $p \in \mathbb{N}$, then for any $n \in \mathbb{N}_0$, S_p^n is a spanning subgraph of the Hamming graph K_p^n.*

While studying the Hamming dimension of Sierpiński graphs, we wanted to improve the lower bound $\mathrm{Hdim}(S_p^n) \geq n$ for arbitrary p, but the construction of the merging labeling (Section 4.3.2) was developed only for $p = 3$. It would be interesting to do something similar for arbitrary p and generalize this approach.

Another interesting topic which arose when we were working on the classification of Sierpiński-type graphs is regularization of Sierpiński triangle graphs. All but extreme vertices have degree $2p-2$, the extreme vertices however are of degree $p-1$. Thus a regularization with an additional one-vertex graph (see Definition 1.7 for the corresponding regularization of S_p^n) would not make sense, but an analogue to Definition 1.9 would give us another family of Sierpiński-like graphs. It would be interesting to investigate this family of graphs since some other ideas related to Sierpiński and Sierpiński triangle graphs might come forward.

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Daljši slovenski povzetek

Grafi tipa Sierpińskega igrajo pomembno vlogo v teoriji grafov kot tudi v drugih vejah matematike. Vsekakor niso pomembni samo v matematiki, saj se pojavljajo tudi v fiziki, psihologiji in verjetno še kje. Vpeljala sta jih Klavžar in Milutinović leta 1997 [40] iz dveh razlogov. Prvi so bile študije topoloških Lipscombovih prostorov (ki so lepo prikazane v [51]), drugi igra Hanojskega stolpa. Slednji je za nas še najbolj zanimiv, saj graf Sierpińskega S_p^n predstavlja različico prvotne igre Hanojskega stolpa, imenovano *zamenjevalni Hanojski stolp*.

Igra Hanojskega stolpa je sestavljena iz treh palic in n diskov, ki so po velikosti urejeni na eni izmed palic. Cilj igre je prestaviti stolp diskov iz ene palice na drugo, tako da pri tem upoštevamo *božansko pravilo*, ki zapoveduje, da ne smemo postaviti večjega diska na manjši disk. Večji izziv predstavlja razširitev igre na p palic. Prvič se takšna razširitev originalnega problema pojavi že leta 1908 v Dudeneyjevi knjigi [10], bolj podrobno pa sta se problema lotila Frame [11] in Stewart [60], ki sta leta 1941 vsak zase objavila domnevni optimalni rešitvi za najmanjše število potez. Optimalnost njunih rešitev, znana pod imenom *Frame-Stewartova domneva*, še dandanes ni dokazana.

Pri zamenjevalnem Hanojskem stolpu imamo na voljo p palic in n diskov. Božansko pravilo priredimo tako, da lahko v eni potezi premaknemo najmanjši disk ali pa, če imamo na vrhu ene izmed palic sestavljen podstolp najmanjših $\delta - 1$ diskov (torej diskov $1, \ldots, \delta - 1$), zamenjamo disk δ, ki leži na neki drugi palici, s celotnim podstolpom diskov $1, \ldots, \delta - 1$.

Ravno ta povezava oziroma podobnost grafov Sierpińskega z igro Hanojskega stolpa je eden poglavitnih razlogov, zakaj preučujemo njihove metrične lastnosti. To pa niso edine lastnosti, ki so jih preučevali na grafih Sierpińskega. Znanih je mnogo rezultatov, ki smo jih povzeli v poglavju 2 in jih sedaj ne bomo posebej obravnavali.

V nadaljevanju se bomo najprej osredotočili na klasifikacijo grafov tipa Sierpińskega. Nato si bomo ogledali nove rezultate o razdaljah v grafih Sierpińskega in zatem še njihove vložitve v različne grafe. Za konec bomo predstavili zanimiv odprt problem, na katerega smo naleteli med raziskovanjem različnih dimenzij grafov Sierpińskega.

Uvod in klasifikacija

V tej doktorski disertaciji predpostavljamo, da so vsi grafi enostavni in povezani. Množico prvih n naravnih števil, $\{1,\ldots,n\}$, označujemo z $[n]$ in podobno $[n]_0 := \{0,\ldots,n-1\}$. Kadar je $n=2$ oziroma $n=3$, govorimo o binarnih oziroma ternarnih številih. Te množice označimo z $B := [2]_0 = \{0,1\}$ in s $T := [3]_0 = \{0,1,2\}$. *Iversonov oklepaj* predstavlja pretvorbo logičnih vrednosti v vrednosti 0 ali 1, in sicer

$$[X] = \begin{cases} 1\,, & \text{če je } X \text{ resnična izjava,} \\ 0\,, & \text{če } X \text{ ni resnična izjava.} \end{cases}$$

Graf Sierpińskega S_p^n je graf na množici vozlišč $[p]_0^n = \{0,\ldots,p-1\}^n$, kjer sta vozlišči $s = s_n \ldots s_1$ in $t = t_n \ldots t_1$ sosednji, če sta oblike $s = \underline{s} s_\delta t_\delta^{\delta-1}$, $t = \underline{s} t_\delta s_\delta^{\delta-1}$ za $\delta \in [n]$, $\underline{s} \in [p]_0^{n-\delta}$ in $s_\delta \neq t_\delta$. Povezava $\{\underline{s} s_\delta t_\delta^{\delta-1}, \underline{s} t_\delta s_\delta^{\delta-1}\}$ predstavlja potezo pri zamenjevalnem Hanojskem stolpu. Zamenjamo namreč disk δ, ki je na palici s_δ, s podstolpom diskov $1,\ldots,\delta-1$, ki so na t_δ.

