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Significance of flats in CAT(0) geometry

Doctoral thesis

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Pomen ravnin v CAT(0) geometriji

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Abstract

Several questions/conjectures in CAT(0) geometry are inspired by analogous theorems that are known to hold for Riemannian manifolds of nonpositive sectional curvature. This thesis deals with the one which was settled by Bangert and Schröder in early nineties for real analytic manifolds, [BS]. It is called *the flat closing problem* and it predicts a copy of $\mathbb{Z}^m$ in any discrete group which acts properly and cocompactly by isometries on a CAT(0) space X containing an isometric copy of $\mathbb{R}^m$. We summarize results from [CM-ST, CM-DS] about the *full isometry group* of a proper, cocompact and geodesically complete CAT(0) space. Then we apply those results to prove the main theorem from [CZ], a very partial answer to the flat closing conjecture.

Theorem. *If a proper CAT(0) space X is a product of m geodesically complete factors, then discrete Γ, which acts properly and cocompactly on X, contains a copy of $\mathbb{Z}^m$.*

Even though the theorem above is far from the full generality of the flat closing problem, its proof uses a deep machinery from the structure theory of the isometry group of the corresponding CAT(0) space. The proof relies in an essential way to the solution of Hilbert's fifth problem, see Theorem A.6. This solution leads to a dichotomy for the isometry group of a nice non Euclidean CAT(0) space – either it is a Lie group or a totally disconnected locally compact group. Applying this dichotomy to the irreducible factors from the theorem, we deal with two separated approaches. The first case is covered by older results from Lie group theory while the second relies to the geometric properties of CAT(0) space with totally disconnected isometry group, see [CM-ST, §6].

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Key words: CAT(0) spaces, isometry group, locally compact groups, flat closing conjecture.

Povzetek

Številna vprašanja v CAT(0) geometriji izvirajo iz izrekov o Riemannovih mnogoterostih nepozitivnih prereznih ukrivljenosti. V tej disertaciji se ukvarjamo z enim izmed njih, s *problemom periodičnih ravnin*. V kontekstu realnih analitičnih mnogoterosti sta ga rešila Bangert in Schröder, [BS]. Problem sprašuje, ali vedno lahko najdemo kopijo proste abelove grupe $\mathbb{Z}^m$ v grupi, ki deluje kokompaktno diskretno z izometrijami na CAT(0) prostoru X, ki vsebuje izometrično vloženo kopijo $\mathbb{R}^m$. V uvodnih poglavjih povzamemo dognanja iz del [CM-ST, CM-DS] o celotni grupi izometrij pravega kokompaktnega geodezično polnega CAT(0) prostora. Nato ta dognanja uporabimo v dokazu glavnega izreka iz [CZ], ki poda delni odgovor na problem periodičnih ravnin.

Izrek. *Naj bo parvi CAT(0) prostor X produkt m geodezično polnih faktorjev. Tedaj poljubna grupa Γ, ki deluje kokompaktno diskretno z izometrijami na X, vsebuje kopijo $\mathbb{Z}^m$.*

Čeprav predpostavke zapisanega izreka močno posežejo v splošnost problema periodičnih ravnin, so za njegov dokaz potrebni globoki izreki iz strukturne teorije grupe izometrij dotičnega CAT(0) prostora. Za dokaz ključna je rešitev Hilbertovega petega problema, izrek A.6, ki zagotavlja dihotomojo za grupe izometrij določenih CAT(0) prostorov. Bodisi je grupa izometrij Liejeva bodisi je popolnoma nepovezana lokalno kompaktna topološka grupa. Glede na to dihotomijo se dokaz izreka razdeli na dva dela. Prvi del sledi iz znanih izrekov iz teorije Liejevih grup, med tem ko se drugi del sklicuje na geometrijo CAT(0) prostora s popolnoma nepovezano grupo izometrij, [CM-ST, §6].

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Ključne besede: CAT(0) prostori, grupa izometrij, lokalno kompaktne grupe, problem periodičnih ravnin.

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Contents

Chapter 1

Introduction

1.1 About the topic

In the field of mathematics called *geometric group theory*, we are studying an interplay between a group Γ and a space X, on which Γ acts by isometries. The existence of such an action implies certain restrictions on the algebraic structure of Γ and on geometry of X. The restrictions are more stringent if the space X or/and the action have additional "nice" properties.

As evident from [Bri], the theory of non-positively curved spaces plays a special role in this area. The generalization of the concept of (sectional) curvature from manifolds to geodesic metric spaces is attributed to A. D. Aleksandrov, E. Cartan and V. A. Toponogov. They introduced their generalizations of sectional curvature already in the twenties of the previous century. The theory culminated in early eighties of the previous century when Mikhail Gromov reproved many difficult results from the theory of non-positively curved manifolds using only the metric condition of curvature. He introduced the notation CAT(κ); letters C, A and T stand in honor of Cartan, Aleksandrov and Toponogov and the real-valued parameter κ gives an upper bound on the curvature.

A special emphasis goes to CAT(0) spaces. This is a class of geodesic metric spaces in which triangles are no "fatter" than Euclidean ones. To be more precise, a geodesic metric space (X, d) is a **CAT(0) space** if for every triple of points $x_0, x_1, x_2 \in X$ and any point p on an arbitrary geodesic segment between x_1 in x_2 the following holds. If $\bar{x}_0, \bar{x}_1, \bar{x}_2$ are points in the Euclidean plane $\mathbb{R}^2$ with $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$ for all $i, j = 0, 1, 2$ and $\bar{p}$ is a point on the segment between $\bar{x}_1$ and $\bar{x}_2$ with $d(x_1, p) = \|\bar{x}_1 - \bar{p}\|$, then $d(p, x_0) \le \|\bar{p} - \bar{x}_0\|$. Similarly, CAT($\kappa$) is defined as the class of spaces where no triangle is fatter than the corresponding comparison triangle in M_κ, the two-dimensional simply connected manifold of constant Gaussian curvature κ. For positive κ, we have to take care of diameters of triangles in X which we like to compare, since M_κ has bounded diameter for positive κ ($M_{\kappa>0}$ is a sphere $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = \frac{1}{\kappa}\}$ with induced length metric from $\mathbb{R}^3$). But in geometric group theory, CAT(κ) spaces for non-positive κ, which we will also refer to as *non-positively curved spaces*, enjoy much more attention than for positive κ. One of the main directions in research is to what extent properties of CAT(κ) spaces, for strictly negative κ (also called *negatively curved spaces*), hold also for CAT(0) spaces. Observe that since triangles in M_κ are "thiner" than those in $M_{\kappa'}$ for $\kappa < \kappa'$, every CAT(κ) space is also CAT(κ'). Presence of flats (i.e. isometrically embedded copies of $\mathbb{R}^n$ for $n > 1$) in CAT(0) spaces is the basic thing making CAT(0) spaces different from negatively curved ones. Hence, the class of CAT(0) spaces contains much more interesting examples as the family of negatively curved spaces, but the price is that less is known for CAT(0) spaces which are not CAT(κ) for some strictly negative κ, see §1.3 below.

Because the class of all CAT(0) spaces is very large and hence there are lots of groups acting on them, we often impose additional conditions on the group actions and CAT(0) spaces. Let G act on a CAT(0) space X. First of all, we assume that X is a **proper space**, i.e. every ball in X has compact closure. Next, we do not wish the group to be too large (or the action to be too degenerate) hence we assume that the action is **proper**. This means that for every ball B in X, there is *not too many* elements $g \in G$ such that gB intersects B. In the case of discrete group G, *not too many* means finite number. This generalizes to (non-discrete) topological groups, where *not too many* means that the set of such elements has compact closure in G. We can pass by that problem if we take G to be a subgroup of $\mathfrak{Iso}(X)$, where the later is equipped with compact open topology, Definition A.1.

Final restriction is that X has some level of homogeneity, or equivalently, that the group G is large enough. There are several notions of this property:

(i) G acts **minimally** if there is no proper closed nonempty convex G-invariant subsets in X;

(ii) G has **full limit set** which means that for every geodesic ray (isometric embedding) $r : [0, \infty) \to X$, there is a sequence $(g_n)_{n \in \mathbb{N}} \subseteq G$ such that geodesic segments $[r(0), g_n r(0)]$ converge uniformly on compact sets to r;

(iii) G acts **cocompactly**, i.e. there is a compact $K \subseteq X$ such that G-translates of K cover X;

(iv) G satisfies **duality condition** which means that for every geodesic line (isometric embedding) $\ell : \mathbb{R} \to X$, there is a sequence $(g_n)_{n \in \mathbb{N}} \subseteq G$ such that geodesic segments $[\ell(0), g_n \ell(0)]$ converge uniformly on compact sets to $\ell|_{[0,\infty)}$ and $[g_n^{-1}\ell(0), \ell(0)]$ converge uniformly on compact sets to $\ell|_{(-\infty,0]}$.

The most popular from the list above is **cocompactness**. A proper and cocompact action of a discrete group is also called a **geometric action**, while a group acting in that way is called a **CAT(0) group**. A geometric action of a group Γ on a space X is denoted $\Gamma \overset{geo}{\curvearrowright} X$.

We will discuss the duality condition later in §1.4 and compare it with an analogue notion in hyperbolic spaces. Concerning the full limit set, in §2.5 we give an example of a weaker conclusion about a geometry of a proper CAT(0) space assuming only full limit set instead of cocompactness. Our example answers the question [CM-DS, Question 7.1].

1.2 Main examples

The situation described above is a direct generalization of compact manifolds of non-positive sectional curvatures. Indeed, if we take such manifold M and denote $\Gamma := \pi_1(M)$, then Γ acts geometrically (by deck transformations) on the universal covering space $\widetilde{M}$, which is CAT(0). That was the starting point of Gromov's revolution.

Among the other manifolds of non-positive sectional curvature, **symmetric spaces** are of special interest. See [Ebe] for smooth (differential-geometric) approach and [BH, Chapter II, §10] for CAT(0) approach. Symmetric spaces appear in many different ways in mathematics. The name "symmetric" describes one possible definition. For any given point x on the Riemannian manifold M, there is a well-defined map $S_x : M \to M$ sending $c(t)$ to $c(-t)$ for each geodesic c with $c(0) = x$. If the symmetries S_x are Riemannian isometries the manifold M is called symmetric. If we assume that such a manifold M has no compact factors, then M is non-positively curved and hence CAT(0). If we prohibit also the Euclidean factor, then M is said to be a *symmetric space of non-compact type* and (the identity component of) its isometry group is a semi-simple Lie group with trivial center and no compact factors.

That property leads to the next possible definition. Let G be a center-free semi-simple Lie group without compact factors and let K denote a maximal compact subgroup of G. Then $M := G/K$, endowed with a G-invariant Riemannian metric, is a symmetric space of non-compact type. The main example which we have in mind is $G = PSL(n, \mathbb{R})$ and $K = SO(n)$. In fact if M is any symmetric manifold of non-compact type there exists a diffeomorphism onto a totally geodesic submanifold of $PSL(n, \mathbb{R})/SO(n)$ for some n. The pull-back metric on M obtained by means of the embedding coincides with the original metric on M up to a constant multiple on each irreducible de Rham factor. See Appendix B for further discussion.

The last description of a symmetric space generalizes further in the following way. Take any field with discrete valuation, see [Bro, Chapter V, §8] for definition or imagine the special linear group over p-adics, $SL(n, \mathbb{Q}_p)$, as a basic example. Then repeat the construction as above. In that case, $SL(n, \mathbb{Z}_p)$ is a maximal compact subgroup and we can associate (in a canonical way) a simplicial structure on the quotient $\Delta_p(n) := SL(n, \mathbb{Q}_p)/SL(n, \mathbb{Z}_p)$. Equipping each of the simplices with Euclidean metric (i.e. take any m-simplex to be isometric to the standard one) and the whole simplicial complex with induced path metric turns $\Delta_p(n)$ into a CAT(0) space, called **Euclidean building**. Euclidean buildings can be threaded as discrete analogues (because of their combinatorial/simplicial structure) of symmetric spaces.

Symmetric spaces and Euclidean buildings are of special interest from the viewpoint of the theory of locally compact groups, since their isometry groups are two extremities – Lie group in the first case and a totally disconnected locally compact topological group in the second. Under the appropriate assumptions on a CAT(0) space, those two cases together with an obvious example of the Euclidean space cover everything from the viewpoint of isometry group, see Theorem 3.1.

The third family of basic examples of CAT(0) spaces is the family of **CAT(0) cube complexes**. Cube complex is a CW complexes with all the cells isometric to Euclidean cubes $[0,1]^n$ with injective characteristic maps, equipped with shortest-path distance. Cube complexes enjoy large attention because one can locally check CAT(0) condition in purely combinatorial terms, see the theorem of Gromov, [BH, Theorem II.5.20]. Thank to the Cartan-Hadamard theorem, [BH, Theorem II.4.1], their universal covering spaces are CAT(0). Because of that reason, they are the main source for modeling concrete CAT(0) groups (fundamental groups of compact locally CAT(0) spaces) with interesting properties. Furthermore, the additional combinatorial structure of CAT(0) cube complexes reflects in several properties which are known to hold for CAT(0) cube complexes, but are still unknown for general CAT(0) spaces. An example is the rank rigidity conjecture, see [CS] and §1.4 below.

Finally, let us just mention that there are also several constructions for building new CAT(0) spaces from existing ones. We can for example take a product of several CAT(0) spaces and equip it with ℓ^2-metric. There are also several constructions of gluing, see [BH, Chapter II, §11] for definitions and connection with group-theoretic operations such as the product with amalgamation and HNN extension. One example of building a CAT(0) space with selected properties is discussed in §2.5.

1.3 Non-positive vs. negative curvature

Beside CAT(κ) spaces, Gromov defined also **δ-hyperbolic spaces** (here δ is some non-negative parameter). This is another – coarser – notion of (strictly) negative curvature in geodesic metric spaces. We say that a geodesic triangle $\triangle x_0x_1x_2$ (a union of three geodesics $\gamma_{i,j}$ between x_i and x_j, $i,j=0,1,2$, $i\neq j$) is δ-thin if for every index i the geodesic $\gamma_{(i-1),(i+1)}$ lies in the δ-neighborhood of the union $\gamma_{i,i+1}\cup\gamma_{i-1,i}$ (the indices are understood modulo 3). The geodesic metric space X is δ-hyperbolic if every triangle in X is δ-thin. When we do not care about the value of δ, we just say that X is a hyperbolic space.

A calculation in the hyperbolic plane $\mathbb{H}^2$ shows that CAT(κ) spaces, for $\kappa<0$, are hyperbolic (because triangles in CAT(κ) spaces are thiner than triangles in an appropriately rescaled hyperbolic plane M_κ).

A finitely presented group is called a **hyperbolic group** if its Cayley graph with word-metric is hyperbolic, or equivalently, if it acts properly and cocompactly by isometries on some hyperbolic space. Note that for strictly negative κ, CAT(κ) groups (i.e. groups acting geometrically on some CAT(κ) space) are hyperbolic, but the converse is unknown. It is known that every hyperbolic group can act geometrically on a contractible polyhedral complex by isometries (but this complex may not carry negatively/non-positively curved metric). Anyway, hyperbolic groups are known to enjoy most important (structural, algorithmic, geometric ...) properties of CAT(-1) groups. Hence hyperbolic group is an appropriate notion of negative curvature in group theory.

An interesting question is about the boundary value $\kappa=0$ – when a CAT(0) group Γ is hyperbolic. It follows from the definition that Γ is hyperbolic if and only if the space X on which it acts geometrically is hyperbolic. On the level of geometry, it is easily describable by the following theorem.

Theorem 1.1 (**[BH, Theorem II.9.33]**). *A CAT(0) space X admitting some geometric group action is hyperbolic if and only if X has no isometrically embedded copies of the Euclidean plane $\mathbb{R}^2$.*

On the level of (algebraic) group theory, there is still no sufficient condition for a CAT(0) group Γ to be hyperbolic. An obvious necessary condition is that Γ does not contain a copy of $\mathbb{Z}^2$, which is explained by the *flat torus theorem*, Theorem 2.2. In this thesis we address the *flat closing conjecture* which says that the absence of $\mathbb{Z}^2$ in Γ is also sufficient for hyperbolicity. Precisely, we prove (under some technical assumption on the proper CAT(0) space X) the following theorem.

Theorem 1.2 (**[CZ, Corollary 1]**). *If X is a product of m factors, then Γ, acting geometrically on X, contains a copy of $\mathbb{Z}^m$.*

The reason why we would like to have a good criterion for distinguishing hyperbolic groups among CAT(0) groups is that hyperbolic groups have "better properties" than general CAT(0) groups. For example, hyperbolic groups are biautomatic (unknown for CAT(0)), they have well defined boundary at infinity (see §2.1 for definition) (CAT(0) groups do not), they satisfy linear isoperimetric inequality (CAT(0) satisfy quadratic), Dehn functions for their finitely presented subgroups are polynomial (may be exponential for CAT(0)), and several other things. We refer the reader to [Bri] for detailed discussion on the topic.

The Tits alternative is also the property that holds for hyperbolic groups. In CAT(0) context, it is only known for CAT(0) cubical groups and for the classical examples mentioned above (from the work of Tits, where he proved that finitely generated subgroups of linear groups satisfy Tits alternative).

The spirit of hyperbolicity in non-hyperbolic CAT(0) groups is given by **rank one isometries**. The behavior of CAT(0) groups containing isometry of rank one is somewhere between hyperbolic and general CAT(0) groups. Recall that an axial isometry is an isometry having an axis, i.e. isometric embedded copy of $\mathbb{R}$ on which it acts by a nonzero translation (Definition 2.4). A rank one isometry can be described as an axial isometry α such that none of the axis of α lies in a half plane (isometric embedded copy of $\mathbb{R} \times [0, \infty)$ with Euclidean metric). Roughly, a rank one isometry is an isometry with non-Euclidean behavior around its axis. See [Bal, BB] for precise definition.

1.4 Hic abundant leones[1]

As it is evident from the previous sections, there are two main sources for building the theory of CAT(0) spaces (admitting some geometric group action or satisfying some other regularity properties). First option is to try to generalize known results from the theory of Riemannian manifolds of non-positive sectional curvature to the context of CAT(0) spaces (in other words – to reprove some Riemannian theory with weaker machinery, as was Gromov's insight). The second approach is to deduce to what extent properties of hyperbolic groups also hold for CAT(0) groups, or at least for CAT(0) groups with rank one isometries.

The most important structural conjecture about a nice CAT(0) space, the **rank rigidity conjecture**, comes from the world of manifolds where it was settled by W. Ballmann. We refer the reader to [Bal, Chapter IV] for the proof in manifold case and discussion on the problem in CAT(0) context. The rank rigidity conjecture predicts that a "nice" CAT(0) space X is one of the main examples from above – symmetric space, Euclidean building or a product of at least two CAT(0) spaces – or otherwise it possesses a rank one isometry. Hence irreducible de Rham factors are either quite well understood (symmetric spaces, Euclidean buildings) or they behave (in a way) similar to hyperbolic spaces and hence we also have better information than in general.

As an example of the hyperbolic behavior of a CAT(0) group containing a rank one isometry we should mention the dynamic of the induced group action on the visual boundary of the space (see §2.1 for definition; in the case of CAT(0) manifold, the visual boundary is just a sphere). Because of that, the following question (open in general) has an affirmative answer for CAT(0) groups containing isometry of rank one.

Question 1.3. *Let $\Gamma \overset{geo}{\curvearrowright} X$. Does Γ satisfy the duality condition?*

This question is motivated from the hyperbolic world and is related to the extreme behavior of the action of a hyperbolic group on its boundary, called *convergence action* or *South-Nord dynamics*, see [Bow].

Another evident hyperbolic property of rank one isometries is about their axis. If you fix constants $a, b \in [1, \infty)$ and start traveling between two points x and y on some axis of rank one isometry α such that the length of your path does not exceed $ad(x, y) + b$, then the whole your path stays $c = c(a, b, \alpha)$ close to the axis of α (note that c does not depend on $d(x, y)$). This is of course not the case in the Euclidean plane with α a translation, an shown by an example of a semi-circle with endpoints on the axis of translation. This "hyperbolic property" is the key property for solving (or deducing bounds for algorithms for) word problems in hyperbolic groups. From geometric point of view, the described property is related to the divergence of geodesics. Given two isometric embeddings $r, r' : [0, \infty) \to X$ with $r(0) = r'(0) = x_0$, the divergence measures the asymptotic behavior ($t \to \infty$) of how far is from $r(t)$ to $r'(t)$ outside the ball of radius t around x_0. But the reader can measure that for t large enough, this already exceeds the scope of the thesis, see Figure 1.1.

[1] See [Bri, p. 963, Figure 1] for explanation.

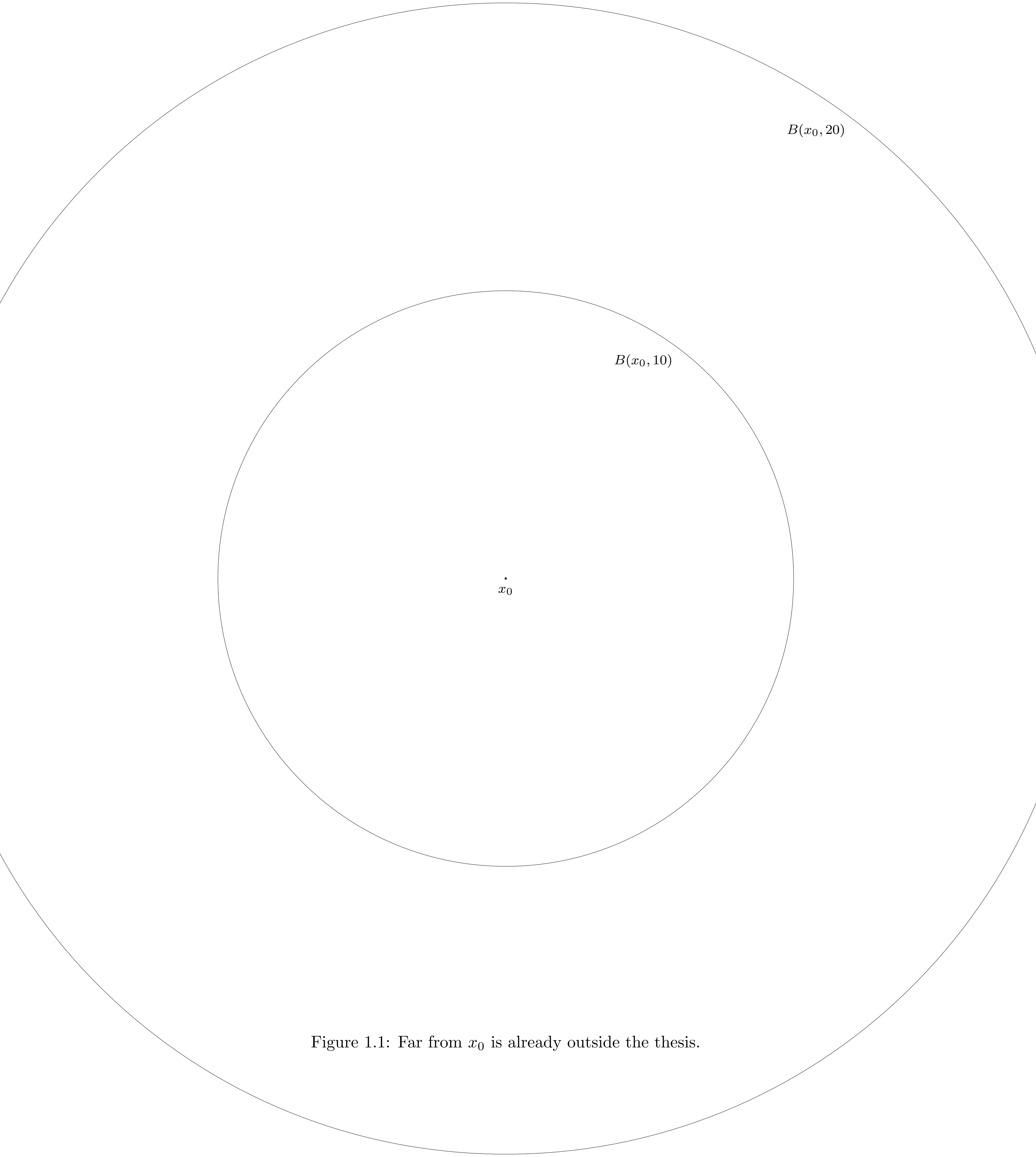

Figure 1.1: Far from x_0 is already outside the thesis.

Chapter 2

CAT(0) geometry

Let us recall the definition of a CAT(0) space.

Let (X, d) be a **geodesic metric space**, i.e. for every pair of points $x_0, x_1 \in X$, there is a (continuous) map $\gamma : I = [0, 1] \to X$ with $\gamma(i) = x_i$ for $i = 0, 1$ and $d(\gamma(t), \gamma(s)) = |t - s| d(x_0, x_1)$. Image of such map will be denoted by $[x_0, x_1]$ and called a **geodesic segment** (or just a **geodesic**). Note that the symbol $[x_0, x_1]$ is in general *not well defined* since geodesic between two points need not be unique, remember an example of a round sphere with $x_0 =$ North pole and $x_1 =$ South pole. A **(geodesic) triangle** in a geodesic metric space X is a choice of three points x_0, x_1, x_2 and some geodesics between them. Abusing the notation again, we denote the triangle as $\triangle x_0x_1x_2 = [x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_0]$. By the triangle inequality for metric d, there always exist points $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in \mathbb{R}^2$ such that $\|\bar{x}_i - \bar{x}_j\| = d(x_i, x_j)$ for all pairs $i, j = 0, 1, 2$, where $\| \cdot \|$ denotes the Euclidean 2-norm. We call the triangle $\triangle \bar{x}_0\bar{x}_1\bar{x}_2$ (which is unique since $\mathbb{R}^2$ is uniquely geodesic metric space) the **comparison triangle** for $\triangle x_0x_1x_2$. For every $z \in [x_i, x_j] \subseteq \triangle x_0x_1x_2$, there is a unique point, called the **comparison point** $\bar{z} \in [\bar{x}_i, \bar{x}_j]$ such that $d(x_i, z) = \|\bar{x}_i - \bar{z}\|$. We say that the **triangle $\triangle x_0x_1x_2$ satisfies CAT(0) inequality** if for every pair of points $z, w \in \triangle x_0x_1x_2$, the inequality $d(z, w) \leq \|\bar{z} - \bar{w}\|$ holds. We say that a **geodesic metric space X satisfies CAT(0) inequality** if every triangle in X satisfy CAT(0) inequality.

In what follows, we define and describe terms that are involved in almost every nontrivial proof of a theorem about CAT(0) geometry or some related topic. We also collect some basic results on CAT(0) geometry and hyperbolicity that are needed later on and/or enlighten a problem of *the flat closing conjecture* we are dealing with in the core.

2.1 Boundary at infinity

To every geodesic metric space, we can associate another topological (or metric) space, called **the boundary at infinity**. It usually turns out that we can deduce more properties of geometry of a CAT(0) space or of a group acting on it by observing also the extended action on its boundary at infinity. In this section, we will describe two most important ways of constructing a boundary (or rather the structure – topology or metric – on it).

A **geodesic line** (**ray**, respectively) in a geodesic metric space is an isometric embedding $\ell : \mathbb{R} \to X$ ($r : [0, \infty) \to X$, respectively). We can define a **boundary at infinity** (also called a **visual boundary**) of X, denoted ∂X, as a quotient of the set of all geodesic rays modulo the following equivalence relation: two rays r, r' are equivalent if $t \mapsto d(r(t), r'(t))$ is a bounded function from $[0, \infty)$ to itself. (In terms of Definition 3.10, this condition is equivalent to saying that images of r and r' are at finite Hausdorff distance.) We denote $r(\infty)$, or sometimes $[r]$, the equivalence class determined by the ray r. Usually, when we do not like to specify the representative r for a point at infinity, we will denote it by some Greek letter.

When X is a proper CAT(0) space, we have for every point $\xi \in \partial X$ and every $x \in X$ a geodesic ray r such that $r(0) = x$ and $r(\infty) = \xi$. To see this, pick some ray q, representing ξ. By CAT(0) inequality and compactness of balls in X, we can deduce that geodesic segments $([x, q(n)])_{n \in \mathbb{N}}$ converge uniformly on compact sets to some geodesic ray r. By construction, $r(0) = x$ and $r(\infty) = q(\infty)$. Invoking CAT(0) inequality again, we deduce that such ray r is unique. We say that a sequence of points $(x_n)_{n \in \mathbb{N}}$ converges to $\xi \in \partial X$, if the sequence of geodesics $([x_0, x_n])_{n \in \mathbb{N}}$ converges uniformly on compact sets to a geodesic ray r representing ξ for some (equivalently, any) point $x_0 \in X$. Note that the sequence of distances $d(x_n, r) = \inf\{d(x_n, r(t)) | t \in [0, \infty)\}$ might be unbounded.

We can topologize the boundary at infinity of a proper CAT(0) space in the following way. Pick a base point $x_0 \in X$ and for every $\xi \in \partial X$ take a ray r_ξ with initial point x_0, pointing towards ξ. We define a basis of neighborhoods of ξ as a family of sets $\{U_{R,\varepsilon} | R, \varepsilon > 0\}$, where

$$U_{R,\varepsilon} := \{q(\infty) \mid q : [0,\infty) \to X \text{ a geodesic ray with } q(0) = x_0, d(q(t), r(t)) < \varepsilon \quad \forall t < R\}.$$

It turns out that this topology is independent of x_0 and with this topology, called the **cone topology**, ∂X becomes a compact Hausdorff space.

Another topology on the boundary is given by an **angular metric** on ∂X, induced from the notion of angle in X. We first introduce some notation.

(i) For three different points $x, y, z \in \mathbb{R}^2$, the symbol $\sphericalangle_x(y, z)$ denotes the angle between segments $[x, y]$ and $[x, z]$ (or vectors $y - x$, $z - x$).

(ii) For three different points $x, y, z \in X$, the symbol $\overline{\sphericalangle}_x(y, z)$ denotes the angle $\sphericalangle_{\bar{x}}(\bar{y}, \bar{z})$, i.e. the angle at $\bar{x}$ in the comparison triangle $\triangle\bar{x}\bar{y}\bar{z}$. It is called the **comparison angle**.

(iii) For three different points $x, y, z \in X$, the symbol $\sphericalangle_x(y, z)$ denotes the limit $\lim_{t\to 0} \overline{\sphericalangle}_x(\gamma_y(t), \gamma_z(t))$, where γ_y, γ_z are parametrizations of geodesics $[x, y]$ and $[x, z]$ with $\gamma_y(0) = \gamma_z(0) = x$. The limit is well defined by CAT(0) inequality. The quantity $\sphericalangle_x(y, z)$ is called **Alexandrov angle between** $[x, y]$ **and** $[x, z]$. It is dominated by the comparison angle $\overline{\sphericalangle}_x(y, z)$.

(iv) For $x \in X$ and $\xi, \eta \in \partial X$, we define $\sphericalangle_x(\xi, \eta) := \sphericalangle_x r_\xi(1), r_\eta(1)$, where r_ξ, r_η are parametrizations of the rays from x to ξ, η, respectively.

(v) For $\xi, \eta \in \partial X$, we define $\sphericalangle\xi, \eta := \sup_{x\in X} \sphericalangle_x(\xi, \eta)$.

It is an exercise in CAT(0) geometry that the map $(\xi, \eta) \mapsto \sphericalangle\xi, \eta$ defines a metric on ∂X. We associate to it an induced length metric, called the **Tits metric** and denoted by d_T. The metric space $(\partial X, d_T)$ is called the **Tits boundary of** X and will also be denoted by $\partial_T X$. Since $\sphericalangle$ (and hence d_T) detects whether any pair of rays representing two points in ∂X diverge one from another, the metric d_T is usually finer than the cone topology which observes (the speed of) divergence only from a chosen base point.

With a geodesic line, we have (in an obvious way) defined two points at infinity, namely $\ell(\infty)$ and $\ell(-\infty)$. We say that two points $\xi, \eta \in \partial X$ are **visible**, if there exists a geodesic line ℓ such that $\ell(\infty) = \xi$ and $\ell(-\infty) = \eta$. Obviously, $\sphericalangle_{\ell(0)}(\xi, \eta) = \sphericalangle\xi, \eta = \pi$, while $d_T(\xi, \eta)$ may be strictly larger than π, or even ∞.

Observe that the action of any group G on X by isometries extends to an action with homeomorphisms (isometries, respectively) of G on the ∂X equipped with the cone topology (the angular or Tits metric, respectively). Indeed, every isometry respects the equivalence relation on geodesic rays and moreover preserves also the Alexandrov angle.

2.1.1 Some properties of the angular metric

The following properties are (among many others) proved in [BH, Chapter II, §9]. Here we will only briefly describe the ideas behind their proofs.

Proposition 2.1. *Let X be a proper CAT(0) space.*

(i) If $\sphericalangle\xi, \eta = \sphericalangle_x(\xi, \eta) < \pi$ for some $x \in X$, then the convex hull of $[x, \xi)$ and $[x, \eta)$ is a flat sector;

(ii) $\sphericalangle\xi, \eta = \lim_{t\to\infty} \overline{\sphericalangle}_x(r_\xi(t), r_\eta(t))$, where $r_\xi, r_\eta : [0, \infty) \to X$ are geodesic rays, pointing to ξ and η and emanating from x.

To prove the second property, one observes first that the right-hand side of the equality does not depend on x. Since $\sphericalangle_x(\xi, \eta) \le \overline{\sphericalangle}_x(r_\xi(t), r_\eta(t))$ for any $t > 0$, the right-hand side is always at least as big as the left-hand side. The opposite inequality is deduced by playing with properties of angles and CAT(0) inequality. One needs to use a trivial fact that the sum of angles between a segment and asymptotic geodesic rays, emanating from endpoints of that segment, is at most π. (It is the lemma saying that the sum of angles of a triangle in CAT(0) space is at most π, generalized to ideal triangles, i.e. triangles with a vertex at the boundary at infinity.) Another tool in the proof is the triangle inequality for the angle, i.e. $\measuredangle_x(y,z)+\measuredangle_x(z,w)\geq\measuredangle_x(y,w)$ for all y,z,w, different from x.

The property (i) is in the spirit of the *flat triangle lemma*, [BH, Proposition II.2.9]. This lemma says that if CAT(0) inequality for any pair of points $u,w\in\triangle xyz$ such that $\{u,w\}\not\subseteq\{x,y,z\}$ is in fact equality, then the triangle $\triangle xyz$ is isometric to its comparison triangle. Flat triangle lemma is equivalent to saying that the comparison angle at x in $\triangle xyz$ equals to Alexandrov angle, i.e. $\overline{\measuredangle}_x(y,z)=\measuredangle_x(y,z)$. Since the function $t\mapsto\overline{\measuredangle}_x(r_\xi(t),r_\eta(t))$ is constant under assumptions of (i), every triangle $\triangle xr_\xi(t)r_\eta(t)$ is flat and non degenerated and hence the whole sector is flat.

2.2 Reasonable subfamily of CAT(0) spaces and main theorems

As mentioned in the introduction, there are too many constructions for building new CAT(0) spaces from existing ones (one example is discussed in §2.5). We are therefore forced to restrict our attention to some subfamily. It turns out that the class of CAT(0) spaces equipped with some geometric group action is large enough to be interesting, but not too large to be unmanageable. In particular, a CAT(0) group is a direct generalization of a fundamental group of a compact Riemannian manifold M of non-positive sectional curvature, acting by deck transformations on the universal cover $\widetilde{M}$. Here we collect some facts about CAT(0) spaces, admitting a geometric group action, which are needed in the sequel.

Theorem 2.2 (**Flat torus theorem**)**.** *Let X be a CAT(0) space and $\Gamma\overset{geo}{\curvearrowright}X$. If there is $A\leq\Gamma$ such that $A\cong\mathbb{Z}^n$, then there is $\mathbb{R}^n$ isometrically embedded into X. Moreover, A acts cocompactly on some n-dimensional flat in X by translations.*

To prove this theorem, we have to divide isometries of CAT(0) spaces into three classes, and associate to each representative of each class some geometrical data. First, we define the **translation length** $|g|$ for isometry g as a quantity $\inf\{d(x,gx)|x\in X\}$, roughly speaking the smallest amount for which g moves points in X. Furthermore, we define $\mathfrak{min}(g)$, the **minimal space of** g, as the set of all points in X, moved by g for $|g|$, i.e. $\mathfrak{min}(g)=\{x\in X|d(x,gx)=|g|\}$. According to those two terms, we first divide isometries of X into two classes – **semi-simple** isometries are those with nonempty minimal space and **parabolic** isometries are all the rest. We further split semi-simple isometries into **elliptic** ones, i.e. those with zero translation length (equivalently, with fixed points) and those with nonzero translation length, called **hyperbolic** isometries.

Lemma 2.3. *Let $\Gamma\overset{geo}{\curvearrowright}X$, where X is CAT(0) space. Every element of Γ acts on X as a semi-simple isometry. Every hyperbolic isometry $g\in\Gamma$ has $\mathfrak{min}(g)\cong\mathbb{R}\times C$ (the right-hand side carries ℓ^2-metric) for some CAT(0) space C and g acts on $\mathfrak{min}(g)$ with translation on the $\mathbb{R}$-part of the decomposition and trivially on the C-part.*

Definition 2.4. *With the notation from the lemma above, a geodesic line $\mathbb{R}\times\{c\}$ for $c\in C$ is called an* axis *of g.*

Sketch of the proof. Suppose there is a parabolic isometry $g\in\Gamma$. Observe first that

$$|h^{-1}gh|=\inf\{d((h^{-1}gh)x,x)|x\in X\}=\inf\{d(g(hx),hx)|x\in X\}=|g|$$

for every $h\in\mathfrak{Iso}(X)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X such that $d(gx_n,x_n)<|g|+n^{-1}$. Pick elements $g_n\in\Gamma$ such that $y_n:=g_nx_n$ remains in a compact set $K\subseteq X$ and let x be some accumulation point of y_n. Since $d((g_ngg_n^{-1})y_n,y_n)=d(gx_n,x_n)<|g|+1$ for all $n\in\mathbb{N}$ and the action of Γ on X is proper and Γ is discrete, it follows that, after passing to a subsequence, $g_ngg_n^{-1}$ is a constant sequence, all terms equal to, let say, $g'\in\Gamma$. But g' is conjugated to g and hence has the same translation length as g and $d(g'y_n,y_n)\overset{n\to\infty}{\longrightarrow}|g|$. Moreover, y_n converges (after passing to a further subsequence) to x, hence $d(g'x,x)=|g|$, contradicting parabolicity of g (and hence g').