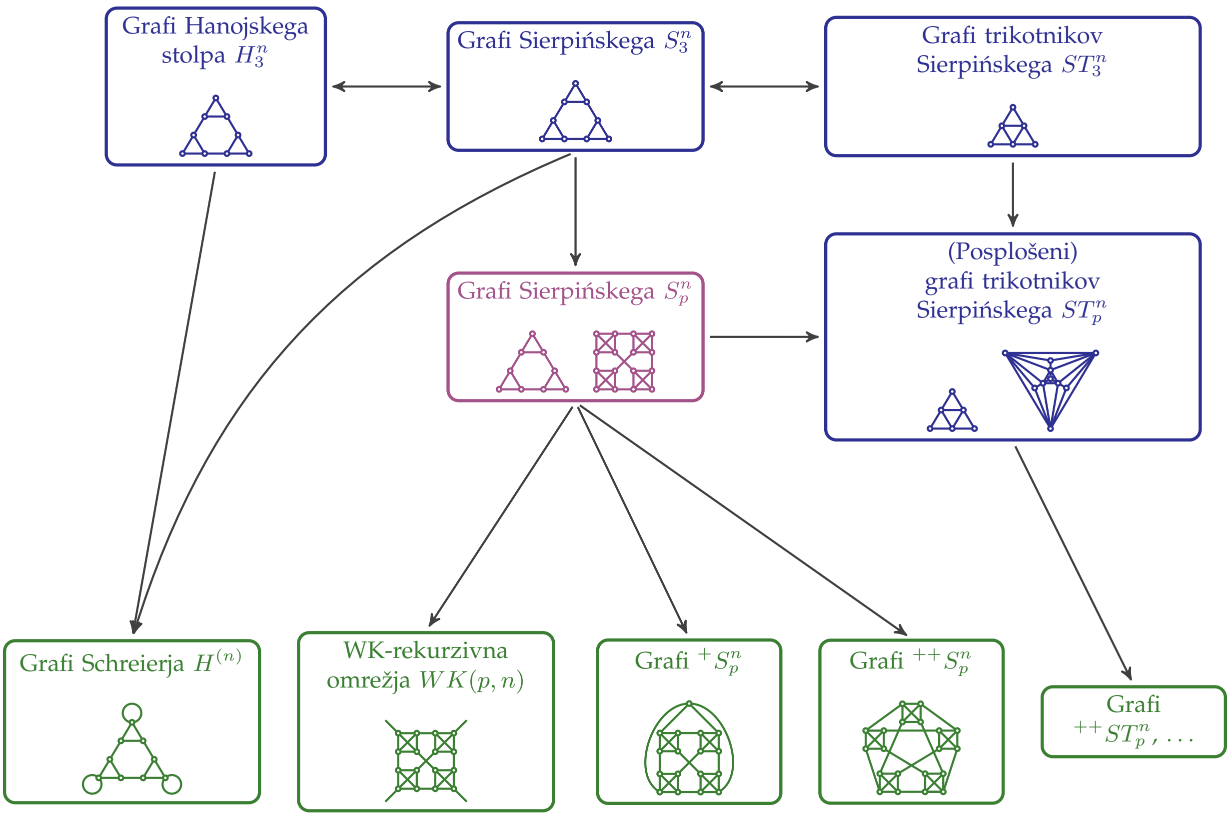
Vozlišča oblike $i \ldots i = i^n$ imenujemo *ekstremna vozlišča* grafa S_p^n. Kasneje bomo videli, da je pot med poljubnima ekstremnima vozliščema i^n in j^n enolična, označimo jo s $P_{ij}^{(n)}$. Strukturo grafov Sierpińskega lahko opišemo tudi rekurzivno. Začnemo z enim vozliščem ($= S_p^0$), naredimo p kopij, ki jih povežemo v polni graf. To lahko ponovimo, ko gradimo graf S_p^n, le da v tem primeru vzamemo p kopij grafa S_p^{n-1}. Tako lahko množico povezav grafov Sierpińskega zapišemo tudi rekurzivno:

$$\begin{aligned} E(S_p^0) &= \emptyset\,, \\ E(S_p^n) &= \{\{is, it\} \mid i \in [p]_0\,, \{s,t\} \in E(S_p^{n-1})\} \cup \\ &\qquad \{\{ij^{n-1}, ji^{n-1}\} \mid i,j \in [p]_0\,, i \neq j\}\,, \quad n \in \mathbb{N}\,. \end{aligned}$$

Podgraf grafa S_p^n, katerega vozlišča imajo skupno predpono $\underline{s} \in [p]_0^{n-\delta}$, $\delta \in [n+1]_0$, je izomorfen grafu S_p^δ in ga označimo z $\underline{s}S_p^\delta$. Povezava med podgrafoma iS_p^{n-1} in jS_p^{n-1} je enolična in jo označujemo z $e_{ij}^{(n)}$. Njeni krajišči sta vozlišči ij^{n-1} in ji^{n-1}. Vsa takšna vozlišča imenujemo *notranja skoraj ekstremna vozlišča*. Analogno vpeljemo *zunanja skoraj ekstremna vozlišča* kot sosednja vozlišča ekstremnih vozlišč. Primer teh vozlišč si lahko ogledamo na sliki 1.2, kjer so ekstremna vozlišča grafa S_5^3 obarvana sivo, zunanja skoraj ekstremna vozlišča rdeče in notranja skoraj ekstremna vozlišča zeleno.

Pri vložitvah, ki jih bomo obravnavali kasneje, se bomo pogosto sklicevali na izometrične cikle $C_{ij\ell}$, kjer so $i,j,\ell \in [p]_0$ paroma disjunktni. Ti cikli so sestavljeni iz poti $iP_{j\ell}$, $jP_{i\ell}$ in ℓP_{ij} ter povezav $e_{ij}^{(n)}$, $e_{j\ell}^{(n)}$ in $e_{i\ell}^{(n)}$. Potrebovali bomo še razdelitev povezav v grafu Sierpińskega. Vse povezave, ki so vsebovane v eni izmed p-klik $\underline{s}S_p^1$, bomo imenovali *klične povezave*, preostale pa *nekične povezave*. Za $p=2$ so sicer vse povezave vsebovane v 2-klikah, ampak niso vse 2-klike oblike $\underline{s}S_2^1$.