To deduce the product decomposition of $\mathfrak{min}(g)$ for g hyperbolic, observe first that for $x\in\mathfrak{min}(g)$, $[g^{-1}x,x]\cup[x,gx]$ equals $[g^{-1}x,gx]$. If not, the midpoint x' of $[g^{-1}x,x]$ and the midpoint $x''=gx'$ of $[x,gx]=g[g^{-1}x,x]$ satisfies $d(x',gx')\leq\frac{1}{2}d(x,g^2x)<\frac{1}{2}\left(d(g^{-1}x,x)+d(x,gx)\right)=|g|$, contradiction. Hence, $\gamma_x:=\bigcup_{n\in\mathbb{Z}}g^n[x,gx]$ is a geodesic line. For any other point $y\in\mathfrak{min}(g)$, the line $\gamma_y:=\bigcup_{n\in\mathbb{Z}}g^n[y,gy]$ is parallel to γ_x in the sense that the Hausdorff distance (Definition 3.10) between them is finite. By the *flat strip theorem*, see [BH, Theorem II.2.13], the union of all γ_z with $z\in\mathfrak{min}(g)$ decomposes as a product $\gamma_x\times\mathrm{Pr}_{\gamma_x}^{-1}(x)$, where Pr_Z denotes the closest point projection onto a convex subset Z of a CAT(0) space, see [BH, Proposition II.2.4]. ★

To prove Theorem 2.2, observe that for any hyperbolic isometry $g \in \Gamma$, the centralizer $\mathfrak{Z}_\Gamma(g)$ preserves $\mathfrak{min}(g)$ and respect its product decomposition. In fact, we have the following theorem due to K. Ruane.

Theorem 2.5 (**Cocompact centralizer, [Rua, Theorem 3.2]**). *Let $\Gamma \overset{geo}{\curvearrowright} X$, where X is a CAT(0) space. For arbitrary $\gamma \in \Gamma$, the centralizer $\mathfrak{Z}_\Gamma(\gamma)$ acts geometrically on $\mathfrak{min}(\gamma)$.*

The proof is an elementary game with CAT(0) geometry and proper cocompact group action.

Proof of Theorem 2.2. The theorem is true for $n = 1$ by the lemma above. Suppose by induction that it is true for $n-1$ and moreover, $\mathfrak{min}(A) = \{x \in X | d(x, ax) = |a| \; \forall a \in A\} = \mathbb{R}^{n-1} \times C$, where every element in A acts as a translation on $\mathbb{R}^{n-1}$-part and trivially on C-part (include this as an induction hypothesis). Add another element t to A such that $\langle A, t\rangle \cong \mathbb{Z}^n$. Since t commutes with A, it preserves $\mathfrak{min}(A)$ and its product decomposition, acting as a translation on $\mathbb{R}^{n-1}$ (see [BH, Theorem 6.8(5)]). Since A already acts cocompactly on $\mathbb{R}^{n-1}$, t must have infinite order, when restricted to C, and is hence a hyperbolic isometry of C. Hence C decomposes as $\mathbb{R} \times C'$. Putting things together, there is $\mathbb{R}^{n-1} \times \mathbb{R} \times \{c'\} \cong \mathbb{R}^n$ on which $\langle A, t\rangle$ acts cocompactly by translations. ★

Surprisingly, the converse of the flat torus theorem is widely open. We are going to enlighten it through the following notion.

2.3 Hyperbolicity

Recall the definition of a (δ-)hyperbolic metric space from the introduction. As we have already mentioned, there is a very nice criterion for distinguishing hyperbolic spaces among proper cocompact CAT(0) spaces. Note that in fact there is no need for existence of Γ acting geometrically on X, just that the full isometry group $\mathfrak{Iso}(X)$ acts cocompactly.

Theorem 2.6 (**[BH, Theorem II.9.33]**). *Let X be a proper cocompact CAT(0) space. Then X is hyperbolic if and only if there is no isometrically embedded copy of $\mathbb{R}^2$ in it.*

It is clear that a hyperbolic space can not contain a flat plane (i.e a copy of $\mathbb{R}^2$) since an equilateral triangle with sides of length $\frac{4n+1}{\sqrt{3}}$ is not n-thin, and we can choose n arbitrary large.

For the other implication, we have to define the following notion.

Definition 2.7. *A CAT(0) space X is said to be* locally visible *if for every $p \in X$ and every $\varepsilon > 0$ there exists $R = R(p, \varepsilon) > 0$ such that if geodesic $[x, y]$ lies entirely outside the ball $B(p, R)$, then $\measuredangle_p(x, y) < \varepsilon$. We say that X is* uniformly visible *if for every $\varepsilon > 0$, $R(\varepsilon) = \sup\{R(p, \varepsilon) | p \in X\} < \infty$.*

This notion seems highly related to hyperbolicity. If you take a look at a geodesic triangle $\triangle pxy$, you see that if p at distance R from the side $[x, y]$, then points x' and y', which are defined to be the points on sides $[p, x]$ and $[p, y]$ at distance $R - \delta$ from p, must be δ-close (since otherwise, $[p, x]$ would not lie in a δ-neighborhood of the other two sides) and hence the angle at p must be small for R large compared to δ. The following lemmas, which also finish the proof of Theorem 2.6, will explain this notion in detail.

Lemma 2.8. *A proper cocompact CAT(0) space is uniformly visible if and only if it does not contain an isometric copy of $\mathbb{R}^2$.*

Proof. Clearly, if $\mathbb{R}^2$ can be found as a subspace of X, then X is not (uniformly) visible since there are arbitrary large equilateral triangles, hence there is no R for $\varepsilon = \frac{\pi}{3}$. For the opposite implication, suppose that X is not visible. Observe that there exists a pair of points $\xi, \eta \in \partial X$ with $\measuredangle \xi, \eta < \pi$. Indeed, take sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and a point p in X and some $\varepsilon > 0$ such that

- $\measuredangle_p(x_n, y_n) > \varepsilon$;
- $\lim_{n\to\infty} d(p, [x_n, y_n]) = \infty$;
- $\lim_{n\to\infty} x_n = \xi$ and $\lim_{n\to\infty} y_n = \xi'$, $\xi \neq \xi' \in \partial X$.

By the second item, the sequence $(p_n)_{n\in\mathbb{N}}$ of projections of p to $[x_n, y_n]$ is unbounded, hence some subsequence of it converges to $\eta \in \partial X$. Suppose $\eta \neq \xi$ (otherwise, $\eta \neq \xi'$). Since $\measuredangle_{p_n}(p, x_n) \geq \frac{\pi}{2}$, we have that comparison angle $\overline{\measuredangle}_{p_n}(p, x_n) \geq \frac{\pi}{2}$ and hence $\overline{\measuredangle}_p(p_n, x_n) \leq \frac{\pi}{2}$. From the properties of the angular metric (Proposition 2.1(ii)), it follows that $\phi := \measuredangle\xi, \eta \leq \frac{\pi}{2}$. Find a sequence $z_n \in X$ such that $\measuredangle_{z_n}(\xi, \eta) > \phi - \frac{1}{n}$ and a sequence $g_n \in \mathfrak{Iso}(X)$ such that $d(g_n z_n, z_0)$ is bounded. Since X is proper, there is a subsequence of $[g_n z_n, g_n\xi)$ and $[g_n z_n, g_n\eta)$ converging to a pair of geodesic rays $[z, \hat{\xi})$ and $[z, \hat{\eta})$ such that $\measuredangle_z(\hat{\xi}, \hat{\eta}) = \phi = \measuredangle\hat{\xi}, \hat{\eta}$. Hence the convex hull of $[z, \hat{\xi})$ and $[z, \hat{\eta})$ bounds a flat sector. In particular, there are arbitrary large flat disks in X. Invoking cocompactness and properness once again, we find a flat plane $\mathbb{R}^2$ in X. ★

Lemma 2.9. *A proper CAT(0) space X is hyperbolic if and only if it is uniformly visible.*

Proof. Hyperbolic space is uniformly visible by a discussion above. For the opposite implication, take $\delta = R(\frac{\pi}{2})$. Let $\triangle xyz \subseteq X$ be arbitrary and take any point $p \in (x, y)$. At least one of the angles $\measuredangle_p(x, z)$ and $\measuredangle_p(y, z)$ is at least $\frac{\pi}{2}$ and then we can apply uniformly visibility condition for either $\triangle xpz$ or $\triangle ypz$ to see that p is δ-close to $[x, z] \cup [y, z]$. ★

2.4 Flat closing conjecture

Let Γ be a CAT(0) group and X a corresponding CAT(0) space. The *flat closing conjecture* predicts that if X contains a m-dimensional flat, then Γ contains a copy of $\mathbb{Z}^m$ (see [Gro, §6.B$_3$]). According to the previous section, this would imply that Γ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z}^2$. This notorious conjecture remains however open as of today. It holds when X is a real analytic manifold of non-positive sectional curvature by the main result of [BS]. In the classical case when X is a non-positively curved symmetric space, it can be established with the following simpler and well known argument: by [BL, Appendix], the group Γ must contain a so called **$\mathbb{R}$-regular** semi-simple element, i.e. a hyperbolic isometry γ whose axes are contained in a unique maximal flat of X. In particular, $\mathfrak{min}(\gamma)$ coincides with the set of all geodesic lines, parallel to some axis of γ. See Apendix B for further discussion on CAT(0) symmetric spaces. By a lemma of Selberg [Sel], the centralizer $\mathfrak{Z}_\Gamma(\gamma)$ is a lattice in the centralizer $\mathfrak{Z}_{\mathfrak{Iso}(X)}(\gamma)$. Alternatively, this also follows from Theorem 2.5. Since the latter centralizer is virtually $\mathbb{R}^m$ with $m = \operatorname{rank}(X) = \operatorname{rank}(\mathfrak{Iso}(X))$, one concludes that Γ contains $\mathbb{Z}^m$, as desired.

It is tempting to try and mimick that strategy of proof in the case of a general CAT(0) space X: if one shows that Γ contains a hyperbolic isometry γ which is **maximally regular** in the sense that its axes are contained in a unique flat of maximal possible dimension among all flats of X, then the flat closing conjecture will follow as above.

The main result of this thesis ensures a copy of $\mathbb{Z}^m$ in a CAT(0) group, when the underlying CAT(0) space contains a special kind of flats. An obvious way of possessing m-dimensional flats is when the space splits as a product of m geodesically complete factors. We can then take a geodesic line in each factor and the product of them is a m-dimensional flat. Corollary 4.3 states that in this case, there is a copy of $\mathbb{Z}^m$ in Γ.

Recall that even the "trivial case" $m = 1$ requires some work. It was done by Swenson and it carries an important idea, which we use in the proof of existence of a *regular hyperbolic isometry* in a product of cocompact geodesically complete CAT(0) spaces with totally disconnected isometry groups, namely Proposition 4.7.

Theorem 2.10 (**[Swe, Theorem 11]**)**.** *If Γ acts geometrically on a CAT(0) space X, then Γ contains an element of infinite order.*

Note that an element of infinite order in a CAT(0) group is necessary hyperbolic, [BH, Proposition II.6.10]. The same idea as Swenson used to prove the above theorem is used later to prove Proposition 4.7, hence we skip the discussion about Theorem 2.10 here.

2.5 Generalization of geometric group action

A possible generalization of the situation $\Gamma \overset{geo}{\curvearrowright} X$ is a proper CAT(0) space with a proper isometric action of a group G with **full limit set**, i.e. for every point $\xi \in \partial X$, there is a sequence $(g_n)_{n\in\mathbb{N}}$ of group elements such that the sequence $(g_n x)_{n\in\mathbb{N}}$ converges to ξ (for some/any point $x \in X$). Ballmann suggested this approach in [Bal, Definition/Exercise/Question III.1.11]. If $\Gamma \overset{geo}{\curvearrowright} X$, the Tits boundary $\partial_T X$ is finite dimensional (geometric dimension, see [Kle] for definition and proof). When one generalizes the setup as above, the natural question is, which consequences of the stronger assumptions still hold.

Question 2.11 (**[CM-DS, Question 7.1]**). *Assume X is a proper CAT(0) space such that its isometry group has full limit set. Is its Tits boundary finite dimensional?*

In the example below, we will construct a space X satisfying assumptions from the question above and with infinite dimensional Tits boundary, hence it answers the question negatively.

Example 2.12. *Take a real line $R \cong \mathbb{R}$ and at each integer point $z \in R$, glue a $|z|$-dimensional Euclidean space $E_z \cong \mathbb{R}^{|z|}$, identifying $z \in R$ and the origin of E_z. We refer to this space as a base space X_0, it is shown at the left part of Figure 2.1. Next, take infinitely many copies of X_0 indexed by all possible $2k$-tuples (where k runs over $\mathbb{N}_0$) of the form $(n_1, w_1, \dots n_k, w_k)$, where $n_1 \neq n_2 \neq \dots \neq n_k$ (non-consecutive n_i and n_j can be equal), $n_i \in \mathbb{Z}\setminus\{0\}$ and $w_i \in \mathbb{Z}^{|n_i|}\setminus\{(0,\dots 0)\}$. Denote such a copy by $X_0^{(n_1,w_1,\dots n_k,w_k)}$; X_0 itself corresponds to a copy of X_0 indexed by a 0-tuple. Denote by $R^{(n_1,w_1,\dots n_k,w_k)}$ a copy of R in $X_0^{(n_1,w_1,\dots n_k,w_k)}$. For $z \in \mathbb{Z}$, denote by $E_z^{(n_1,w_1,\dots n_k,w_k)}$ a copy of $|z|$-dimensional Euclidean space that is glued at $z \in R^{(n_1,w_1,\dots n_k,w_k)}$. Denote a disjoint union of all $X_0^{(n_1,w_1,\dots n_k,w_k)}$ by $\widetilde{X}$.*

Next, define an equivalence relation on $\widetilde{X}$, generated by the following rule. Say that two points $x, x' \in \widetilde{X}$ are equivalent if for some $k \in \mathbb{N}$, some $n_1 \neq \dots \neq n_k \in \mathbb{Z}$, some $w_i \in \mathbb{Z}^{|n_i|}\setminus\{(0,\dots 0)\}$ we have $x \in E_{n_k}^{(n_1,w_1,\dots n_k,w_k)} \subseteq X_0^{(n_1,w_1,\dots n_k,w_k)}$, $x' \in E_{n_k}^{(n_1,w_1,\dots n_{k-1},w_{k-1})} \subseteq X_0^{(n_1,w_1,\dots n_{k-1},w_{k-1})}$ and $x' = x - w_k$.

The quotient X of $\widetilde{X}$ under this equivalence relation is the space with desired property.

Graphically, the above construction means that at each point $w \in E_{|z|} \subseteq X_0$ with integer coordinates (except at the origin) we glue another copy X_0' of X_0, identifying $E_{|z|}$ and $E'_{|z|}$ via a translation $x' = x - w$. This produces a subspace $X_1 \subseteq X$ consisting of X_0 and all $X_0^{(n_1,w_1)}$. Now, keep growing your space by gluing new copies of X_0 along Euclidean subspaces of $X_0^{(n_1,w_1)}$ (execpt along $E_{n_1}^{(n_1,w_1)}$) to produce a subspace $X_2 \subseteq X$ consisting of X_0, all $X_0^{(n_1,w_1)}$ and all $X_0^{(n_1,w_1,n_2,w_2)}$ ... The final space X equals to the union $\bigcup_{n=0}^{\infty} X_n$.

Obviously, X is a CAT(0) space since it is produced by gluing construction, see [BH, Chapter II, §11]. Since X_0 is proper and every ball in X meets only finitely many copies of X_0 (by construction since $n_i \neq n_{i+1}$ in index $2k$-tuples), the whole space X is proper. To see that $\mathfrak{Iso}(X)$ has full limit set, take any geodesic ray $r : [0,\infty) \to X$ with initial point $r(0) = 0 \in R$. By construction of X, for any $t > 0$, there is $T > t$ and $w \neq (0,\dots 0)$ with integer coordinates in some $E_n^{(n_1,w_1,\dots n_{k-1},w_{k-1})} \subseteq X_0^{(n_1,w_1,\dots n_{k-1},w_{k-1})}$, $n \neq n_{k-1}$, not contained in the image of r, such that $d(w, r(T)) \leq 1$. We can apply an isometry $\alpha \in \mathfrak{Iso}(X)$ which maps X_0 to $X_0^{(n_1,w_1,\dots n_{k-1},w_{k-1},n,w)}$. Now the geodesic segment $[r(0), \alpha r(0)]$ is 1-close to a geodesic ray r up to time t. Since t was arbitrary and the metric on X is convex, we are done. Since there are round spheres of arbitrary high dimensions in $\partial_T X$, the Tits boundary of X is not finite dimensional.

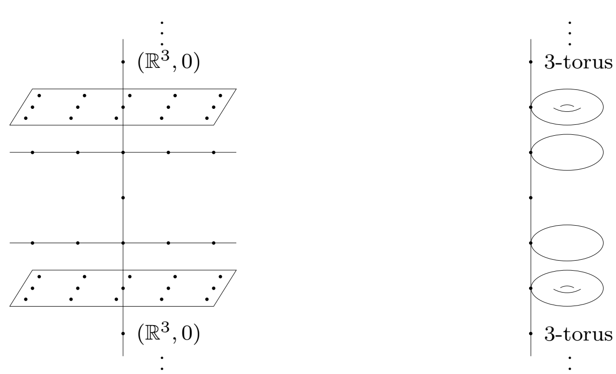

Figure 2.1: Base space X_0 and the fundamental domain for $\mathfrak{Iso}(X)$-action

Chapter 3

Structure theory for CAT(0) space's isometry group

A remarkably deep properties about nice CAT(0) spaces and their isometry groups can be deduced using an additional tool – the structure theory of locally compact topological groups. One can equip the full isometry group G of a proper metric space with a compact open topology (Definition A.1), which turns G into a locally compact Hausdorff topological group (Theorem A.2). For a proper cocompact irreducible geodesically complete non-Euclidean CAT(0) space, Theorem 3.16 presents the dichotomy for its isometry group – either it is a semi-simple virtually connected Lie group with a trivial center or is a totally disconnected topological group. In the later case, there are some theorems (e.g. Alexandrov angle rigidity, open stabilizers) describing further properties of isometries – they behave much like in a discrete case, while in the Lie group case, the situation is quite well understood already from older theory (see [Pra, PR, Sel]). In the next sections, we are going to present some results of this spirit, which are mostly due to [CM-ST, CM-DS] and which we will use later to prove a special case of the *flat closing conjecture*.

According to Theorem A.2, isometry group of a proper CAT(0) space X is a locally compact Hausdorff topological group. Under additional assumptions on X, we will deduce finer structure results about $\mathfrak{Iso}(X)$.