Eden pomembnejših rezultatov doktorske disertacije je klasifikacija *grafov tipa Sierpińskega*, ki je prikazana na diagramu spodaj. Na vrhu diagrama lahko najdemo družine grafov Hanojskega stolpa H_3^n, grafov Sierpińskega S_3^n in grafov trikotnikov Sierpińskega ST_3^n. Ti predstavljajo izvor (splošnih) grafov Sierpińskega, ki jih najdemo v sredini diagrama. Spodnja vrsta diagrama predstavlja grafe, ki so podobni grafom Sierpińskega in jih izpeljemo iz njih kot regularizacije (grafi ${}^{+}S_p^n$ in ${}^{++}S_p^n$) ali pa so bili neodvisno vpeljani (grafi $H^{(n)}$ in $WK(p,n)$). Grafi ${}^{++}ST_p^n$, ki se nahajajo skrajno desno spodaj v diagramu, še niso bili vpeljani in so ena izmed motivacij za prihodnje raziskovanje. Grafe trikotnikov Sierpińskega lahko regulariziramo na podoben način kot grafe Sierpińskega. Zanimivo bi bilo pogledati nekatere lastnosti teh grafov.



Metrične lastnosti

Oglejmo si nekatere osnovne definicije, potrebne za obravnavo metričnih lastnosti, in že znane rezultate s tega področja za grafe Sierpińskega.

Razdalja med dvema vozliščema u in v grafa G je dolžina najkrajše u,v-poti in jo označujemo z $d_G(u,v)$. Manj znana je *(celotna) razdalja* vozlišča u grafa G, ki je definirana kot vsota vseh razdalj do u:

$$d_G(u) = \sum_{v \in V(G)} d_G(u,v)\,.$$

Za začetne primere grafov Sierpińskega ni težko določiti razdalje med poljubnima vozliščema, saj so izomorfni polnim grafom. Že leta 1997, ko sta Klavžar in Milutinović vpeljala družino grafov Sierpińskega [40], sta obravnavala razdalje v teh grafih. Podala sta ključno lemo z eksplicitno formulo za izračun razdalje od poljubnega do ekstremnega vozlišča v grafu S_p^n:

Lema 1. [Lemma 3.1] *Če je $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem za poljuben $j \in [p]_0$ in poljubno vozlišče $s = s_n \dots s_1$ grafa S_p^n velja*

$$d(s, j^n) = \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1},$$

najkrajša pot med s in j^n pa je enolična. V posebnem primeru, razdalja med poljubnima različnima ekstremnima vozliščema i^n in j^n znaša $2^n - 1$.

Ena izmed posledic leme 1 je določitev premera grafov S_p^n, kar je dokazal Parisse v svojem članku o metričnih lastnostih grafov Sierpińskega [52].

Trditev 2. [Proposition 3.4] *Če sta $n \in \mathbb{N}_0$, $p \in \mathbb{N}$ in $p \geq 2$, potem je premer grafa S_p^n enak $2^n - 1$.*

Prav tako iz leme 1 sledi, da razdalja med poljubnima vozliščema grafa Sierpińskega ni odvisna od skupne predpone teh vozlišč. Natančneje:

Posledica 3. [Corollary 3.3] *Če je $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem za poljubni vozlišči js in jt grafa S_p^{n+1} velja*

$$d_{S_p^{n+1}}(js, jt) = d_{S_p^n}(s,t)\,.$$

Sedaj ko vemo, da so najkrajše poti do ekstremnih vozlišč enolične, lahko uporabimo to dejstvo skupaj z rekurzivno strukturo grafov Sierpińskega za iskanje vseh možnih kandidatk za najkrajšo pot med poljubnima vozliščema grafa S_p^n. Obstaja natanko $p-1$ takšnih kandidatk:

Definicija 4. *Naj bosta $n, p \in \mathbb{N}$ in naj bosta $i, j \in [p]_0$ različna. Nadalje naj bosta $s = \underline{s}i\overline{s}$ in $t = \underline{s}j\overline{t}$ vozlišči grafa S_p^n, za kateri sta $\overline{s}, \overline{t} \in [p]_0^{\delta-1}$ in je $\underline{s} \in [p]_0^{n-\delta}$ za $\delta \in [n]$. Potem definiramo*

$$\begin{aligned} & d_i(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_j(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_{S_p^{n-1}}(\overline{s}, j^{\delta-1}) + 1 + d_{S_p^{n-1}}(\overline{t}, i^{\delta-1})\,, \\ \forall \ell \in [p]_0 \setminus \{i,j\}: \quad & d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d_{S_p^{n-1}}(\overline{s}, \ell^{\delta-1}) + 1 + 2^{\delta-1} + d_{S_p^{n-1}}(\overline{t}, \ell^{\delta-1})\,. \end{aligned}$$

Razdalji $d_i(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ in $d_j(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ imenujemo direktni razdalji *med s in t.*

Navadno bomo uporabljali in navajali le eno od obeh direktnih razdalj, saj sta enaki. Direktni razdalji pripadajočo s,t-pot imenujemo *direktna s,t-pot*. Očitno je najkrajša pot med poljubnim in ekstremnim vozliščem vedno direktna. Na sliki 3.1 si lahko ogledamo graf S_4^4, v katerega smo vrisali poti, ki ustrezajo razdaljam $d_\ell(0231, 2301)$ za $\ell \in [4]_0$. Direktna pot, t. j. pot, ki pripada razdalji $d_0(0231, 2301) = d_2(0231, 2301)$, je obarvana rdeče, pot za $d_1(0231, 2301)$ zeleno in pot za $d_3(0231, 2301)$ modro. Očitno je najkrajša pot med vozliščema 0231 in 2301 direktna pot in $d_{S_4^4}(0231, 2301) = 9$.

Z zgornjo definicijo lahko navedemo rezultat za razdaljo med poljubnima vozliščema grafa Sierpińskega.

Izrek 5. [Theorem 3.6] *Naj bosta $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$. Če sta $s = \underline{s}i\overline{s}$ in $t = \underline{s}j\overline{t}$ vozlišči grafa S_p^n, kjer sta $i, j \in [p]_0$ različna ter $\delta \in [n]$, $\overline{s}, \overline{t} \in [p]_0^{\delta-1}$ in $\underline{s} \in [p]_0^{n-\delta}$, potem velja*

$$d_{S_p^n}(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p]_0\right\} . \tag{5.1}$$

Minimum (5.1) je lahko dosežen kvečjemu pri dveh razdaljah d_ℓ, $\ell \in [p]_0 \setminus \{i\}$, kar pomeni, da sta med poljubnima vozliščema grafa Sierpińskega kvečjemu dve najkrajši poti, (glej [40, Theorem 6] ali alternativni nedavni dokaz tega dejstva [26, Corollary 1.1]). Nadalje velja še, da če imamo dve najkrajši poti med vozliščema, potem je ena izmed njiju direktna pot.

Poglavitni problem splošne formule za razdaljo med poljubnima vozliščema grafa S_p^n je, da je enaka minimumu $p - 1$ vrednosti. Zato težimo k temu, da bi razdaljo izrazili z eksplicitno formulo. Glede na to, da imamo eksplicitno formulo za razdaljo do ekstremnih vozlišč, smo se lotili raziskovanja razdalj do skoraj ekstremnih vozlišč in izpeljali naslednjo formulo za zunanja skoraj ekstremna vozlišča.

Trditev 6. [Proposition 3.14] *Če sta $n, p \in \mathbb{N}$ in je $j^n k$ zunanje skoraj ekstremno vozlišče grafa S_p^{n+1}, potem lahko za poljuben $i \in [p]_0 \setminus \{j\}$ razdaljo med poljubnim vozliščem is grafa S_p^{n+1} in vozliščem $j^n k$ zapišemo kot*

$$d_{S_p^{n+1}}(is, j^n k) = d(s, j^n) + 2^n - [i = k] .$$

S pomočjo dokaza te trditve lahko določimo tudi vsa tista vozlišča grafa S_p^{n+1}, ki imajo dve najkrajši poti do zunanjega skoraj ekstremnega vozlišča jk^n:

Trditev 7. [Proposition 3.16] *Če sta $n, p \in \mathbb{N}$ in je $j^n k$ zunanje skoraj ekstremno vozlišče grafa S_p^{n+1}, potem obstajata dve najkrajši poti med poljubnim vozliščem s grafa S_p^{n+1} in vozliščem $j^n k$ natanko tedaj, ko je $s = j^{n-m} i k^m$ za $m \in [n]$ in $i \in [p]_0 \setminus \{j, k\}$.*

S pomočjo razdalje do zunanjih skoraj ekstremnih vozlišč grafa S_p^n (trditev 6) lahko izračunamo tudi razdaljo zunanjega skoraj ekstremnega vozlišča.

Izrek 8. [Theorem 3.20] *Če sta $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem za različna $j, k \in [p]_0$, velja*

$$d_{S_p^{n+1}}(j^n k) = \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right) p^{n+1} + \frac{p}{p-1} .$$

Za dokaz tega izreka potrebujemo tudi razdalje ekstremnih vozlišč:

Lema 9. [Lemma 3.19] *Če sta $n, p \in \mathbb{N}$, potem za poljuben $i \in [p]_0$, velja*

$$d_{S_p^n}(i^n) = p^{n-1}(p-1)\,(2^n - 1)\ .$$

Podobne rezultate, kot smo jih dokazali za zunanja skoraj ekstremna vozlišča, smo izpeljali tudi za notranja skoraj ekstremna vozlišča. Dokazi so tukaj težavnejši, saj se ta vozlišča nahajajo precej globlje v grafih kot zunanja skoraj ekstremna vozlišča. Slednja so sosednja ekstremnim vozliščem, ki se nahajajo na skrajnem robu grafov Sierpińskega. Kljub temu nam je uspelo izračunati razdaljo od poljubnega do notranjega skoraj ekstremnega vozlišča.

Da bi izrazili to razdaljo z eksplicitno formulo, potrebujemo definiciji za direktna in posebna vozlišča. Vozlišče s je *direktno* za notranje skoraj ekstremno vozlišče jk^n (v S_p^{n+1}), če velja: $s_d = k$ velja natanko tedaj, ko je $d = n+1$, ali ko obstaja $\delta \in [n+1] \setminus [d]$, tako da velja $s_\delta = j$. Podobno je vozlišče s *posebno* za jk^n (v S_p^{n+1}), če obstaja tak $\delta \in [n]$, da je $s = \underline{s}kj^{\delta-1}$, $\underline{s} \in ([p]_0 \setminus \{j,k\})^{n+1-\delta}$. Na sliki 3.4 so označena posebna (oranžna) in direktna (zelena) vozlišča grafa S_6^3 za vozlišče 144. Imena teh vozlišč smo izbrali zato, ker so direktna vozlišča za j^nk natanko tista vozlišča, za katera je direktna pot enolična najkrajša pot do j^nk. Podobno so posebna vozlišča za j^nk natanko tista vozlišča, ki imajo dve najkrajši poti do j^nk. Sedaj lahko navedemo naslednji rezultat o razdalji do notranjega skoraj ekstremnega vozlišča.