3.1 Decompositions

The following theorem due to Caprace and Monod, [CM-ST, Theorem 1.1], classify the factors of a nice CAT(0) space.

Theorem 3.1. *Let X be a proper, cocompact and geodesically complete CAT(0) space such that $\mathfrak{Iso}(X)$ has no global fixed points at infinity. Then there is a canonical (unique, preserved by all isometries) splitting*

$$X = \mathbb{R}^n \times M \times Y,$$

(each of the factors may be trivial) with M a symmetric space of noncompact type and Y a CAT(0) space with totally disconnected isometry group. $\mathfrak{Iso}(M)$ is a semi-simple Lie group and

$$\mathfrak{Iso}(X) = \mathfrak{Iso}(\mathbb{R}^n) \times \mathfrak{Iso}(M) \times \mathfrak{Iso}(Y).$$

We will omit the proof in full generality, but rather present simpler arguments, which are enough for our setting in the *flat closing conjecture*. Interested reader can find all the details in [CM-ST, Theorems 1.1 and 1.6]. For the purposes of this thesis, we will refer to the solution of Hilbert's fifth problem (see Appendix A.3), which goes back to Montgomery and Zippin, [MZ2]. After accepting it and some other general result on geodesic metric spaces due to Förtsch and Lytchak [FL], here collected in Theorem 3.4, we deduce the proof of our version of the flat closing conjecture in more straight-forward way than originally in [CZ]. We proceed with giving terminology and stating necessary theorems in appropriate forms.

Definition 3.2. *Let X be a geodesic metric space. An* Euclidean piece *in X is an image of an isometric embedding of a convex subset of a real Hilbert space into X. To each affine piece $\varphi(K)$, where K is a* *convex subset of a real Hilbert space and* $\varphi : K \to X$ *is an isometric embedding, we can associate its dimension as an algebraic dimension of the linear span of* K*. An* Euclidean rank of a geodesic metric space *is the supremum of dimensions of Euclidean pieces in* X*.*

Remark 3.3. *The dimension of a Euclidean piece* P *is well-defined quantity since it equals to the supremum of the number* n *of points* $x_1, \dots x_n \in P$ *for which there exist* $x_0 \in P$ *and* $d > 0$ *such that* x_i *is at distance* d *from* x_0 *and all the distances between two different* x_i, x_j *are* $d\sqrt{2}$*.*

The following theorem about the de Rham docomposition is originally stated for general geodesic metric spaces of *finite affine rank*, see [FL, Theorem 1.1 and Corollary 1.3], i.e. in definition above we allow pieces of X to be subspaces of X isometric to a convex subsets of normed real vector spaces. But we do not need that generality because affine pieces in CAT(0) spaces are in fact Euclidean. This follows from the characterization of Hilbert spaces as Banach spaces where the parallelogram inequality is equality, see note [AN], and the fact that the strict parallelogram inequality violates CAT(0) inequality.

Theorem 3.4 (**The de Rham decomposition**)**.** *Let* X *be a CAT(0) space of finite Euclidean rank. Then there is a unique decomposition*

$$X \cong \mathbb{R}^n \times X_1 \times \cdots \times X_m, \quad n, m \geq 0,$$

where $\cong$ *denotes the isometry for the right-hand side equipped with* ℓ^2*-metric. Each* X_i *is nontrivial, non-isometric to the real line and indecomposable in that way. Furthermore, the isometry group of* X *is a finite extension of the direct product of* $\mathfrak{Iso}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$ *and the* $\mathfrak{Iso}(X_i)$*'s for* $i = 1, \dots m$*.*

Analysis of $\mathfrak{Iso}(X_i)$ from the theorem above will refer to the solution of Hilbert's fifth problem. Further results will be deduced using finer structure of X (namely, geodesic completeness and presence of a group acting on X geometrically) in the next sections. We conclude this section with the proposition which ensures that in our setting, the assumptions of Theorem 3.4 are satisfied.

Proposition 3.5. *Let* X *be a geodesically complete CAT(0) space and let* $\Gamma \overset{geo}{\curvearrowright} X$*. Then* X *has finite Euclidean rank.*

Proof. We have to bound the dimension of pieces in X. Let $n \in \mathbb{N}$ be any number such that there is a positive constant $d \leq 1$ and points $x_0, x_1, \dots x_n \in X$ with $d(x_0, x_i) = d$ for all $i = 1, \dots n$ and $d(x_i, x_j) = d\sqrt{2}$ for all $i, j = 1, \dots n$ with $i \neq j$. By geodesic completeness, we can extend geodesic segment $[x_0, x_i]$ over x_i and choose a point y_i on it at distance 1 from x_0. By CAT(0) inequality, we have $d(y_i, y_j) \geq \sqrt{2}$ for all $i, j = 1, \dots n$ with $i \neq j$. On the other hand, there is a bound on the number of points in any ball of radius 1, which are pairwise at least $\sqrt{2}$ apart, by the properness and cocompactness of X. Hence Euclidean rank is bounded from above. ★

3.2 Minimality

As cocompactness forces the isometry group (or its subgroup) to be large enough, there are also another notions of largeness of a group acting on some space. In what follows, we will use minimality.

Definition 3.6. *Let* $G \leq \mathfrak{Iso}(X)$*, where* X *is CAT(0) (or, more generally, any geodesic metric) space. We say that* G *acts* minimally *if there is no proper nonempty closed convex* G*-invariant subset in* X*.*

In particular, the closed convex hull $\overline{\mathrm{conv}(Gx)}$ of any G-orbit equals to the whole X. The following proposition is a special case of [CM-ST, Proposition 1.5] and [CM-DS, Theorem 3.11].

Proposition 3.7. *Let* $\Gamma \overset{geo}{\curvearrowright} X$*, where* X *is a geodesically complete CAT(0) space. Then* $\mathfrak{Iso}(X)$ *(and also* Γ*) acts minimally. Furthermore,* $\mathfrak{Iso}(X)$ *has no fixed points at* ∂X*.*

Proof. Suppose for contradiction that Γ does not act minimally. Then there is a Γ-invariant closed convex subset $Y \subsetneq X$. Pick $x \in X \setminus Y$ and let x' be the projection of x on Y. Extend the geodesic segment $[x', x]$ over x to get a ray $r : [0, \infty) \to X$ with $r(0) = x'$. Observe that $d(r(t), Y) = d(r(t), x') = t$ since otherwise we get a triangle with two right (or even bigger) angles, see [BH, Proposition II.2.4(3)], which is an absurd. Hence there are points arbitrary far away from Γ-orbit of any point from Y, which contradicts the cocompactness. Since $\mathfrak{Iso}(X)$ is even bigger group than Γ, it also acts minimally.

Next part of the proposition follows by similar, but easier arguments as in [CM-DS, Theorem 3.11]. Suppose for contradiction that there is a point $\xi \in \partial X$, fixed by $\mathfrak{Iso}(X)$. In particular, ξ is Γ-fixed. Take any geodesic line $c : \mathbb{R} \to X$ with $c(\infty) = \xi$ (it exists since X is geodesically complete) and denote $x_0 := c(0)$. Let $\gamma_n \in \Gamma$ be elements such that $d(\gamma_n c(-n), x_0)$ is bounded. Let $\{s_1, \dots s_m\}$ be a generating set of Γ. Recall that Γ is finitely generated by Švarc-Milnor Lemma, [BH, Proposition I.8.19]. Since $c(\infty)$ is Γ-fixed, the set $\{d(\gamma_n^{-1} s_i \gamma_n x_0, x_0) | n \in \mathbb{N}\}$ is bounded for every $i = 1, \dots m$. Since Γ acts properly and is discrete, the sequence $(\gamma_n^{-1} s_i \gamma_n)_{n \in \mathbb{N}}$ is (up to passing to a subsequence) constant for every $i = 1, \dots m$. Pick $\widetilde{\gamma} \in \Gamma$ such that $\gamma_n^{-1} s_i \gamma_n = \widetilde{\gamma}^{-1} s_i \widetilde{\gamma}$ for all $n \in \mathbb{N}$ and all $i = 1, \dots m$. In particular, $\widetilde{\gamma}^{-1} s_i \widetilde{\gamma} c$ is at a bounded distance from c, hence parallel to c. Since s_i's generate Γ, $\widetilde{\gamma}^{-1} \gamma \widetilde{\gamma} c$ and hence γc is at bounded distance from c for every $\gamma \in \Gamma$. Let P be the union of all geodesics parallel to c. By what is written above, P is Γ-invariant and by minimality of the Γ-action, it must be the whole X. By [BH, Theorem 2.14], $X = P \cong \mathbb{R} \times X'$ with ξ being the endpoint of the $\mathbb{R}$-factor. By the uniqueness from Theorem 3.4, the factor $\mathbb{R}$ must be a subspace of $\mathbb{R}^n$. But the group $\mathbb{R}^n \rtimes O(n) \leq (\mathbb{R}^n \rtimes O(n)) \times \mathfrak{Iso}(X_1) \times \dots \times \mathfrak{Iso}(X_m) \leq \mathfrak{Iso}(X)$ has no fixed point at infinity because it contains reflection across the origin, $\vec{x} \mapsto -\vec{x}$, which is fixed point free at $\partial \mathbb{R}^n$. This finishes the proof. ★

Remark 3.8. *Note that for minimality of the action of $\mathfrak{Iso}(X)$ the assumption of the existence of a group Γ acting geometrically on X is not necessary. However, non-existence of $\Gamma \overset{geo}{\curvearrowright} X$ might cause presence of $\mathfrak{Iso}(X)$-fixed points at ∂X, as shown by an example in [Hei].*

In [CM-ST, Theorem 1.10] it is stated that under the assumptions of Proposition 3.7 every nontrivial normal subgroup $N \trianglelefteq \mathfrak{Iso}(X)$ also acts minimally and without fixed points at ∂X. This property is called **geometric density** (of normal subgroups). For our purposes, it is unnecessary for N to be without fixed points in ∂X. A proof of the proposition below uses some nontrivial results from [BaL, Kle].

Proposition 3.9. *For an irreducible CAT(0) space satisfying conclusions of Proposition 3.7, any normal subgroup N of $\mathfrak{Iso}(X)$ still acts minimally or it is trivial.*

In the proof of that proposition and for Lemma 3.12, we will need the following definition. Recall that for a subset A of a metric space, $\mathcal{N}_r(A)$ denotes the union all open balls of radii r with centers in A.

Definition 3.10. *Let X be a metric space. The* Hausdorff distance *is a map $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ that associates to each pair $A, B \subseteq X$ the number $\inf\{r \in [0, \infty) | A \subseteq \mathcal{N}_r(B)$ and $B \subseteq \mathcal{N}_r(A)\}$; infimum of the empty set is ∞.*

Most often, it is only important for us whether the Hausdorff distance between two subsets of a metric space is finite or not.

Proof of Proposition 3.9. Suppose first that there is no minimal closed convex N-invariant subset of X. Then there is a descending chain of closed convex N-invariant subsets $Z_1 \supseteq Z_2 \supseteq \dots$ with $\bigcap_{m \in \mathbb{N}} Z_m = \emptyset$. Fix a base point $x_0 \in X$ and let $x_m := \mathrm{Pr}_{Z_m}(x_0)$, the nearest point projections of x_0 onto Z_m. Passing to a subsequence, we may assume that $(x_m)_{m \in \mathbb{N}}$ converges to $\xi \in \partial X$. First observe that ξ does not depend on the choice of x_0 since for some other base point at distance R from x_0, its projection to Z_m is at distance at most R from x_m (since projection on a closed convex subset of a CAT(0) space is 1-Lipschitz map). We claim that ξ is $\mathfrak{Iso}(X)$-fixed. To see that, let $g \in \mathfrak{Iso}(X)$ be arbitrary. Since N is normal in $\mathfrak{Iso}(X)$, the set gZ_m is also N-invariant subset of X for every $m \in \mathbb{N}$ and $g \in \mathfrak{Iso}(X)$. Without loss of generality, we may assume that $Z_m = \overline{\mathrm{conv}(N z_m)}$ for some point $z_m \in Z_m$. Hence

$$gZ_m = g\overline{\mathrm{conv}(N z_m)} = \overline{\mathrm{conv}(gN z_m)} = \overline{\mathrm{conv}(g(g^{-1}Ng) z_m)} = \overline{\mathrm{conv}(N g z_m)}.$$

But the Hausdorff distance between Ngz_m and Nz_m is finite (it is at most $d(gz_m, z_m)$), hence the Hausdorff distance $d_H(Z_m, gZ_m)$ is finite by Lemma 3.12 below. Hence Z_m and gZ_m have the same boundary at infinity, which means that g preserves ∂Z_m for any m. In particular, $\mathfrak{Iso}(X)$ preserves $\bigcap_{m \in \mathbb{N}} \partial Z_m$. But the circumradius (in the metric $\sphericalangle$) of this intersection is at most $\pi/2$; indeed, for any $\eta \in \bigcap_{m \in \mathbb{N}} \partial Z_m$, we can choose a sequence $y_m \in Z_m$ such that y_m converges to η. By Proposition 2.1(ii), $\sphericalangle \xi, \eta = \lim_{m \to \infty} \overline{\sphericalangle}_{x_0}(x_m, y_m)$, but each of the angles appearing in the limit is smaller than $\pi - \sphericalangle_{x_m}(x, y_m) \leq \pi/2$ by [BH, Proposition II.2.4(3)]. Since X is equipped with a geometric group action, its Tits boundary is finite dimensional by [Kle], hence we can apply [BaL, Proposition 1.4] to ensure the canonical circumcenter of $\bigcap_{m\in\mathbb{N}} \partial Z_m$, which is $\mathfrak{Iso}(X)$-fixed point on ∂X. This is a contradiction with Proposition 3.7.

Let now Z be a minimal closed convex N-invariant subset of X. As above, each gZ for $g \in \mathfrak{Iso}(X)$ is at finite Hausdorff distance from Z and is still minimal closed convex N-invariant subset of X. Let $\mathcal{Z}$ be the family of all minimal closed convex subsets of X at finite Hausdorff distance from Z. We claim that the function $d(\cdot, Z)$ is constant on each $Z' \in \mathcal{Z}$. If it is not, then there exists $r \in [0,\infty)$ such that $Z'_r := \{z' \in Z' | d(z', Z) \le r\}$ is a proper subset of Z'. Since the metric d is convex, Z'_r is closed convex subset of Z'. It is clearly N-invariant since moving elements from Z' by isometries from N does not change the distance to Z. But this contradicts minimality of Z'.

Let $x_0 \in Z$ continue to denote a base point. Let Y be a subset of X consisting of the nearest point projections of x_0 to elements of $\mathcal{Z}$. By the Sandwich lemma (see [BH, Exercise II.2.12(2)]), $\mathrm{conv}(Z_1 \cup Z_2)$ is isometric to $Z \times [0, d_H(Z_1, Z_2)]$ for any pair $Z_1, Z_2 \in \mathcal{Z}$. Hence Y is convex and $\bigcup \mathcal{Z}$ is isometric to $Z \times Y$. As in the first paragraph, we find out that $\bigcup \mathcal{Z}$ is $\mathfrak{Iso}(X)$-invariant. By minimality, it is the whole space X. Since X is irreducible, either Z or Y is trivial. In the first case, N acts trivially on X and is hence trivial. In the later case, $X = Z$ and hence N acts minimally. ★

Remark 3.11. *The same proof as above works in the situation where we take a normal subgroup N of any subgroup $G \le \mathfrak{Iso}(X)$, acting minimally and without fixed points at infinity instead of the whole $\mathfrak{Iso}(X)$ in Proposition 3.9.*

Lemma 3.12. *Let U and W be two subsets of a CAT(0) space X at finite Hausdorff distance. Then the Hausdorff distance between $\overline{\mathrm{conv}(U)}$ and $\overline{\mathrm{conv}(W)}$ is also finite.*

Proof. We use the following characterization of a convex hull of a set U. Let $U_0 := U$ and $U_n := \bigcup_{x,y\in U_{n-1}} [x,y]$ for $n \ge 1$. Then $\mathrm{conv}(U) = \bigcup_{n\in\mathbb{N}} U_n$.

Since the metric on CAT(0) space is convex, it follows that

$$d_H(U_n, W_n) \le d_H(U_{n-1}, W_{n-1}) \le \cdots \le d_H(U, W) \text{ for all } n \in \mathbb{N}.$$

This can be proven by induction. For any $u \in U_n$, there exist $x, y \in U_{n-1}$ such that $u \in [x,y]$. Let $x', y' \in W_{n-1}$ be points such that $d(x,x'), d(y,y') \le d_H(U_{n-1}, W_{n-1})$. By convexity of the metric on CAT(0) space, there is u' on $[x',y'] \subseteq W_n$ with $d(u,u') \le \max\{d(x,x'), d(y,y')\}$ and hence $U_n \subseteq \mathcal{N}_{d_H(U_{n-1},W_{n-1})}(W_n)$. By symmetry, $d_H(U_n, W_n) \le d_H(U_{n-1}, W_{n-1})$.

We conclude that $d_H(\mathrm{conv}(U), \mathrm{conv}(W)) \le d_H(U,W)$. Since the closure does not change the Hausdorff distance, we are done. ★

The next step on the way to the dichotomy requires the notion of the amenable and the solvable radical of a locally compact topological group, which is (as well as a definition of amenability) given in §A.2. Some properties of the behavior of amenable subgroups in CAT(0) context are taken from [AB].

Corollary 3.13. *Under the assumptions of Proposition 3.9, N has trivial amenable radical.*

Proof. If N is trivial, it has trivial amenable radical. Otherwise N acts minimally. Suppose that the amenable radical A of N is nontrivial. The main result of Adams and Ballmann, namely [AB, Theorem], states that A either preserves a flat F in X or fixes a point at ∂X. The former case is impossible. To see this, observe that the set of all flats F' at finite Hausdorff distance from F splits as a product $F \times Y$ (see the last part of the proof of Proposition 3.9). Since A is a normal subgroup of N, the later space is N-invariant, hence equals X by minimality of the N-action. But X is indecomposable (hence Y is a point) and not isometric to a flat, a contradiction.

The second case (A-fixed point at ∂X) can be excluded as follows. By amenability of A, there is an A-invariant probability measure μ on ∂X (see Appendix A.2). From the proof of [AB, Theorem], we deduce that the support of μ is contained in the set of *flat points*. In geodesically complete case, the set of flat points coincide with the boundary of the Euclidean factor. Since X is non-Euclidean and irreducible, the set of flat points is empty, hence the support of a probability measure μ is empty, which is absurd. Hence the amenable radical of $\mathfrak{Iso}(X)$ is trivial.