Trditev 10. [Proposition 3.26] *Če sta $n, p \in \mathbb{N}$ in je jk^n notranje skoraj ekstremno vozlišče grafa S_p^{n+1}, potem lahko za poljuben $i \in [p]_0 \setminus \{j\}$ razdaljo med poljubnim vozliščem is grafa S_p^{n+1} in vozliščem jk^n zapišemo kot*

$$d_{S_p^{n+1}}(is, jk^n) = \begin{cases} d(s, j^n) + 2^n - [i = k](2^n - 1), & \textit{če je } is \textit{ direktno za } jk^n\,, \\ d(s, k^n) + 2^n + 1, & \textit{sicer}\,. \end{cases}$$

S pomočjo tega rezultata lahko podobno kot prej izpeljemo razdaljo notranjega skoraj ekstremnega vozlišča.

Izrek 11. [Theorem 3.27] *Če sta $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem za poljubna različna $j, k \in [p]_0$ velja*

$$d_{S_p^{n+1}}(jk^n) = \frac{p^2-2}{p(p+2)}(2p)^{n+1} - \frac{p-2}{2p}p^{n+1} - \frac{p}{2(p+2)}(p-2)^{n+1}\ .$$

K osnovnim metričnim lastnostim sodi tudi metrična dimenzija grafa. Ta je bila vpeljana v letih 1974–1975. Neodvisno so jo vpeljali Harary in Melter [22] ter Slater [59]. Pred nekaj leti sta Bailey in Cameron objavila članek [2], kjer lahko najdemo podrobno zgodovino razvoja metrične dimenzije, prav tako pa tudi povezave te dimenzije z drugimi grafovskimi invariantami. Drugi izčrpen pregledni članek na to temo sta napisala Goddard in Oellermann [14].

Preden se lotimo metrične dimenzije grafov Sierpińskega, si oglejmo potrebne osnovne definicije. Podmnožica vozlišč $R = \{u_1, \ldots, u_k\} \subseteq V(G)$, $k \in \mathbb{N}$, je *resolventna množica* (grafa

G), če za poljubni različni vozlišči x, y grafa G velja

$$\big(d(x,u_1),\ldots,d(x,u_k)\big) \neq \big(d(y,u_1),\ldots,d(y,u_k)\big)\,.$$

Metrična dimenzija grafa G, $\mu(G)$, je velikost najmanjše resolventne množice. Za grafe Sierpińskega smo dokazali naslednji izrek.

Izrek 12. [Theorem 3.33] *Če sta $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem je*

$$\mu(S_p^{n+1}) = p - 1\,.$$

Še več, če je R najmanjša resolventna množica, potem za poljuben $j \in [p]_0$ velja $|R \cap V(jS_p^n)| \leq 1$.

Drugi del izreka z drugimi besedami pove, da je v vsakem podgrafu jS_p^n grafa S_p^{n+1} kvečjemu eno vozlišče neke minimalne resolventne množice.

Dokaz tega izreka poteka konstruktivno. Najprej pokažemo, da je množica ekstremnih vozlišč grafa S_p^{n+1} resolventna. Zatem dokažemo, da ta množica ostane resolventna tudi, če iz nje odstranimo poljubno ekstremno vozlišče. Torej je množica

$$R_{p-1}^{n+1} := \{i^{n+1} \mid i \in [p-1]_0\}$$

resolventna za graf S_p^{n+1}. Za ugotovitev, da je R_{p-1}^{n+1} tudi minimalna resolventna množica, potrebujemo še direktno posledico trditve 6:

Posledica 13. [Corollary 3.32] *Če sta $n \in \mathbb{N}_0$ in $p \in \mathbb{N}$, potem za poljubne paroma disjunktne $i, j, k \in [p]_0$ in za $s \in [p]_0^n$ velja*

$$d_{S_p^{n+1}}(is, j^n k) = d_{S_p^{n+1}}(is, j^{n+1})\,.$$

Vložitve

Pri obravnavanju vložitev bomo potrebovali nekatere definicije. Teoretično ozadje vložitev lahko najdemo v knjigah [33] in [21]. Pogosto bomo obravnavali vložitve v grafovske produkte. *Kartezični produkt* grafov G in H, $G \,\square\, H$, je graf, definiran kot

$$\begin{aligned} &V(G \,\square\, H) = V(G) \times V(H)\,,\\ &E(G \,\square\, H) = \{\{(g,h),(g',h')\} \mid g = g', \{h,h'\} \in E(H) \text{ ali } \{g,g'\} \in E(G), h = h'\}\,. \end{aligned}$$

Hammingovi grafi so definirani kot kartezični produkti polnih grafov. Hammingov graf z n faktorji, izomorfnimi polnemu grafu K_p, označujemo s K_p^n. *Vložitev* grafa G v graf H je injektivni homomorfizem, t. j. injektivna preslikava $f : V(G) \to V(H)$, za katero velja: če je $\{u,v\}$ povezava grafa G, potem je $\{f(u), f(v)\}$ prav tako povezava grafa H. *Slika* $f(G)$ grafa G glede na vložitev f je graf, definiran kot $V(f(G)) = f(V(G))$ in $E(f(G)) = \{\{f(u), f(v)\} \mid \{u,v\} \in$

$E(G)\}$. Pripomnimo, da ni nujno vsaka praslika povezave grafa H s krajišči v množici $f(V(G))$ tudi povezava v $f(G)$. *Izometrična vložitev* je vložitev, ki ohranja razdalje. Vložitev $G \to H$ je *inducirana*, če je slika grafa G induciran podgraf grafa H. Očitno je vsaka izometrična vložitev tudi inducirana, obrat ne velja. Na primer P_3 je induciran podgraf grafa C_5, ampak ni izometričen v C_5 (glej sliko 4.1).

Oglejmo si še definicijo kvocientnega grafa. Naj bo $\mathcal{F} = \{F_1, \dots, F_r\}$ particija množice povezav grafa G. Potem je *kvocientni graf* G/F_i, $i \in [r]$, graf, katerega množica vozlišč so povezane komponente grafa $G \setminus F_i$, kjer sta komponenti C_i in C_j sosednji (v G/F_i), če obstaja povezava v grafu G, ki ima eno krajišče v C_i in drugo v C_j.

Vložitev grafa G v kartezični produkt grafov $H = \mathop{\square}\limits_{i=1}^{k} H_i$ je *neredundantna*, če nima nobenih odvečnih vozlišč in neuporabljenih faktorjev. To pomeni, da se vsako vozlišče faktorjev kartezičnega produkta pojavi kot koordinata v sliki vsaj enega vozlišča grafa G in v vsakem faktorju imamo vsaj dve vozlišči. V tem primeru rečemo, da je G *neredundantni* podgraf grafa H.

Najprej si oglejmo vložitve grafov Sierpińskega v grafe Hanojskega stolpa. Vemo, da velja $S_3^n \cong H_3^n$, zato smo se vprašali, ali je mogoče posplošiti ta rezultat. Glede na grafe S_p^n imajo grafi H_p^n za $p > 3$ precej več povezav med podgrafi, izomorfnimi H_p^{n-1}, zato izomorfizem teh grafov ni več mogoč. Ker so definirani na isti množici vozlišč, smo se vprašali, ali so grafi Sierpińskega podgrafi grafov Hanojskega stolpa. Tudi to ni vedno res:

Izrek 14. [Theorem 4.5] *Če sta $n, p \in \mathbb{N}$, potem lahko graf S_p^n vložimo v graf H_p^n natanko tedaj, ko je p liho število ali $n = 1$.*

Oglejmo si sedaj izometrične vložitve grafov Sierpińskega v kartezične produkte grafov. Klasična teorija Grahama in Winklerja [15] pravi, da je takšna neredundantna vložitev poljubnega grafa G enolična, če zahtevamo, da ima kartezični produkt največje možno število faktorjev. Imenujemo jo kanonična metrična reprezentacija grafa G. Za opis vložitve potrebujemo relacijo Θ: dve povezavi $e = uv$ in $f = xy$ grafa G sta v relaciji Θ natanko tedaj, ko velja

$$d(u,x) + d(v,y) \neq d(u,y) + d(v,x)\,.$$

Relacija Θ je refleksivna in simetrična, ne pa nujno tranzitivna. Da bi dobili ekvivalenčno relacijo, tvorimo tranzitivno ovojnico relacije Θ in jo označimo s Θ^*. Particijo, ki jo dobimo ob delovanju relacije Θ^* na množico povezav grafa G, označimo z $\mathcal{E} = \{E_1, \dots, E_\rho\}$. Potem definiramo *kanonično metrično reprezentacijo* grafa G kot vložitev

$$\begin{aligned} &\alpha : V(G) \to V(G/E_1) \,\square\, \cdots \,\square\, V(G/E_\rho)\,, \\ &\alpha(v) = (\alpha_1(v), \dots, \alpha_\rho(v))\,, \end{aligned}$$

kjer je $\alpha_i : V(G) \to V(G/E_i)$ naravna projekcija, ki preslika $v \in V(G)$ v povezano komponento grafa G/E_i, v kateri se v nahaja. Kanonična metrična reprezentacija je *trivialna*, če je $\rho = 1$. To pomeni, da povezave grafa G tvorijo en sam Θ^*-razred in imajo posledično tudi en sam faktor v vložitvi.

Za večino grafov Sierpińskega je kanonična metrična reprezentacija trivialna:

Trditev 15. [Proposition 4.12] *Če je $p \in \mathbb{N}$ in $p \geq 4$, potem je za poljuben $n \in \mathbb{N}$ kanonična metrična reprezentacija grafa S_p^n trivialna.*

Preostaneta nam samo dve možnosti za netrivialno kanonično metrično reprezentacijo, namreč $p = 2$ in $p = 3$. Graf S_2^n je izomorfen poti na 2^n vozliščih. Za poti (in splošneje drevesa) je znano, da vsaka povezava tvori svoj Θ^*-razred. Torej je kanonična metrična reprezentacija grafa S_2^n izometrična vložitev v kocko Q_{2^n-1}. Za $p = 3$ definirajmo

$$\begin{aligned} F_i^n &:= \big\{\{i^n, i^{n-1}j\}, \{i^n, i^{n-1}\ell\}\big\} \cup \big\{i^{n-m}e_{j\ell}^{(m)} \mid m \in [n]\big\}\,, \\ \widetilde{F^n} &:= E(S_3^n) \setminus (F_0^n \cup F_1^n \cup F_2^n)\,, \end{aligned}$$

kjer je $\{i, j, \ell\} = T$. Potem velja

Izrek 16. [Theorem 4.14] *Če je $n \in \mathbb{N}$ in $n \geq 2$, potem so Θ^*-razredi grafa S_3^n naslednji: F_0^n, F_1^n, F_2^n in $\widetilde{F^n}$.*

Na sliki 4.4 so predstavljeni Θ^*-razredi grafov S_3^2 in S_3^3, slika 4.5 pa prikazuje kvocientni graf $S_3^4/\widetilde{F^4}$.

Čeprav ima graf S_3^n netrivialno kanonično metrično reprezentacijo, nam le-ta ne pomaga veliko. Ima namreč samo štiri Θ^*-razrede, od katerih je $\widetilde{F^n}$ skoraj tako velik kot graf S_3^n. To je razlog za preučevanje (preostalih) induciranih vložitev grafov Sierpińskega. V ta namen vpeljemo novo dimenzijo, imenovano Hammingova dimenzija, ki je največje število faktorjev Hammingovega grafa, v katerega neredundantno in inducirano vložimo neki graf.

Definicija 17. *Naj bo G graf.* Hammingova dimenzija, $\mathrm{Hdim}(G)$, *grafa G je maksimalna dimenzija Hammingovega grafa, v katerega vložimo G kot neredundanten induciran podgraf. Če graf G ni induciran podgraf nobenega Hammingovega grafa, potem je* $\mathrm{Hdim}(G) = \infty$.