Since every solvable group is amenable, the solvable radical of $\mathfrak{Iso}(X)$ is trivial as well. ★

Proposition 3.14. *For an irreducible CAT(0) space X, not isometric to $\mathbb{R}$, any group $G \le \mathfrak{Iso}(X)$ acting minimally on X has trivial centralizer $\mathfrak{Z}_{\mathfrak{Iso}(X)}(G)$.*

Proof. Suppose there is a nontrivial $g \in \mathfrak{Iso}(X)$, commuting with each element of G. Then the function $\tau_g : X \to [0, \infty)$, defined by $\tau_g(x) = d(x, gx)$ is clearly G-invariant. If there are $0 \le r < r'$ in the image of τ_g, then $\tau_g^{-1}([0, r])$ is closed (since τ_g is continuous) convex (since metric on X is convex) G-invariant (since τ_g is G-invariant) subset of X. This contradicts the minimality of the G-action. Hence τ_g must be constant, in particular, g is a Clifford translation [BH, Definition II.6.14]. By [BH, Theorem II.6.15(1)], X splits as $\mathbb{R} \times X'$. Since X is not a real line, X' is nontrivial, which contradicts irreducibility of X. ★

3.3 Dichotomy for the full isometry group

To prove the dichotomy for the isometry group of a cocompact, geodesically complete CAT(0) space, admitting some geometric action of a discrete group, we refer to the following well known results.

Lemma 3.15. *An outer automorphism group of a Lie group with trivial amenable radical is finite.*

Proof. See [Mon, Theorem 11.3.4] ★

Theorem 3.16 (**[Cap, Corollary III.3]**). *Let X be a geodesically complete CAT(0) space admitting some geometric group action. Then $\mathfrak{Iso}(X)$ is virtually a product of $\mathbb{R}^n \rtimes O(n)$, virtually connected semi-simple Lie groups without compact factors and totally disconnected topological groups (which may as well be discrete).*

Proof. By Theorem 3.4, X splits as $\mathbb{R}^n \times X_1 \times \cdots \times X_m$ with X_i indecomposable and $\mathfrak{Iso}(X)$ virtually splits as $(\mathbb{R}^n \rtimes O(n)) \times \mathfrak{Iso}(X_1) \times \cdots \times \mathfrak{Iso}(X_m)$. Hence Γ acting geometrically on X has a finite index subgroup Γ', which respects the product decomposition. As deduced at the second part of the proof of Proposition 3.7, Γ'-fixed points at ∂X are contained at the boundary of the Euclidean factor. Hence induced Γ'-action on each ∂X_i is fixed point free. Note also that each X_i is minimal (i.e. $\mathfrak{Iso}(X_i)$ acts minimally) since minimality of the $\mathfrak{Iso}(X)$-action passes to a finite index subgroup and hence to factors. Hence Proposition 3.9 applies. Let G_i be the identity component of $\mathfrak{Iso}(X_i)$. It is closed normal subgroup of $\mathfrak{Iso}(X_i)$, hence by Proposition 3.9, it is either trivial in which case $\mathfrak{Iso}(X_i)$ is totally disconnected or it acts minimally on X_i. To see that in the later case, G_i is a Lie group, it is enough to prove that there is no non-trivial compact normal subgroup of G_i and then apply Theorem A.6. Suppose that $K \trianglelefteq G_i$ is compact. Then K has nontrivial fixed point set in X_i, which is G_i-invariant by normality. But G_i acts minimally, hence fixed point set of K must be the whole X_i. In particular, $K = \{1\}$. This also proves the absence of compact factors in G_i.

Since G_i has trivial solvable and amenable radical by Corollary 3.13, G_i is semi-simple by Definition A.9. By Lemma 3.15, $\mathrm{Out}(G_i)$ is finite. Hence the kernel of $\phi : \mathfrak{Iso}(X_i) \to \mathrm{Out}(G_i)$ (conjugation) is a normal subgroup of $\mathfrak{Iso}(X_i)$ of finite index. It is clear that $\mathfrak{Ker}(\phi) = G_i \cdot \mathfrak{Z}_{\mathfrak{Iso}(X_i)}(G_i)$. By Proposition 3.14, $\mathfrak{Z}_{\mathfrak{Iso}(X_i)}(G_i) = \{1\}$, hence $\mathfrak{Ker}(\phi) = G_i$ is a finite index subgroup of $\mathfrak{Iso}(X_i)$. ★

3.4 Totally disconnected isometry group

The behavior of a totally disconnected isometry group of a CAT(0) space, similar to discrete groups, will be an important tool in the proof of a version of the flat closing conjecture for reducible CAT(0) spaces. To deduce all the properties we need, we have to assume geodesic completeness of a space. We are going to prove the three important theorems, [CM-ST, Theorem 6.1, Corollary 6.3(iii) and Proposition 6.8].

Theorem 3.17 (**Open stabilizers**). *Let X be a proper cocompact geodesically complete CAT(0) space with a totally disconnected isometry group. Then the stabilizer of every bounded subset of X is open in $\mathfrak{Iso}(X)$.*

Proof. By Proposition 3.7 and Remark 3.8, $\mathfrak{Iso}(X)$ acts minimally. Let C denote the set of all points in X having an open stabilizer in $\mathfrak{Iso}(X)$. By Theorem A.7, there exists a compact open subgroup $K \le \mathfrak{Iso}(X)$, which obviously fixes a point of X, hence C is nonempty. Furthermore, C is $\mathfrak{Iso}(X)$-invariant since $\mathrm{Stab}_{\mathfrak{Iso}(X)}(gx) = g\mathrm{Stab}_{\mathfrak{Iso}(X)}(x)g^{-1}$ and conjugation by $g \in \mathfrak{Iso}(X)$ is a homeomorphism of $\mathfrak{Iso}(X)$. Given two points $x, y \in C$, $\mathrm{Stab}_{\mathfrak{Iso}(X)}(x)$ and $\mathrm{Stab}_{\mathfrak{Iso}(X)}(y)$ are compact open subgroups and hence $S_{x,y} := \mathrm{Stab}_{\mathfrak{Iso}(X)}(x) \cap \mathrm{Stab}_{\mathfrak{Iso}(X)}(y)$ is a compact open subgroup. Note that $S_{x,y}$ fixes x and y, hence fixes the whole geodesic segment $[x, y]$. Thus C is convex. By minimality of the $\mathfrak{Iso}(X)$-action, it is dense.

By Corollary A.8, C can be expressed as a union of increasing chain of closed convex sets $(C_n)_{n\in\mathbb{N}}$, where C_n is a fixed point set of $\bigcap_{i=1}^n K_i$, where $(K_n)_{n\in\mathbb{N}}$ is a basis of neighborhoods of the identity of $\mathfrak{Iso}(X)$, consisting of compact open subgroups. By the following lemma, C equals X and every bounded subset is contained in some C_n, hence has open stabilizer. ★

Lemma 3.18 (**[CM-ST, Lemma 6.4]**)**.** *Let $C_1 \subseteq C_2 \subseteq \dots$ be an increasing chain of closed convex subsets of a proper geodesically complete CAT(0) space X, whose union is dense in X. Then every bounded subset of X is contained in C_n for some n.*

Proof. Suppose for contradiction that there is a bounded subset $B \subseteq X$, which contains point y_n outside C_n for every $n \in \mathbb{N}$. Since B is bounded and the sequence C_n is increasing, there is a bound $R < \infty$ such that $d(y_n, C_n) \leq R$ for all $n \in \mathbb{N}$. We will construct a sequence z_n of points in some neighborhood of B with $d(z_n, z_m) \geq 1$ for all $m \neq n$. Let x_n be the projection of y_n to C_n and let $r_n : [0, \infty) \to X$ be a geodesic ray emanating from x_n through y_n. Notice that $d(r_n(t), C_n) = t$. Since $\bigcup_{n\in\mathbb{N}} C_n$ is dense in X, for each $x \in X$ there is n such that $d(x, C_n) < 1$.

Let $z_1 = r_1(2)$ and $n_1 = 1$. Suppose we have constructed $z_1, \dots z_k$ with the following properties. There is a sequence $1 = n_1 < \dots < n_k$ such that $d(z_i, C_{n_i}) = 2$ for $i = 1, \dots k$ and $d(z_i, C_{n_{i+1}}) < 1$ for $i = 1, \dots k-1$. Let n_{k+1} be the natural number such that $d(z_k, C_{n_{k+1}}) < 1$ and let $z_{k+1} = r_{n_{k+1}}(2)$. We add a new term z_{k+1} to our sequence, still satisfying required properties. Observe that by the construction, for $i < j$ we have

$$d(z_i, z_j) \geq d(z_j, C_{n_j}) - d(z_i, C_{n_j}) \overset{C_{n_{i+1}} \subseteq C_{n_j}}{\geq} 2 - d(z_i, C_{n_{i+1}}) \geq 1$$

and that

$$d(z_i, B) \leq d(z_i, y_{n_i}) \leq d(z_i, x_{n_i}) + d(x_{n_i}, y_{n_i}) \leq 2 + R.$$

Hence assuming that $B \nsubseteq C_n$ for every $n \in \mathbb{N}$, we can construct arbitrary many points in $\mathcal{N}_{R+2}(B)$ at pairwise distances at least 1. But this contradicts properness of X. ★

Theorem 3.19 (**Semi-simple isometries**)**.** *Let X be a proper cocompact geodesically complete CAT(0) space with a totally disconnected isometry group. Then every element of $\mathfrak{Iso}(X)$ acts on X as a semi-simple isometry.*

Proof. Suppose for a contradiction that there is a parabolic $g \in \mathfrak{Iso}(X)$. Let $x_n \in X$ be a sequence such that $d(gx_n, x_n) < |g| + n^{-1}$. Since $\mathfrak{Iso}(X)$ is cocompact, we can take a sequence of $g_n \in \mathfrak{Iso}(X)$ such that $y_n := g_n x_n$ lies in a prescribed compact set $K \subseteq X$. Then $g_n g g_n^{-1}$ is a sequence that moves $y_n \in K$ for less than $|g| + n^{-1} = |g_n g g_n^{-1}| + n^{-1}$. By properness of the $\mathfrak{Iso}(X)$-action, it sub-converges to some $g' \in \mathfrak{Iso}(X)$ with $d(g'y, y) = |g'| = \lim_{n\to\infty} |g_n g g_n^{-1}| = |g|$ for an accumulation point y of a sequence $(y_n)_{n\in\mathbb{N}}$. By Theorem 3.17, the stabilizer $S_y \leq \mathfrak{Iso}(X)$ of y is open, hence $g'S_y$ contains all but finitely many $g_n g g_n^{-1}$. But each element of $g'S_y$ moves y to $g'y$, hence $d(g_n g g_n^{-1} y, y) = d(g'y, y) = |g_n g g_n^{-1}|$, which contradicts parabolicity of g. ★

Theorem 3.20 (**Alexandrov angle rigidity, [CM-ST, Proposition 6.8]**)**.** *Let X be a proper cocompact geodesically complete CAT(0) space with a totally disconnected isometry group. Then there is $\varepsilon > 0$ such that for any elliptic isometry $g \in \mathfrak{Iso}(X)$ and any $x \in X$ not fixed by g, we have $\sphericalangle_c(gx, x) \geq \varepsilon$, where c denotes the projection of x on the set of g-fixed points.*

Proof. Suppose there is no ε with desired properties. Hence we can find a sequence $(g_n)_{n\in\mathbb{N}}$ of elliptic isometries of X, a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X such that $g_n x_n \neq x_n$ and $(c_n)_{n\in\mathbb{N}}$, the sequence of projections of x_n to g_n-fixed point set with the property that $\lim_{n\to\infty} \sphericalangle_{c_n}(x_n, g_n x_n) = 0$. Without loss of generality, we may assume that $d(x_n, c_n) = 1$ since otherwise, we can replace x_n with a point on a geodesic ray from c_n through x_n at distance 1 from c_n.

Since X is cocompact, there is a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathfrak{Iso}(X)$ such that $h_n c_n$ (sub)converges to some $c \in X$. Pass to another subsequence such that $h_n x_n$ converges to some $x \in X$. By definition of the topology on $\mathfrak{Iso}(X)$, $h_n g_n h_n^{-1}$ has an accumulation point $g' \in \mathfrak{Iso}(X)$. Note that $\sphericalangle_{h_n c_n}(h_n x_n, (h_n g_n h_n^{-1}) h_n x_n) = \sphericalangle_{c_n}(x_n, g_n x_n)$. By CAT(0) property, $d(h_n x_n, h_n g_n h_n^{-1}(h_n x_n))$ tends to 0 as n tends to infinity. Hence x, as well as c, is g'-fixed. By Theorem 3.17, the stabilizer G_x of x is open and hence contains $h_n g_n h_n^{-1}$ for all but finitely many n. Since $h_n x_n$ converges to x and x is $h_n g_n h_n^{-1}$-fixed, it is not $h_n c_n$ which is the projection of $h_n x_n$ to $(h_n g_n h_n^{-1})$-fixed point set, since $1 = d(h_n c_n, h_n x_n)$ for all n, but $d(h_n x_n, x)$ tends to 0 as n tends to infinity. ★

Chapter 4

Existence of regular elements

The flat closing conjecture in its most general form can be stated as follows.

Conjecture 4.1 (**Flat closing**). *Let $\Gamma \overset{geo}{\curvearrowright} X$, where X is a proper CAT(0) space, containing an isometric embedded copy of $\mathbb{R}^m$. Does Γ contain a copy of $\mathbb{Z}^m$?*

The name for the conjecture comes from the flat torus theorem – if the conjecture holds, this means that if X contains a flat, it also contains a *closed flat* or a *periodic flat*, i.e. a flat that is preserved by some free abelian subgroup of Γ.

In that form, Swenson answered the question affirmatively for $m = 1$, see Theorem 2.10 or [Swe, Theorem 11]. For higher m, it is believed that the conjecture is wrong, although there is no candidate for the counterexample. The evidence for that belief comes from work of Wise, [W1, W2], where he constructed a proper and cocompact CAT(0) space with isometrically embedded copy of $\mathbb{R}^2$, which is not periodic, and even an example with an embedded copy of $\mathbb{R}^2$, which is not the limit of periodic flats. (But still, some other flats in Wise's examples are periodic.)

Anyway, there are some special cases when the flat closing conjecture holds. In early nineties, Bangert and Schröder proved it for real analytic manifolds of non-positive sectional curvature, see [BS]. In [SW2], Sageev and Wise proved it for CAT(0) cube complexes under some additional assumptions on combinatorial-geometrical structure.

In this chapter, we will prove the following theorem, which implies a version of flat closing conjecture, namely Corollary 4.3.

Theorem 4.2. *Assume that X is geodesically complete and $\Gamma \overset{geo}{\curvearrowright} X$. Then Γ contains a hyperbolic element which acts as a hyperbolic isometry on each indecomposable de Rham factor of X.*

Every CAT(0) space X as in the theorem admits a canonical de Rham decomposition, see Theorem 3.4 and Proposition 3.5. Notice that the number of indecomposable de Rham factors of X is a lower bound on the dimension of all maximal flats in X, although two such maximal flats need not have the same dimension in general. As expected, we deduce a corresponding lower bound on the maximal rank of free abelian subgroups of Γ.

Corollary 4.3 ([**CZ**, **Corollary 1**]). *If X is a product of m geodesically complete factors, then discrete Γ, which acts properly and cocompactly on X, contains a copy of $\mathbb{Z}^m$.*

The first step of the proof consists in applying Proposition 3.5 and Theorem 3.4, which ensures that X splits as

$$X \cong \mathbb{R}^n \times X_1 \times \cdots \times X_m.$$

Invoking Theorem 3.4 once again, $\mathfrak{Iso}(X)$ possesses a finite index subgroup which splits as a direct product of $\mathbb{R}^n \rtimes O(n)$ and $\mathfrak{Iso}(X_i)$'s. Hence also Γ has a finite index subgroup preserving the product decomposition of X. Each $\mathfrak{Iso}(X_i)$ is by Theorem 3.16 either a semi-simple virtually connected Lie group or a totally disconnected locally compact group. The next essential point is that, by Theorem A.12, the group Γ virtually splits as $\mathbb{Z}^n \times \Gamma'$, and the factor Γ' (resp. $\mathbb{Z}^n$) acts properly and cocompactly on $X_1 \times \cdots \times X_m$ (resp. $\mathbb{R}^n$). Therefore, our main theorem is a consequence of the following.

Proposition 4.4. *Let $X = M \times Y_1 \times \cdots \times Y_q$, where $\mathfrak{Iso}(M)$ is a semi-simple virtually connected Lie group with trivial center and no compact factors and Y_i's are geodesically complete locally compact CAT(0) spaces with totally disconnected isometry groups. Any discrete cocompact group of isometries of X contains an element $(\gamma_M, \gamma_1, \ldots \gamma_q)$, acting as an element with $\mathfrak{Z}_{\mathfrak{Iso}(M)}(\gamma_M) \cong \mathbb{R}^{\mathrm{rank}(\mathfrak{Iso}(M))}$, and with γ_i a hyperbolic isometry of Y_i for all i.*

As in the discussion in the introduction, this yields a lower bound on the rank of maximal free abelian subgroups of Γ, from which Corollary 4.3 follows.

Corollary 4.5. *Let $X = M \times Y_1 \times \cdots \times Y_q$ be as in Proposition 4.4. Then any discrete cocompact group of isometries of X contains a copy of $\mathbb{Z}^{\mathrm{rank}(\mathfrak{Iso}(M))+q}$.*

Proof. Let $\Gamma \leq \mathfrak{Iso}(X)$ be a discrete subgroup acting cocompactly. Upon replacing Γ by a subgroup of finite index, we may assume that Γ preserves the given product decomposition of X. Let $\gamma \in \Gamma$ be as in Proposition 4.4 and let γ_M (resp. γ_i) be its projection to $\mathfrak{Iso}(M)$ (resp. $\mathfrak{Iso}(Y_i)$). Then $\mathfrak{min}(\gamma_M) = \mathbb{R}^{\mathrm{rank}(\mathfrak{Iso}(M))}$ and for all i we have $\mathfrak{min}(\gamma_i) \cong \mathbb{R} \times C_i$ for some CAT(0) space C_i, see [BH, Theorem II.6.8(5)]. Hence the desired conclusion follows from the following lemma. ★

Lemma 4.6. *Let $X = X_1 \times \cdots \times X_p$ be a proper CAT(0) space and Γ a discrete group acting properly cocompactly on X. Let also $\gamma \in \Gamma$ be an element preserving some n_i-dimensional flat in X_i on which it acts by translation, for all i. Then Γ contains a free abelian group of rank $n_1 + \cdots + n_p$.*

Proof. By assumption γ preserves the given product decomposition of X. We let γ_i denote the projection of γ on $\mathfrak{Iso}(X_i)$. Observe that

$$\mathfrak{min}(\gamma) = \mathfrak{min}(\gamma_1) \times \cdots \times \mathfrak{min}(\gamma_p).$$

By hypothesis, we have $\mathfrak{min}(\gamma_i) \cong \mathbb{R}^{n_i} \times C_i$ for some CAT(0) space C_i. Therefore $\mathfrak{min}(\gamma) \cong \mathbb{R}^{n_1+\cdots+n_p} \times C_1 \times \cdots \times C_p$. By Theorem 2.5, the centralizer $\mathfrak{Z}_\Gamma(\gamma)$ acts geometrically on $\mathfrak{min}(\gamma)$. Therefore, invoking Theorem A.12, we infer that $\mathbb{Z}^{n_1+\cdots+n_p}$ is a (virtual) direct factor of $\mathfrak{Z}_\Gamma(\gamma)$. ★

It remains to prove Proposition 4.4. We proceed in three steps. The first one provides an element $\gamma_Y \in \Gamma$ acting as a hyperbolic isometry on each Y_i. This combines an argument of Swenson, Theorem 2.10, with the phenomenon of Alexandrov angle rigidity, Theorem 3.20. The latter requires the hypothesis of geodesic completeness. The second step uses that Γ has subgroups acting properly and cocompactly on M, and thus contains an element γ_M acting as an $\mathbb{R}$-regular isometry of M by [BL]. The last step uses a result from [PR] ensuring that for all elements δ' in some Zariski open subset of $\mathfrak{Iso}(M)$ and all sufficiently large $n > 0$, the product $\gamma_M^n \delta'$ is $\mathbb{R}$-regular. Invoking the Borel density theorem, i.e. Theorem A.11, we finally find an appropriate element $\delta \in \Gamma$ such that the product $\gamma = \gamma_M^n \delta \gamma_Y$ has the requested properties. We now proceed to the details.

Proposition 4.7. *Let $Y = Y_1 \times \cdots \times Y_q$, where Y_i is a geodesically complete locally compact CAT(0) space with totally disconnected isometry group, and let $G \leq \mathfrak{Iso}(Y)$ be cocompact. Then G contains an element acting on Y_i as a hyperbolic isometry for all i.*

Proof. Upon replacing G by a finite index subgroup, we may assume that G preserves the given product decomposition of Y, see Theorem 3.4. Let $\rho : [0, \infty) \to Y$ be a geodesic ray which is **regular**, in the sense that its projection to each Y_i is a ray (in other words, if we decompose ρ according to the given product decomposition of Y, we have $\rho = (\rho_1, \ldots \rho_q)$ with none of ρ_i being a constant map).

Since G is cocompact, we can find a sequence $(g_n)_{n\in\mathbb{N}}$ in G and a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive integers such that the sequence of maps

$$\rho_n \colon [-t_n, \infty) \to Y, \quad t \mapsto g_n.\rho(t + t_n)$$

converges uniformly on compact subsets of $\mathbb{R}$ to a geodesic line $\ell : \mathbb{R} \to Y$. Set $h_{i,j} = g_i^{-1} g_j \in G$ and consider the angle

$$\theta = \measuredangle_{\rho(t_i)} \left(h_{i,j}^{-1}.\rho(t_i), h_{i,j}.\rho(t_i) \right).$$

Observe that by the construction of $h_{i,j}$, the angle θ is arbitrarily close to π for $i < j$ large enough. In [Swe, Theorem 11], it is proven in the case of discrete group acting geometrically on Y that if the angle θ is close enough to π, then the isometry $h_{i,j}$ is hyperbolic. The proof uses the uniform upper bound on the order of torsion elements in CAT(0) group. There is no such bound in our setting (since G

may be non-discrete), but still we are able to apply a geometric argument, Alexandrov angle rigidity (see Theorem 3.20). If $h_{i,j}$ is elliptic, let c be the projection of $\rho(t_i)$ onto $h_{i,j}$-fixed point set. We can bound an angle between $[\rho(t_i), c]$ and $[h_{i,j}.\rho(t_i), c]$ in terms of θ. Indeed, the isosceles triangles $\triangle\big(c, h_{i,j}^{-1}.\rho(t_i), \rho(t_i)\big)$ and $\triangle\big(c, \rho(t_i), h_{i,j}.\rho(t_i)\big)$ are congruent, hence

$$\begin{array}{rcl} \measuredangle_c\left(\rho(t_i), h_{i,j}.\rho(t_i)\right) & \leq & \pi - \measuredangle_c\left(\rho(t_i), h_{i,j}.\rho'(t_i)\right) - \measuredangle_c\left(\rho(t_i), h_{i,j}^{-1}.\rho(t_i)\right) \\ & \leq & \pi - \measuredangle_{\rho(t_i)}\left(h_{i,j}^{-1}.\rho(t_i), h_{i,j}.\rho'(t_i)\right) \\ & = & \pi - \theta. \end{array}$$

Now, if θ is close enough to π, this contradicts Alexandrov angle rigidity.

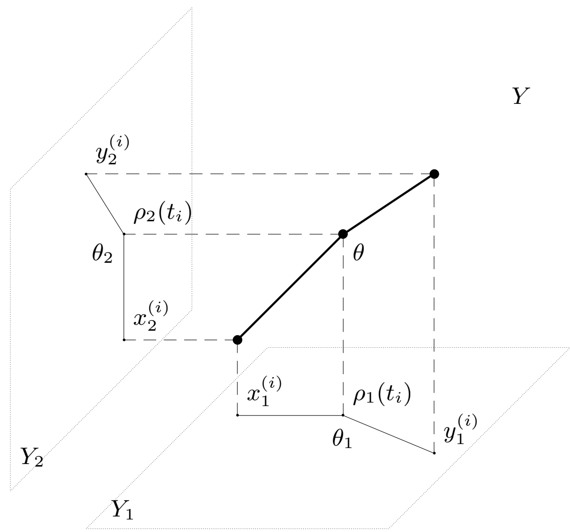

Figure 4.1: Projections of $[h^{-1}.\rho(t_i), \rho(t_i)] \cup [\rho(t_i), h.\rho(t_i)]$ to the factors.

Since $h_{i,j}$ respects the product decomposition of Y, we can apply this procedure to each Y_k-component of $h_{i,j}$. Fix some small $\delta > 0$. Let $x^{(i)}$ (resp. $y^{(i)}$) be the point at distance δ from $\rho(t_i)$ and lying on the geodesic segment $[h_{i,j}^{-1}.\rho(t_i), \rho(t_i)]$ (resp. $[\rho(t_i), h_{i,j}.\rho(t_i)]$). By construction, for $i < j$ large enough, the union of the two geodesic segments $[x^{(i)}, \rho(t_i)] \cup [\rho(t_i), y^{(i)}]$ lies in an arbitrary small tubular neighborhood of the geodesic ray ρ. Since the projection $Y \to Y_k$ is 1-Lipschitz, it follows that the Y_k-component of $[x^{(i)}, \rho(t_i)] \cup [\rho(t_i), y^{(i)}]$, which we denote by $[x_k^{(i)}, \rho_k(t_i)] \cup [\rho_k(t_i), y_k^{(i)}]$, is uniformly close to ρ_k, the Y_k-component of ρ, which is nontrivial by construction. Therefore, the angle

$$\theta_k = \measuredangle_{\rho_k(t_i)}(x_k^{(i)}, y_k^{(i)})$$

is arbitrarily close to π for $i < j$ large enough. Pick $i < j$ so large that $\theta_k > \pi - \varepsilon_k$ for all $k = 1, \dots q$, where $\varepsilon_k > 0$ is the constant from Alexandrov angle rigidity for Y_k. Set $h = h_{i,j}$ and let h_k be the projection of h on $\mathfrak{Iso}(Y_k)$. By construction h_k is hyperbolic for all k and we are done. ★

Proof of Proposition 4.4. Let Γ be a discrete group acting properly and cocompactly on X. First observe that (after passing to a finite index subgroup) we may assume that Γ preserves the given product decomposition of X, see Theorem 3.4.

Let G be the projection of Γ to $\mathfrak{Iso}(Y_1) \times \cdots \times \mathfrak{Iso}(Y_q)$. Then G acts cocompactly on $Y = Y_1 \times \cdots \times Y_q$. Therefore it contains an element g acting as a hyperbolic isometry on Y_i for all i by Proposition 4.7.

Let $\gamma = (\alpha, h)$ be the decomposition of γ along the splitting $\mathfrak{Iso}(X) = \mathfrak{Iso}(M) \times \mathfrak{Iso}(Y)$. By construction h acts as a hyperbolic isometry on Y_i for all i. Pick some $y \in \mathsf{min}(h) \subseteq Y$.

Let $U \leq \mathfrak{Iso}(Y)$ be the pointwise stabilizer of $\{y, \gamma y\}$. Notice that every element of $\mathfrak{Iso}(Y)$ contained in the coset Uh maps $h^{-1}y$ to y and y to $h.y$, and therefore acts also as a hyperbolic isometry on Y_i for all i.

On the other hand U is a compact (since $\mathfrak{Iso}(Y)$ acts properly) open (by Theorem 3.17) subgroup of $\mathfrak{Iso}(Y)$. Set $\Gamma_U = \Gamma \cap (\mathfrak{Iso}(M) \times U)$. Notice that Γ_U acts properly and cocompactly on M by Theorem A.13. In other words the projection of Γ_U to $\mathfrak{Iso}(M)$ is a cocompact lattice. Abusing notation slightly, we shall denote this projection equally by Γ_U.

By the appendix from [BL] (see also [Pra] for an alternative argument), the group Γ_U contains an element γ_M acting as an element with a centralizer of maximal possible dimension, i.e. as an $\mathbb{R}$-regular element on M. By [PR, Lemma 3.5] there is a Zariski open set $V = V(\gamma_M)$ in $\mathfrak{Iso}(M)$ with the following property. For any $\delta \in V$ there exists n_δ such that an element $\gamma_M^n \delta$ is $\mathbb{R}$-regular for any $n \geq n_\delta$. By the Borel density theorem (see Theorem A.11), the intersection $\Gamma_U \cap V\alpha^{-1}$ is nonempty. Pick an element $\delta \in \Gamma_U \cap V\alpha^{-1}$. Then $\delta\alpha \in V$ which means by definition that $\gamma_M^n \delta\alpha$ is $\mathbb{R}$-regular for all $n \geq n_0$ for some integer n_0.

Pick an element $\gamma'_M \in \Gamma$ (resp. $\delta' \in \Gamma$) which lifts γ_M (resp. δ). Set

$$\gamma = (\gamma'_M)^{n_0} \delta' \gamma_Y \in \Gamma.$$

The projection of γ to $\mathfrak{Iso}(M)$ is $\gamma_M^{n_0} \delta\alpha$ and is thus $\mathbb{R}$-regular. The projection of γ to $\mathfrak{Iso}(Y)$ belongs to the coset Uh, and therefore acts as a hyperbolic isometry on Y_i for all i. ★

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Appendices

Appendix A

Generalities about isometry groups

A.1 Topology on isometry group

A natural topology on the group of transformations of a metric space is a compact open topology.

Definition A.1. *Let X be a topological space. The* compact open topology *on the set of continuous maps $X \to X$ is the topology with sub-basis consisting of sets*

$$W_{K,U} := \{f : X \to X \mid f(K) \subseteq U\},$$

where K runs over compact subsets of X and U runs over open subsets of X.

Equipped with this topology, the isometry group of a proper metric space turns out to be a nice topological group.

Theorem A.2. *Let X be a proper metric space. Then the full isometry group $\mathfrak{Iso}(X)$, equipped with the compact open topology, is a locally compact (Hausdorff) topological group, and the natural action of $\mathfrak{Iso}(X)$ on X is continuous and proper.*

Proof. See [Cap, Exercise III.1]. ★

Note that the compact open topology on the isometry group of X as in the theorem above coincide with the point open topology, i.e. the topology with a sub-basis $\{W_{F,U} | U \text{ open and } F \text{ finite}\}$, [Cap, Exercise III.1(iv)]. Since X is a proper metric space, it is second countable, hence it is enough to take only countably many sets $W_{K,U}$ in a sub-basis for compact open (or point open) topology, namely those with U being a member of some countable basis of X and K finite.

A.2 Amenability, radical

In measure theory, there is a famous statement, the Banach-Tarski paradox, which says that it is possible to cut a unit ball $B^3 \subseteq \mathbb{R}^3$ into finitely many pieces (i.e. we can express B^3 as a finite disjoint union $\coprod_{i=1}^n A_n$) and then on each A_n, we can apply an isometry $\varphi_n \in \mathbb{R}^3 \rtimes O(3)$ such that $\varphi_n(A_n)$ are pairwise disjoint and $\coprod_{i=1}^n \varphi_n(A_n)$ is isometric to a disjoint union of two unit balls in $\mathbb{R}^3$. This paradox is a consequence of the fact that $O(3)$ contains a copy of a free group of rank 2, denoted by F_2, and that F_2 admits a paradoxical decomposition, i.e. there exist a decomposition $F_2 = X_1 \sqcup X_2 \sqcup Y_1 \sqcup Y_2$ and elements $a_1, a_2, b_1, b_2 \in F_2$ such that $F_2 = a_1X_1 \sqcup a_2X_2 = b_1Y_1 \sqcup b_2Y_2$.

According to the Banach-Tarski paradox, one defines a class of groups that behave non-paradoxically, i.e. do not allow a paradoxical decomposition as F_2 does. It turns out that there are several equivalent definitions (which, at first glance, seem quite different one from another) for that class of groups. After one of the definitions, the class of non-paradoxical groups is called amenable groups.

Definition A.3 (Amenable group, two definitions). *A (locally compact topological) group G is called amenable if it admits a finitely additive G-invariant probability measure, called a mean. Equivalently, G is amenable if for every continuous action of G on any compact Hausdorff topological space X, there is a G-invariant (countably additive) probability measure on X.*

The following notion is important in the proof of the dichotomy for isometry groups of CAT(0) spaces of our interest. We refer the reader to [Fur, §3] for characterization of an amenable radical.

Definition A.4 (**Amenable (solvable) radical**). *An amenable (solvable) radical of a topological group is the maximal normal amenable (solvable) subgroup.*

There are several references on amenable groups. We refer the reader to [T2] for some basic properties and examples and for further references, also on the Banach-Tarski paradox.

A.3 Hilbert's fifth problem

In this section we recall the original statement/question of the Hilbert's fifth problem, one of the twenty three problems posed at the 1900's International Congress of Mathematics in Paris. We apply it to prove Theorem 3.16, the structure theorem for the isometry group of a CAT(0) space with some nice properties. The topic is summarized from [T1, Chapter 1] and [Cap, Lecture III]

Question A.5 (**Hilbert's 5th problem**). *Is every topological group, which is locally Euclidean, necessary a Lie group?*

The solution of that problem was published in 1952, see [Gla, MZ1]. A bit later, the following restatement of it was proven by the same mathematicians.

Theorem A.6 (**Glaeson; Montgomery-Zippin**). *Let G be a connected locally compact topological group. Then any identity neighborhood in G contains a compact normal subgroup $K \trianglelefteq G$ such that G/K is a Lie group.*

The theorem says that every connected locally compact topological group can be *approximated* by Lie groups. In other words, G as in the theorem above is isomorphic to an inverse limit of a sequence of Lie groups. Think of terms in a sequence as the quotients of G by smaller and smaller compact normal subgroups. This point of view is crucial to establish the fact that Theorem A.6 answers Question A.5. See [T1, §1.6.3] for details.

The following theorem deals with the opposite extreme of connected topological groups – totally disconnected topological groups. In principle, this is all we need to know, since for any topological group G, the connected component of identity, denoted by G°, is (by definition) a connected topological group which is normal in G and G/G° is totally disconnected.

Theorem A.7 (**Van Dantzig, [Dan] and [T1, Theorem 1.6.7]**). *Let G be a totally disconnected locally compact topological group. Then any neighborhood of the identity contains a compact open subgroup.*

A similar version of this theorem has also been stated in Bourbaki's Elements of Mathematics.

Corollary A.8. *Let X be a proper metric space with totally disconnected isometry group. Then the identity element of $\mathfrak{Iso}(X)$ has a countable basis of neighborhoods, consisting of compact open subgroups.*

Proof. Since $\mathfrak{Iso}(X)$ is second countable, it is also first countable. Let $(U_n)_{n\in\mathbb{N}}$ be a basis of neighborhoods of $\mathfrak{id}_X$. For each U_n, there is a compact open subgroup $K_n \leq \mathfrak{Iso}(X)$ by Theorem A.7. ★

A.4 Lie groups

Let us recall that the main examples of Lie-type CAT(0) spaces (i.e. CAT(0) spaces X with $\mathfrak{Iso}(X)$ being Lie group) are symmetric spaces of non-compact type. See Appendix B for detailed discussion. Note that without geodesic completeness, there are also exotic examples of CAT(0) spaces with Lie isometry group, see [MP].

Note that we are dealing with semi-simple virtually connected real Lie groups with trivial center and without compact factors. Everything is self-explaining, except the semi-simplicity, which has several equivalent definitions. We use the following one.

Definition A.9 (**Semi-simple Lie group**). *A Lie group G is semi-simple if it has trivial solvable radical.*

The following result is used several times in the proof of Theorem 4.4 in order to find elements with appropriate properties in discrete subgroups of Lie group. It is proven in [Bor, Statement (ii)]. Recall the definition of a lattice.

Definition A.10. *Let G be a topological group. A* lattice Γ *in G is a discrete subgroup such that G/Γ carries a G-invariant probability measure. A lattice is called* uniform *if the quotient G/Γ is compact.*

Observe that for discrete $\Gamma \leq G$, the compactness of the quotient G/Γ implies the existence of a G-invariant probability measure on that quotient. Hence since $\mathfrak{Iso}(X)$ acts properly on X, we know that for $\Gamma \overset{geo}{\curvearrowright} X$, the group Γ is a uniform lattice in $\mathfrak{Iso}(X)$.

Recall that the Zariski topology on matrix groups is the topology whose closed sets are zeros of polynomial equations in matrix coefficients. We use the Zariski topology in the proof of the existence of regular elements in CAT(0) groups via the results from [PR] and the following theorem.

Theorem A.11 (**Borel density theorem, [Bor]**)**.** *A lattice in a semi-simple Lie group without compact factors is Zariski dense.*

A.5 Products

Here we present two tools from the theory of locally compact groups that allow us to find an isometry γ in Γ acting geometrically on a product $\mathbb{R}^n \times X \times Y$, where X is a Lie-type factor and Y has a totally disconnected isometry group, such that γ acts hyperbolically on all factors.

First, we split off Euclidean factor on the algebraic level. The following generalization of Bieberbach's theorem is due to Caprace and Monod. For the original (slightly more general) statement the reader can look at [CM-DS, Theorem 3.8]. Originally, Bieberbach conjectured that any lattice in isometry group of $\mathbb{R}^n$ is virtually $\mathbb{Z}^n$. The proof can be found in [Aus]. The following theorem deals with lattices in a product of $\mathbb{R}^n$ with some other CAT(0) space.

Theorem A.12. *Let $\Gamma \overset{geo}{\curvearrowright} X$, where X splits as a product $\mathbb{R}^n \times X'$, where X' is geodesically complete and without Euclidean factor. Then Γ virtually splits as $\mathbb{Z}^n \times \Gamma'$, where $\mathbb{Z}^n$ acts by translations on $\mathbb{R}^n$-factor and trivially on X' and $\Gamma' \overset{geo}{\curvearrowright} X'$.*

To prove that theorem, one needs to analyze lattices in the product $(\mathbb{R}^n \rtimes O(n)) \times G \times H$, where G is a semi-simple Lie group with trivial center and has no compact factors and H is a totally disconnected locally compact group. Note that the product $G \times H$ appears as a finite index subgroup of $\mathfrak{Iso}(X')$.

Once we split off $\mathbb{R}^n$, there remain two factors, one with semi-simple Lie isometry group and another with a totally disconnected isometry group. To deal with them, we need the following.