Očitno je $\mathrm{Hdim}(G) = 1$ natanko tedaj, ko je G poln graf. Da bi si lažje predstavljali Hammingovo dimenzijo, si jo oglejmo na nekaterih znanih družinah grafov. Za pot na n vozliščih je $\mathrm{Hdim}(P_n) = n - 1$. Naslednji lep primer so zvezde, kjer velja $\mathrm{Hdim}(K_{1,n}) = n$. Vseh grafov ne moremo inducirano vložiti v Hammingov graf. Dve takšni družini grafov so kolesa W_n in "skoraj polni grafi" K_n^-. Za te grafe je $\mathrm{Hdim}(W_n) = \mathrm{Hdim}(K_n^-) = \infty$.

Za določanje oziroma ocenjevanje Hammingove dimenzije nekega grafa je zelo uporabna teorija, ki sta jo Klavžar in Peterin razvila o induciranih podgrafih Hammingovih grafov [43]. V ta namen vpeljemo dva pogoja za označitve povezav:

Pogoj A. *Označitev (povezav) grafa G zadošča pogoju A, če za poljuben trikotnik grafa G velja, da imajo njegove povezave isto oznako.*

Pogoj B. *Označitev (povezav) grafa G zadošča pogoju B, če za poljubni nesosednji vozlišči u in v grafa G velja, da obstajata dve različni oznaki i in j, ki se pojavita na vsaki inducirani u, v-poti.*

Pogoja A in B sta uporabni orodji za preučevanje Hammingove dimenzije, saj sta avtorja v [43] dokazala naslednji izrek, ki ga bomo izrazili s Hammingovo dimenzijo.

Izrek 18. [Theorem 4.16] *Če je G povezan graf, potem je $\mathrm{Hdim}(G) < \infty$ natanko tedaj, ko obstaja označitev povezav grafa G, ki zadošča pogojema A in B.*

Dokaz izreka je konstruktiven. Zaenkrat omenimo samo to: če imamo označitev povezav grafa G z ℓ različnimi oznakami, ki zadošča pogojema A in B, potem nam ta porodi vložitev grafa G v Hammingov graf dimenzije ℓ.

Graf S_p^n lahko vložimo v Hammingov graf dveh dimenzij s pomočjo $(1|2)$-označitve: vse klične povezave grafa S_p^n označimo z 1, neklične pa z 2. Očitno takšna označitev zadošča pogoju A, saj so vsi polni grafi označeni z 1. Pogoj B za to označitev pa sledi iz konstrukcije grafov Sierpińskega, saj poljubni dve neklični povezavi nista incidenčni. S tem dobimo prve meje za Hammingovo dimenzijo grafov S_p^n

$$2 \leq \mathrm{Hdim}(S_p^n) < \infty\,. \tag{5.2}$$

To oceno bomo v nadaljevanju poskusili izboljšati.

Za začetek definirajmo še eno označitev. *Označitev trikotnikov Sierpińskega* grafa S_p^n konstruiramo induktivno. Povezave grafa S_p^1 označimo z 1. Sedaj predpostavimo, da je S_p^{n-1} že označen, in označimo povezave vsakega podgrafa iS_p^{n-1}, $i \in [p]_0$, grafa S_p^n enako kot S_p^{n-1}. Preostalim povezavam $e_{ij}^{(n)}$ damo oznako n. Velja naslednja lema.

Lema 19. [Lemma 4.20] *Če sta $n, p \in \mathbb{N}$ in je $p \geq 3$, potem označitev trikotnikov Sierpińskega grafa S_p^n izpolnjuje pogoja A in B.*

Primer te označitve lahko vidimo na sliki 4.7. S pomočjo te označitve lahko tudi opišemo inducirano vložitev grafov Sierpińskega v kartezični produkt grafov trikotnikov Sierpińskega.

Izrek 20. [Theorem 4.21] *Če sta $n, p \in \mathbb{N}$ in je $p \geq 3$, potem obstaja inducirana vložitev*

$$S_p^n \to ST_p^{n-1} \,\square\, ST_p^{n-2} \,\square\, \cdots \,\square\, ST_p^0\,.$$

Očitno označitev trikotnikov Sierpińskega porabi n oznak, kar prejšnjo spodnjo mejo (5.2) za Hammingovo dimenzijo precej izboljša. Za $n = 3$ bomo s posebno označitvijo to mejo še izboljšali. Konstruiramo jo tako, da porabi čim več oznak, vendar pa kljub temu zadošča pogojema A in B.

Podobno kot pri označitvi trikotnikov Sierpińskega tudi *združevalno označitev* definiramo induktivno. Za S_3^2 uporabimo $(1|2)$ označitev. Sedaj predpostavimo, da je graf S_3^{n-1} že označen z združevalno označitvijo. Potem vsak podgraf iS_3^{n-1} označimo na enak način kot S_3^{n-1}, vendar tako, da za poljubna različna $i, j \in T$ iS_3^{n-1} in jS_3^{n-1} dobita popolnoma različne oznake. Preostalim povezavam $e_{01}^{(n)}$, $e_{12}^{(n)}$ in $e_{02}^{(n)}$ dodelimo enake oznake, kot jih imajo njim nasproti ležeči trikotniki $2e_{01}^{(n-1)}$, $0e_{12}^{(n-1)}$ in $1e_{02}^{(n-1)}$. Na tem mestu naj pripomnimo, da takšna označitev ne bi zadoščala pogoju B, saj se nekatere oznake na ciklu $C_{012}^{(n)}$ pojavijo samo enkrat. Zato si oglejmo naslednje usmerjene poti[1] na ciklu $C_{012}^{(n)}$:

$$[01P_{12}^{(n-2)}, 21P_{10}^{(n-2)}];$$
$$[02P_{12}^{(n-2)}, 12P_{02}^{(n-2)}];$$
$$[20P_{01}^{(n-2)}, 10P_{02}^{(n-2)}].$$

Potujemo hkrati po poteh $01P_{12}^{(n-2)}$ in $21P_{10}^{(n-2)}$. Brž ko na poti $01P_{12}^{(n-2)}$ naletimo na oznako ℓ_0, ki se pojavi samo enkrat na celotni poti $0P_{12}^{(n-1)}$, jo združimo s pripadajočo oznako ℓ_2 na poti $21P_{10}^{(n-2)}$. (Zaradi konstrukcije označitve in grafov Sierpińskega se oznaka ℓ_2 prav tako samo enkrat pojavi na poti $2P_{10}^{(n-1)}$.) Združevanje oznak v tem primeru pomeni, da zamenjamo vsakršno pojavitev oznake ℓ_2 v grafu S_3^n z oznako ℓ_0. Isti postopek naredimo za preostala dva para poti.