Theorem A.13 (**[CM-DS, Lemma 3.2]**)**.** *Let $G = S \times D$, where S is a semi-simple Lie group and D is a totally disconnected locally compact topological group. Let Γ be a lattice in G and let $U \leq D$ be a compact open subgroup. Then $\Gamma_U := \Gamma \cap (S \times U)$ is a lattice in $S \times U$. In particular, the projection of Γ_U to S is a lattice in S.*

Appendix B

On symmetric spaces

The example of non-positively curved symmetric space is the space $P_1(n,\mathbb{R})$ of positive definite $n\times n$ matrices of determinant 1 equipped with the Riemannian metric $\langle X,Y\rangle_P = \mathrm{Tr}(XP^{-1}YP^{-1})$. (The tangent space at $P\in P_1(n,\mathbb{R})$ can readily be identified with the space of symmetric matrices X with $\mathrm{Tr}(XP^{-1}) = \mathrm{Tr}(\sqrt{P^{-1}}X\sqrt{P^{-1}}) = 0$.) In fact if M is any symmetric manifold of non-compact type there exists a diffeomorphism onto a totally geodesic submanifold of some $P_1(n,\mathbb{R})$. The pull-back metric on M obtained by means of the embedding coincides with the original metric on M up to a constant multiple on each irreducible de Rham factor. See Eberlein [Ebe] for a more detailed account of symmetric manifolds.

An important aspect in the study of a Riemannian manifold is the investigation of isometries as well as the group of all isometries. Here we address the problem of classification of the Riemannian isometries of $P_1(n,\mathbb{R})$. Our vantage point, however, is that of CAT(0) geometry as it affords greater flexibility and lucidity by neglecting the differentiable structure where it is possible.

The classification of all isometries of $P_1(n,\mathbb{R})$ is by no means trivial. Recently, Fujiwara, Nagano, and Shioya [FNS] classified the isometries and their fixed point sets for the connected component of the identity in the full group of isometries of $P_1(3,\mathbb{R})$. To some extent that achievement was an application of their more general investigation of parabolic isometries of CAT(0) spaces. Here we classify the isometries of $P_1(3,\mathbb{R})$ in the connected component of the inversion $\sigma(P) = P^{-1}$. In particular, we note that there are parabolic isometries in that component and we determine their fixed point set at infinity, thereby solving a problem posed by Fujiwara, see [Fuj, Problem 4.1].

To every matrix $g\in SL(n,\mathbb{R})$ we can associate the Riemannian isometry $g: P_1(n,\mathbb{R})\to P_1(n,\mathbb{R})$ sending each P to gPg^T. The resulting representation $SL(n,\mathbb{R})\to \mathfrak{Iso}(P_1(n,\mathbb{R}))$ induces an isomorphism of $PSL(n,\mathbb{R})$ and the identity component of $\mathfrak{Iso}(P_1(n,\mathbb{R}))$. (See [BH, Chapter II, §10] for details.) By virtue of that isomorphism we view $PSL(n,\mathbb{R})$ as a subgroup of $\mathfrak{Iso}(P_1(n,\mathbb{R}))$. Similarly we associate to every element $g\in SL(n,\mathbb{R})$ the Riemannian isometry $\widetilde{g}\colon P_1(n,\mathbb{R})\to P_1(n,\mathbb{R})$ sending each P to $gP^{-1}g^T$. This results in a diffeomorphism of $PSL(n,\mathbb{R})$ and the component of inversion in $\mathfrak{Iso}(P_1(n,\mathbb{R}))$ which we denote by $PSL(n,\mathbb{R})\sigma$. Note that for odd n we can identify $SL(n,\mathbb{R}) = PSL(n,\mathbb{R})$.

The isometry group $\mathfrak{Iso}(P_1(n,\mathbb{R}))$ has exactly two components for each $n>2$. To see this, observe that the transvections (ie. composition of two symmetries S_x and S_y) form a finite index normal subgroup of the whole isometry group. Since the map $P\mapsto P^{-T}$ is the only nontrivial outer automorphism of $PSL(n,\mathbb{R})$ for $n>2$ we have that index $[\mathfrak{Iso}(P_1(n,\mathbb{R})) : PSL(n,\mathbb{R})]$ is two. Hence we now have a technique to determine geometric objects (minimal/fixed point set and fixed points at infinity) associated to any isometry of $P_1(n,\mathbb{R})$. We apply it to illustrate the advantage of CAT(0) over the differential-geometric approach.

Notation. For a CAT(0) space X and $\alpha\in\mathfrak{Iso}(X)$, let us denote by $\mathfrak{fix}(\alpha)$ the fixed point set of α (which makes sense if α is elliptic) and by $\mathfrak{fix}_\infty(\alpha)$ the set of α-fixed points at the boundary at infinity, ∂X.

We will use the following definition of the simplicial structure on $\partial_T P_1(n,\mathbb{R})$ (which coincides with the standard one from [BH, Chapter II, §10]). A simplex of dimension m, $m=0,1,\dots n-2$ in $\partial_T P_1(n,\mathbb{R})$ is determined with the following data: an ordered orthonormal basis $(e_1,\dots e_n)$ (or, equivalently, a matrix $O\in O(n)$ whose columns are $e_1,\dots e_n$) and a subset $\{i_1,\dots i_{m+1}\}$ of $\{1,2,\dots n-1\}$ of power $m+1$. That simplex consists of all matrices $X\in\mathcal{S}$ such that $OXO^T = \mathrm{diag}(\lambda_1,\dots\lambda_n)$ where

$$\lambda_1 = \cdots = \lambda_{i_i} \geq \lambda_{i_1+1} = \cdots = \lambda_{i_2} \geq \cdots \geq \lambda_{i_m+1} = \cdots = \lambda_{i_{m+1}} \geq \lambda_{i_{m+1}+1} = \dots \lambda_n.$$

Note that for two different orthogonal matrices, we may get the same m-simplex.

B.1 Additional properties of CAT(0) space's isometries

The following theorem and its consequence Corollary B.3 are the key tools for classifying isometries from the nonidentity component of $\mathfrak{Iso}(P_1(n,\mathbb{R}))$. Indeed, some power of any group element in a group of isometries with finitely many connected components lies in the identity component. In case $n=3$, we have a complete characterization of the identity component of $\mathfrak{Iso}(P_1(3,\mathbb{R}))$ by [FNS, §6.3]

Recall that for an isometry α of a CAT(0) space X, the minimal space of α, denoted $\mathfrak{min}(\alpha)$, is the set of points that are translated for the minimal distance $|\alpha| = \inf\{d(x,\alpha(x))|x \in X\}$. In case of an elliptic isometry α, its minimal space is also denoted by $\mathfrak{fix}(\alpha)$. We denote by $\mathfrak{fix}_\infty(\alpha)$ the set of fixed points of the induced α-action on ∂X.

Theorem B.1. *Let (X,d) be a proper CAT(0) space. An isometry α of X has the same type (elliptic, hyperbolic or parabolic) as its powers and the translation lengths relate as $|\alpha^n| = n|\alpha|$. Moreover, $\mathfrak{fix}_\infty(\alpha) \subseteq \mathfrak{fix}_\infty(\alpha^n)$ and in the semi-simple case $\mathfrak{min}(\alpha) \subseteq \mathfrak{min}(\alpha^n)$.*

Proof. Let $n \in \mathbb{N}$. Recall that

- if α is elliptic, then so is α^n, since $\mathfrak{fix}(\alpha) \subseteq \mathfrak{fix}(\alpha^n)$;
- if α^n is elliptic, then α itself is elliptic, because for $x \in \mathfrak{fix}(\alpha^n)$ the orbit of x under α is finite, hence its circumcentre is a fixed point for α (see [BH, Proposition II.2.7]);
- if α is hyperbolic, so is α^n, since it acts as translation by $n|\alpha|$ on $\mathfrak{min}(\alpha)$;
- if α^n is hyperbolic, then α is hyperbolic (see [BH, Theorem II.6.8]).

Consequently, α^n is parabolic if and only if α is parabolic, and the first statement of the theorem follows. For the second part let us first observe that the limit

$$\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x)$$

exists and is independent of x (see [BH, Exercise II.6.6 (1)]). The existence follows from the fact that for a fixed x the function $f(n) = d(x,\alpha^n x)$ is subadditive. It is well known that for such functions the limit $\lim_{n\to\infty} \frac{f(n)}{n}$ exists. To show independence of x take another point y. The triangle inequality yields

$$d(x,\alpha^n x) - d(x,y) - d(\alpha^n x,\alpha^n y) \leq d(y,\alpha^n y) \leq d(x,\alpha^n x) + d(x,y) + d(\alpha^n x,\alpha^n y).$$

Note that $d(x,y) = d(\alpha^n x,\alpha^n y)$. Hence dividing by n and taking the limit we obtain the equality

$$\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x) = \lim_{n\to\infty} \frac{1}{n} d(y,\alpha^n y).$$

The evaluation of this limit is also a part of the cited exercise, but only for the semi-simple case where the proof is easier because one can take $x \in \mathfrak{min}(\alpha)$. We give here a proof of the general case. The triangle inequality implies

$$d(x,\alpha^n x) \leq d(x,\alpha x) + d(\alpha x,\alpha^2 x) + \cdots + d(\alpha^{n-1}x,\alpha^n x). \tag{$\diamondsuit$}$$

Choose an arbitrary $\varepsilon > 0$. Let x be such that $d(x,\alpha x) \leq |\alpha| + \varepsilon$. It follows from ($\diamondsuit$) that $\frac{1}{n} d(x,\alpha^n x) \leq |\alpha| + \varepsilon$, hence $\lim_{n\to\infty} \frac{1}{n} d(x,\alpha^n x) \leq |\alpha|$. For the reverse inequality, let x' and $x'' = \alpha x'$ be the midpoints of geodesic segments $[x,\alpha x]$ and $[\alpha x,\alpha^2 x]$. It follows (by the convexity of metric d on X) that $d(x,\alpha^2 x) \geq 2d(x',x'') = 2d(x',\alpha x')$. Applying this inductively we note that for each n there exists a point $\widetilde{x}_n \in X$ so that

$$d(x,\alpha^{2^n} x) \geq 2^n d(\widetilde{x}_n,\alpha\widetilde{x}_n).$$

Consequently $2^{-n} d(x,\alpha^{2^n} x) \geq |\alpha|$ for each n. The asserted inequality follows by taking the limit.

To conclude the proof, note that

$$|\alpha^n| = \lim_{m\to\infty} \frac{1}{m} d(x,(\alpha^n)^m x) = n \lim_{m\to\infty} \frac{1}{nm} d(x,\alpha^{nm} x) = n|\alpha|.$$

If $\xi \in \mathfrak{fix}_\infty(\alpha)$, then obviously ξ is already contained in $\mathfrak{fix}_\infty(\alpha^n)$. In the hyperbolic case, every axis of α is also an axis for α^n, because α^n acts on it as a translation by $n|\alpha|$. In the elliptic case, if $\alpha x = x$ then also $\alpha^n x = x$. ★

Example B.2. *The inclusions in the statement of the theorem can be strict as shown by the following examples.*

For the semi-simple case let β be a rotation of order $n \geq 2$ on the Euclidean space $\mathbb{R}^3$, and let τ be a translation in the direction of the axis of β. Then $\alpha = \tau\beta$ is a semi-simple isometry (elliptic if τ is trivial and a hyperbolic glide-rotation if τ is nontrivial). On the one hand, the only fixed points at infinity of α are the ends of the axis of β. On the other hand, $\alpha^n = \tau^n$ is a translation, hence it fixes the whole of $\partial\mathbb{R}^3$.

For the parabolic case let $X = \mathbb{R} \times \mathbb{H}^2$ and let α act as reflection across the origin on $\mathbb{R}$ and as an arbitrary parabolic isometry τ on $\mathbb{H}^2$ (for instance $\tau(x,y) = (x+1,y)$ in the upper halfplane model). Then $\mathfrak{fix}_\infty(\alpha)$ is a point, but $\mathfrak{fix}_\infty(\alpha^2) = S^0 * \mathfrak{fix}_\infty(\tau) \approx [0,\pi]$.

Theorem B.1 yields the following corollary.

Corollary B.3. *Let $\widetilde{g} \in PSL(n,\mathbb{R})\sigma$. Then $\widetilde{g}^2 = gg^{-T} \in PSL(n,\mathbb{R})$, hence the isometry $\widetilde{g}$ has the same type as the isometry $gg^{-T} \in PSL(n,\mathbb{R})$ and $|\widetilde{g}| = \frac{1}{2}|gg^{-T}|$.*

Proof. A trivial computation shows that in $\mathfrak{Iso}(P_1(n,\mathbb{R}))$, $(\widetilde{g})^2 = gg^{-T}$. Hence Theorem B.1 applies. ★

This means that the isometry $\widetilde{g} \in PSL(n,\mathbb{R})\sigma$ is semi-simple if and only if the matrix gg^{-T} is diagonalizable over $\mathbb{C}$ (see [BH, Proposition II.10.61]), and is elliptic if and only if gg^{-T} is conjugate (in $SL(n,\mathbb{R})$) to an orthogonal matrix.

By using the classification of isometries in the identity component of $\mathfrak{Iso}(P_1(n,\mathbb{R}))$ in [BH, Chapter II, §10], Corollary B.3 can be used to determine the type of any isometry $\widetilde{g} \in PSL(n,\mathbb{R})\sigma$.

The next lemma shows a nice relation between the fixed point set of an elliptic isometry α of a complete CAT(0) space and the fixed point set of the induced action of α at infinity (see also [Swe, Lemma 10]). We are going to apply it in the proof of Theorem B.9 below.

Lemma B.4. *Let α be a semi-simple isometry of a complete CAT(0) space X and let $F = \mathfrak{min}(\alpha)$. Then $\mathfrak{fix}_\infty(\alpha) = \partial F$.*

Proof. Let us denote $F_\infty := \mathfrak{fix}_\infty(\alpha)$. Because of convexity of F the inclusion $\partial F \subseteq F_\infty$ is obvious. For the reverse inclusion, take an element $\xi \in F_\infty$. For an arbitrary point $x \in F$ let $c\colon ([0,\infty), 0, \infty) \to (X, x, \xi)$ be the unique geodesic ray with initial point x in the class of geodesic rays representing ξ. As $\xi \in F_\infty$, the geodesic ray $\alpha \circ c$ is asymptotic to c, which means that $f(t) = d(\alpha(c(t)), c(t))$ is a bounded function of t. As the metric of a CAT(0) space is convex, f is itself convex and therefore decreasing. On the other hand, $f(t) \geq d(\alpha(c(0)), c(0)) = |\alpha|$, hence $t \mapsto f(t)$ is constant. This means that the image of c lies entirely in F, hence $\xi \in \partial F$. ★

B.2 The non-identity component of $\mathfrak{Iso}(P_1(3,\mathbb{R}))$

In this section, we dive into $PSL(n,\mathbb{R})\sigma$ to explore the machinery needed for our main result, Theorem B.9.

B.2.1 Jordan forms

Recall that the geometric properties of an isometry of a given CAT(0) space X behave nicely under conjugation. In particular, for given $\alpha, \beta \in \mathfrak{Iso}(X)$, the isometries α and $\beta\alpha\beta^{-1}$ have the same type (elliptic, hyperbolic or parabolic) and their translation lengths are the same. Furthermore, $\mathfrak{min}(\beta\alpha\beta^{-1}) = \beta.\mathfrak{min}(\alpha)$. The following result about conjugation in $PSL(n,\mathbb{R})\sigma$ will be of use to us.

Lemma B.5. *Isometries $\widetilde{g}, \widetilde{h} \in PSL(n,\mathbb{R})\sigma$ are conjugate if there exists $A \in SL(n,\mathbb{R})$ such that $g = AhA^T$.*

Proof. Let $g = AhA^T$. For $P \in P_1(n,\mathbb{R})$ we have

$$\widetilde{g}.P = gP^{-1}g^T = (AhA^T)P^{-1}(Ah^TA^T) = A.(\widetilde{h}.(A^{-1}.P)).$$

★

To analyze the nonidentity component of the isometry group $\mathfrak{Iso}(P_1(3,\mathbb{R}))$, it is enough to classify all the isometries of the form $gg^{-T} \in SL(3,\mathbb{R})$ by Corollary B.3. Following the classification of isometries in $SL(3,\mathbb{R})$ from [FNS, §6.3], we have to determine the real Jordan form of gg^{-T} for each $g \in SL(3,\mathbb{R})$. Observe that conjugation of gg^{-T} by $A \in SL(n,\mathbb{R})$ corresponds to conjugating the isometry $\widetilde{g}$ by A. Since conjugation in $SL(3,\mathbb{R})$ does not change the isometry type and the translation length of gg^{-T}, we can restrict ourselves to solving the equation $g = Ag^T$ for all possible real Jordan matrices A that correspond to the isometries in $SL(3,\mathbb{R})$. We can solve that equation as a homogeneous system of linear equations and then scale to land at $g \in SL(3,\mathbb{R})$. By some lengthy but straightforward linear algebra, we get four families of solutions which we list below. The conjugacy relation between different solutions in each family is deduced by Lemma B.5 and is given in the last column of the table below. We employ the following notation.

$$A_{x,y} = \operatorname{diag}\left(\frac{y}{x}, \sqrt{\frac{x}{y}}, \sqrt{\frac{x}{y}}\right) \quad \text{and} \quad B_{x,y} = \begin{bmatrix} 1 & 1 & \frac{x-y}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad C_{x,y} = \operatorname{diag}\left(\sqrt{\frac{x}{y}}, \sqrt{\frac{x}{y}}, \frac{y}{x}\right)$$

	Possible real Jordan form A for matrix in $SL(3,\mathbb{R})$	Solutions g_x of $g = Ag^T$	Conjugacy relations among solutions
(1)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4x^2} & 0 & 0 \\ 0 & x & 2x \\ 0 & -2x & 0 \end{bmatrix}$, $x \neq 0$	$g_x = A_{x,y} g_y A_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(2)	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} x & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $x \in \mathbb{R}$	$g_x = B_{x,y} g_y B_{x,y}^T$ for any x, y
(3)	$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $a^2 + b^2 = 1, b \neq 0$	$\begin{bmatrix} x & \frac{xb}{1+a} & 0 \\ \frac{-xb}{1+a} & x & 0 \\ 0 & 0 & \frac{1+a}{2x^2} \end{bmatrix}$, $x \neq 0$	$g_x = C_{x,y} g_y C_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(3’)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	g is any symmetric matrix	g and g' are conjugated iff either both are positive or both have two negative eigenvalues
(4)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & a \end{bmatrix}$, $a \notin \{0, 1\}$	$\begin{bmatrix} -\frac{a}{x^2} & 0 & 0 \\ 0 & 0 & \frac{x}{a} \\ 0 & x & 0 \end{bmatrix}$, $x \neq 0$	$g_x = A_{x,y} g_y A_{x,y}^T$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$
(5)	all the rest	no solutions	

B.2.2 Minimal spaces

As Example B.2 shows, there is no straightforward way to determine the minimal space of $\widetilde{g}$, given $\mathtt{min}(gg^{-T}) = \mathtt{min}(\widetilde{g}^2)$. Hence we have to calculate $\mathtt{min}(\widetilde{g})$ by hand. To this end, we first retrieve some information about semi-simple isometries in $\mathfrak{Iso}(P_1(n,\mathbb{R}))\sigma$ for general n. We use that information to determine all possible shapes of minimal spaces of semi-simple isometries in $SL(3,\mathbb{R})\sigma$.