S pomočjo združevalne označitve dobimo naslednjo spodnjo mejo za $\mathrm{Hdim}(S_3^n)$:

Izrek 21. [Theorem 4.26] *Če je $n \in \mathbb{N}$ in $n \geq 4$, potem velja*

$$\mathrm{Hdim}(S_3^n) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2}n - \frac{9}{4}\,.$$

Seveda moramo za uporabo združevalne označitve najprej dokazati, da zadošča pogojema A in B. Dokaz ni trivialen in ga bomo tukaj izpustili. Prav tako bomo izpustili podrobnosti izračuna števila oznak v združevalni označitvi S_3^n. Poteka namreč tako, da preštejemo, koliko oznak združimo v posameznem koraku konstrukcije označitve.

S pomočjo združevalne označitve lahko določimo še naslednje natančne vrednosti Hammingove dimenzije:

Trditev 22. [Proposition 4.25] $\mathrm{Hdim}(S_3^2) = 3$ *in* $\mathrm{Hdim}(S_3^3) = 6$.

Za konec omenimo še zgornjo mejo ter nekatere natančne vrednosti Hammingove dimenzije za poljuben p.

[1] Usmerjena pot je pot, ki ima začetno in končno vozlišče ter je vrstni red pomemben.

Izrek 23. [Theorem 4.28]

$$(i) \quad \mathrm{Hdim}(S_3^n) \leq 5 \cdot 3^{n-3} + 1 \quad (n \geq 3)\,.$$
$$(ii) \quad \mathrm{Hdim}(S_p^n) \leq \frac{2}{p-1}p^{n-2} + \frac{2p-4}{p-1} \quad (p \geq 4 \textit{ in } n \geq 2)\,.$$

Izrek dokažemo tako, da preštejemo, najmanj koliko oznak moramo združiti. S pomočjo te meje pa lahko določimo še naslednje vrednosti Hammingove dimenzije za $n = 2$ in $n = 3$.

Trditev 24. [Proposition 4.27] *Če je $p \in \mathbb{N}$ in $p \geq 4$, potem velja*

$$(i) \quad \mathrm{Hdim}(S_p^2) = 2\,.$$
$$(ii) \quad \mathrm{Hdim}(S_p^3) = 4\,.$$

Na sliki 4.10 sta predstavljeni dve optimalni označitvi, ki zadoščata pogojema A in B ter porabita 12 oznak, kar je tudi zgornja meja po izreku 23.

Motivacija za nadaljnje delo

Doktorsko disertacijo smo zaključili z nekaj vprašanji, ki so med raziskovanjem ostala neodgovorjena. Tu omenimo le zanimiv problem.

Med preučevanjem metrične dimenzije smo pomislili, da bi raziskali še Wienerjevo dimenzijo grafov Sierpińskega. *Wienerjeva dimenzija*, $\dim_W(G)$, grafa G je vpeljana kot število različnih (celotnih) razdalj v grafu G [1]. Torej, če je $\{d_G(u) \mid u \in V(G)\} = \{\delta_1, \ldots, \delta_k\}$, potem je Wienerjeva dimenzija grafa G enaka k. Za nekatere začetne grafe Sierpińskega ni težko določiti Wienerjeve dimenzije (s pomočjo računalnika):

$p \setminus n$	2	3	4	5	6	7	8	9	10
2	2	4	8	16	32	64	128	256	512
3	2	4	13	40	120	356	1084	3268	9832
4	2	5	15	50	187	715	2793	?	?
5	2	5	15	52	201	854	?	?	?
6	2	5	15	52	203	?	?	?	?
7	2	5	15	52	203	?	?	?	?

Ti rezultati porodijo naslednjo trditev:

Trditev 25. [Proposition 5.1] *Če sta $n, p \in \mathbb{N}$ in $p \geq 2$, potem velja*

$$\dim_W(S_p^2) = 2 \quad \textit{in} \quad \dim_W(S_2^n) = 2^{n-1}\,.$$

Izrek 2.28 za $n = 2$ pove, da imamo v grafu S_p^2 samo dve orbiti vozlišč. Eno orbito tvorijo ekstremna vozlišča, drugo pa skoraj ekstremna vozlišča. S tem dobimo zgornjo mejo $\dim_W(S_p^2) \leq 2$ za $p \geq 2$. Razdalje ekstremnih in skoraj ekstremnih vozlišč so enake

$$d_{S_p^2}(ij) = (p-1) + (2p-1) + (p-2) \cdot (2p + d_{S_p^1}(i)) = p(3p-4)\,,$$
$$d_{S_p^2}(i^2) = (p-1) + (p-1) \cdot (2p + d_{S_p^1}(i)) = p(3p-3) < p(3p-4) = d_{S_p^2}(ij)\,.$$

Prva enakost trditve torej velja. Druga enakost sledi iz dejstva $S_2^n \cong P_{2^n}$.

Čeprav smo določili Wienerjeve dimenzije nekaterih grafov Sierpińskega, v splošnem ta dimenzija še vedno prestavlja odprt problem.

Problem 26. *Naj bosta $n, p \in \mathbb{N}$ in $n, p \geq 3$. Določi Wienerjevo dimenzijo grafa S_p^n.*