Assume first that $\widetilde{g}$ is hyperbolic. Without loss of generality take $I \in \mathtt{min}(\widetilde{g})$ (otherwise we can conjugate $\widetilde{g}$ by $\sqrt{R}^{-1}$ for $R \in \mathtt{min}(\widetilde{g})$). Let $X \in S_0(n,\mathbb{R})$ with $\|X\|_2 = 1$ be such that $\widetilde{g}$ acts as a translation on $\exp(\mathbb{R}X)$. We can as well assume that X (or equivalently, $\exp(X)$) is diagonal since otherwise we can conjugate $\widetilde{g}$ by an orthogonal matrix O for which the O-conjugate of X is diagonal. For an arbitrary $t \in \mathbb{R}$ and $t_0 := |\widetilde{g}|$ this means

$$\widetilde{g}.\exp(tX) = g\exp(-tX)g^T = \exp((t+t_0)X).$$

Acting by $\exp\left(-\frac{t_0 X}{2}\right) \in PSL(n,\mathbb{R})$ on this equality gives

$$\exp\left(-\frac{t_0 X}{2}\right) g \exp(-tX) g^T \exp\left(-\frac{t_0 X}{2}\right) = \exp(tX), \qquad (\heartsuit)$$

which implies $\exp\left(-\frac{t_0 X}{2}\right) g = O \in O(n)$ (it fixes I), hence $g = \exp\left(\frac{t_0 X}{2}\right) O$ is a polar decomposition for g. Inserting $t - t_0$ in place of t in the equation ($\heartsuit$) gives that $g \exp\left(\frac{t_0 X}{2}\right) = O' \in O(n)$ and hence $g = O' \exp\left(-\frac{t_0 X}{2}\right)$ is another polar decomposition for g. A simple application of the above equalities yields $O = O'$:

$$O \exp\left(\frac{-t_0 X}{2}\right) O^T = \exp\left(\frac{t_0 X}{2}\right) = O' \exp\left(\frac{-t_0 X}{2}\right) O^T,$$

where the last equality comes from both polar decompositions.

From this we can derive additional information in case $n = 3$.

Lemma B.6. *The minimal space $\mathfrak{min}(\widetilde{g})$ of a hyperbolic isometry $\widetilde{g} \in SL(3,\mathbb{R})\sigma$ is isometric to $\mathbb{R}$.*

Proof. As explained above, we may assume that $I \in \mathfrak{min}(\widetilde{g})$ and $X = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in S_0(3,\mathbb{R})$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \neq 0$ is such that $\widetilde{g}$ acts as a translation on $\exp(\mathbb{R}X)$. Suppose that also $P = \exp(Y) \in \mathfrak{min}(\widetilde{g})$ where $Y \in S_0(3,\mathbb{R})$ is linearly independent of X. As $\exp(Y) \in \mathfrak{min}(\widetilde{g})$, there is a geodesic parallel to $c : t \mapsto \exp(tX)$ through $\exp(Y)$. We borrow the notation of [BH, Proposition II.10.67]: let $F(b)$ denote the union of geodesics, parallel to b. With this notation, it means that $\exp(Y) \in F(c)$ hence $\exp(Y)$ commutes with $\exp(X)$ (by the same proposition) and this implies $[X, Y] = 0$, i.e. X and Y are diagonalizable in some common basis.

As above, we show that $g = \exp(\frac{t_0 X}{2}) O$ is a polar decomposition for g, where $t_0 = |\widetilde{g}|$. Regarding O we know $O \exp(-tX) O^T = \exp(tX)$ or, equivalently, $OXO^T = -X$. Hence the spectrum of X must satisfy $\sigma(X) = -\sigma(X)$. Because X is nonzero, the only possibility is that $\sigma(X) = \{\lambda, 0, -\lambda\}$ for positive λ and that O is just a "permutation" of the basis, swapping $\operatorname{Lin}\{e_1\}$ and $\operatorname{Lin}\{e_3\}$ and leaving $\operatorname{Lin}\{e_2\}$ invariant. From $[X, Y] = 0$ we get also $[OYO^T, X] = 0$, hence X and OYO^T are diagonalizable in a common basis. But X has 3 different eigenvalues, hence is diagonalizable in only one basis, which means that the three matrices X, Y and OYO^T are diagonalizable in that basis. Hence, Y is a diagonal matrix. The convexity of $\mathfrak{min}(\widetilde{g})$ implies that $\exp(tY) \in \mathfrak{min}(\widetilde{g})$ for all $0 \leq t \leq 1$ and we can calculate

$$\begin{aligned} t_0 &= d(\exp(tY), g \exp(-tY) g^T) = \\ &= d\left(I, \exp\left(-\frac{tY}{2}\right) \exp\left(\frac{t_0 X}{2}\right) O \exp(-tY) O^T \exp\left(\frac{t_0 X}{2}\right) \exp\left(-\frac{tY}{2}\right)\right) = \\ &= d\left(I, \exp\left(-\frac{tY}{2} + \frac{t_0 X}{2} - tOYO^T + \frac{t_0 X}{2} - \frac{tY}{2}\right)\right) = \\ &= d\left(I, \exp(t_0 X - tY - tOYO^T)\right) = \|t_0 X - tY - tOYO^T\|_2. \end{aligned}$$

Because Y is supposed to be linearly independent of X and the length of the vector $t_0 X - tY - tOYO^T$ is independent of t, we have $OYO^T = -Y$. If we write $Y = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$, the last equality means $0 \neq \mu_1 = -\mu_3$ and $\mu_2 = 0$. Hence Y is linearly dependent of X, and the contradiction completes the proof. ★

We proceed to the elliptic case where again we start with general n. If $\widetilde{g}$ is elliptic, then it fixes some $P \in P_1(n,\mathbb{R})$. This means $P = gP^{-1}g^T$ which we rewrite as

$$I = \sqrt{P^{-1}} g \sqrt{P^{-1}} \left(\sqrt{P^{-1}} g \sqrt{P^{-1}}\right)^T.$$

Therefore $\widetilde{g}$ is conjugate to $\widetilde{h}$ where $h = \sqrt{P^{-1}} g \sqrt{P^{-1}} = \sqrt{P^{-1}} g \sqrt{P^{-1}}^T \in SO(n)$. Conversely, if $g \in SO(n)$, then $\widetilde{g}.I = I$ and obviously $\widetilde{g}$ is elliptic.

Suppose now that for $g \in SO(n)$, the isometry $\widetilde{g}$ fixes some $P = \exp(X) \neq I$. As in the proof of Lemma B.6, the property $gXg^T = -X$ implies that the spectrum of X is symmetric about 0 and that $\exp(\mathbb{R}X) \subseteq \mathfrak{fix}(\widetilde{g})$. If X has n different eigenvalues, then g acts as an involution on the set of n different eigenspaces of X. Hence, $\widetilde{g}$ has order either 2 or 4 (since g^2 may be minus the identity on each eigenspace for nonzero eigenvalue). Hence for $n = 3$ we have the following lemma.

Lemma B.7. *The fixed point set of an elliptic isometry from $SL(3,\mathbb{R})\sigma$ is either a single point or a hyperbolic plane.*

Proof. Let $\widetilde{g}$ be an elliptic isometry. Without loss of generality suppose $I \in \mathfrak{fix}(\widetilde{g})$, hence $g \in SO(3)$. If there is a nonzero X with $\exp(X) \in \mathfrak{fix}(\widetilde{g})$, we have that $gXg^T = -X$. In particular, the spectrum of X equals $\{\lambda, 0, -\lambda\}$ for some nonzero λ, and g swaps the eigenspaces corresponding to the nonzero eigenvalues and preserves the eigenspace of the eigenvalue 0. After another conjugation, we may assume that $X = \mathrm{diag}(\lambda, 0, -\lambda)$ and hence

$$g = \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \mp 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

A computation shows that in each case, there is another linearly independent $Y \in S_0(3,\mathbb{R})$ such that $gYg^T = -Y$. Then for any linear combination $S = tX + sY$ we have $\exp(S) \in \mathfrak{fix}(\widetilde{g})$. Furthermore, Y and X do not commute and hence $\mathfrak{fix}(\widetilde{g})$ is not a flat. But it still has constant curvature since it is homogeneous: $\exp(-\frac{tX+sY}{2})$ conjugates $\widetilde{g}$ to itself by Lemma B.5, hence it preserves $\mathfrak{fix}(\widetilde{g})$, but it also moves $\exp(tX + sY)$ to I. We conclude that $\mathfrak{fix}(\widetilde{g})$ is a scaled hyperbolic plane. ★

B.2.3 Boundary at infinity

Recall from Section **??** that $\partial_T P_1(n,\mathbb{R})$ is a simplicial complex.

Lemma B.8. *The inversion σ acts as a simplicial map on $\partial_T P_1(n,\mathbb{R})$.*

Proof. Let $\xi \in \partial_T P_1(n,\mathbb{R})$ be represented by a geodesic ray $[t \mapsto \exp(tX)]_{t>0}$ for $X \in S_0(n,\mathbb{R})$. Then $\sigma.\xi$ is represented by $[t \mapsto \exp(-tX)]_{t>0}$. This means that σ maps the simplex, determined by the ordered orthonormal basis $(e_1, \dots e_n)$ and $\{i_1, \dots i_{m+1}\} \subseteq \{1, 2, \dots n-1\}$ to the simplex determined by $(e_n, \dots e_1)$ and $\{n - i_{m+1}, \dots n - i_1\}$. ★

If we take an apartment $A \approx S^{n-2}$, which is the boundary of a flat containing I, then σ acts as a reflection across the center of S^{n-2}.

We know (see e.g. [BH, Proposition II.10.75]) that for an isometry α in $PSL(n,\mathbb{R})$, the set $\mathfrak{fix}_\infty(\alpha)$ is a simplicial subcomplex of $\partial_T P_1(n,\mathbb{R})$ but for α in $PSL(n,\mathbb{R})\sigma$ that is generally not true, see the classification theorem for $SL(3,\mathbb{R})\sigma$ below.

The tools developed above together with [FNS, Theorem 6.1] make the classification of isometries in $SL(3,\mathbb{R})\sigma$ quite easy. In the next theorem, we use c_i, $i = 1, 2, \dots 6$, for the chambers consisting of equivalence classes of rays $t \mapsto \exp(tX)$ for diagonal matrices $X = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in S_0(3,\mathbb{R})$. More accurately,

$$\begin{aligned}
c_1 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_1 \geq \lambda_2 \geq \lambda_3\}, \\
c_2 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_2 \geq \lambda_1 \geq \lambda_3\}, \\
c_3 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_2 \geq \lambda_3 \geq \lambda_1\}, \\
c_4 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_3 \geq \lambda_2 \geq \lambda_1\}, \\
c_5 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_3 \geq \lambda_1 \geq \lambda_2\}, \\
c_6 &:= \{\text{equivalence classes of rays } t \mapsto \exp(t \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \mid \lambda_1 \geq \lambda_3 \geq \lambda_2\}.
\end{aligned}$$

Furthermore, let v_i denote the common vertex of c_i and c_{i-1} (indices modulo 6) such that the simplex $[v_i, v_{i+1}]$, i.e. the simplex spanned on v_i and v_{i+1}, equals c_i. Let C_i denote the barycenter of the simplex c_i.

B.2.4 Classification

Theorem B.9. *Let $\widetilde{g} \in SL(3,\mathbb{R})\sigma$ and let gg^{-T} have a (real) Jordan form as in the table above. We have*

(1) $\widetilde{g}$ is parabolic, $\mathfrak{fix}_\infty(\widetilde{g}) = C_2$ and $|\widetilde{g}| = 0$;

(2) $\widetilde{g}$ is parabolic, $\mathfrak{fix}_\infty(\widetilde{g}) = C_1$ and $|\widetilde{g}| = 0$;

(3) $\widetilde{g}$ is elliptic, $\mathfrak{fix}(\widetilde{g})$ is a single point and $\mathfrak{fix}_\infty(\widetilde{g}) = \emptyset$;

(3') $\widetilde{g}$ is elliptic and

- *(a) if g is positive, $\mathfrak{fix}(\widetilde{g})$ is a single point g and $\mathfrak{fix}_\infty(\widetilde{g}) = \emptyset$;*
- *(b) if g is not positive, $\mathfrak{fix}(\widetilde{g})$ is a hyperbolic plane and $\mathfrak{fix}_\infty(\widetilde{g})$ is its boundary;*

(4) $\widetilde{g}$ is semi-simple and

- *(a) if $a = -1$, $\widetilde{g}$ is elliptic, $\mathfrak{fix}(\widetilde{g})$ is a hyperbolic plane and $\mathfrak{fix}_\infty(\widetilde{g})$ is its boundary;*
- *(b) if $a \neq -1$, $\widetilde{g}$ is hyperbolic, $|\widetilde{g}| = \sqrt{2}|\log(|a|)|$, $\mathfrak{fix}(\widetilde{g})$ is a single axis and $\mathfrak{fix}_\infty(\widetilde{g})$ consists of ends of this axis.*

Proof. We analyze each case separately.

Case (1). By [FNS, Theorem 6.1], $|gg^{-T}| = 0$ and $\mathfrak{fix}_\infty(gg^{-T}) = c_1 \cup c_2 \cup c_3$, hence by Theorem B.1 and Corollary B.3, $|\widetilde{g}| = 0$ and $\mathfrak{fix}_\infty(\widetilde{g}) \subseteq c_1 \cup c_2 \cup c_3$. For arbitrary $x \in \mathbb{R} \setminus \{0\}$ we calculate

$$d\left(\widetilde{g}_x.\exp\left(\operatorname{diag}(-t,2t,-t)\right), \exp\left(\operatorname{diag}(t,t,-2t)\right)\right) =$$

$$= d\left(\exp\left(\operatorname{diag}\left(-\frac{t}{2},-\frac{t}{2},t\right)\right).g_x.\exp\left(\operatorname{diag}(t,-2t,t)\right), I\right) =$$

$$= d\left(\begin{bmatrix} \frac{1}{16x^4} & 0 & 0 \\ 0 & x^2(e^{-3t}+4) & -2x^2e^{-3t/2} \\ 0 & -2x^2e^{-3t/2} & 4x^2 \end{bmatrix}, I\right)$$

which is bounded when $t \to \infty$. This means that $\widetilde{g}.v_3 = v_2$ (geodesic ray $\exp\left(\operatorname{diag}(-t,2t,-t)\right)_{t>0}$ represents v_3 and $\exp\left(\operatorname{diag}(t,t,-2t)\right)_{t>0}$ represents v_2 in $\partial_T P_1(3,\mathbb{R})$). Similarly we get $\widetilde{g}.v_2 = v_3$. Because $\mathfrak{fix}_\infty(\widetilde{g})$ is connected and nonempty (see [FNS, §1]), the only fixed point of $\widetilde{g}$ at infinity is the barycenter C_2 of $[v_2, v_3] = c_2$.

Case (2). Similarly as in Case (1), from $|gg^{-T}| = 0$ we get $|\widetilde{g}| = 0$, from $\mathfrak{fix}_\infty(gg^{-T}) = c_1$ we get $\mathfrak{fix}_\infty(\widetilde{g}) \subseteq c_1$ and for arbitrary $x \in \mathbb{R}$ we calculate that $\widetilde{g}_x.v_1 = v_2$ and $\widetilde{g}_x.v_2 = v_1$, hence the only fixed point of $\widetilde{g}$ at infinity is the barycenter C_1 of c_1.

Case (3). The matrix gg^{-T} is orthogonal and $\mathfrak{fix}(gg^{-T}) = \{\exp(\operatorname{diag}(t,t,-2t)) | t \in \mathbb{R}\}$. Since for any $x \neq 0$, $\mathfrak{fix}(\widetilde{g}_x) \subseteq \{\exp(\operatorname{diag}(t,t,-2t)) | t \in \mathbb{R}\}$ by Theorem B.1, Lemma B.7 gives that the fixed point set of $\widetilde{g}_x$ is a single point that can be calculated using the conjugacy relation from the table above and the fact that g_x is orthogonal exactly when $x = \pm\sqrt{\frac{1+a}{2}}$ in which case $\widetilde{g}_x$ fixes $\{I\}$. By Lemma B.4, $\mathfrak{fix}_\infty(\widetilde{g}) = \emptyset$.

Case (3'a). Note that $\widetilde{g}$ is conjugate to $\widetilde{I}$ because $I = \sqrt{g^{-1}}g\sqrt{g^{-1}}^T$. Inversion on $P_1(3,\mathbb{R})$ acts as a reflection around I on any line $t \to \exp(tX)$, hence $\mathfrak{fix}(\widetilde{I}) = \{I\}$ and $\mathfrak{fix}_\infty(\widetilde{I}) = \partial\{I\} = \emptyset$. Conjugating again to get back $\widetilde{g}$ yields $\mathfrak{fix}(\widetilde{g}) = \{g\}$.

Case (3'b). Since the matrix g is symmetric, not positive, and has determinant 1, it has exactly two negative eigenvalues, and thus by Lemma B.5 the isometry $\widetilde{g}$ is conjugate to $\widetilde{g}'$ where $g' = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$. Observe that $\widetilde{g}'$ fixes two geodesic lines through I, namely

$$\exp(\operatorname{diag}(t,0,-t))_{t\in\mathbb{R}} \quad \text{and} \quad \exp\left(t\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)_{t\in\mathbb{R}}$$

(because $g'Xg'^T = -X$ for both possibilities X noted above), and by Lemma B.7, $\mathfrak{fix}(\widetilde{g}')$ and hence $\mathfrak{fix}(\widetilde{g})$ is isometric to a hyperbolic plane. By Lemma B.4, $\mathfrak{fix}_\infty(\widetilde{g}') = \partial\mathfrak{fix}(\widetilde{g}')$.

Case (4a). The isometry $gg^{-T} = \operatorname{diag}(1,-1,-1)$ has a large (3-dimensional) fixed point set, parametrized as

$$P_{s,t,u} := \begin{bmatrix} e^{-2u} & 0 & 0 \\ 0 & e^u e^s \cosh(t) & e^u \sinh(t) \\ 0 & e^u \sinh(t) & e^u e^{-s} \cosh(t) \end{bmatrix}, \quad s,t,u \in \mathbb{R}.$$

For arbitrary $x \neq 0$, the solution of the equation $g_x P_{s,t,u}^{-1} g_x^T = P_{s,t,u}$ is $u = \log(|x|)$. Hence the fixed point set for $\widetilde{g_x}$ is a hyperbolic plane by Lemma B.7 and $\mathfrak{fix}_\infty(\widetilde{g})$ is its boundary.

Case (4b). The isometry gg^{-T} and hence $\widetilde{g}$ is hyperbolic,

$$|\widetilde{g}| = \frac{1}{2}|gg^{-T}| = \frac{1}{2}2\sqrt{(\log|a|)^2 + \left(\log\frac{1}{|a|}\right)^2} = \sqrt{2}|\log(|a|)|.$$

Since for $x = \pm\sqrt{|a|}$ we have $\widetilde{g}_x.I = g_x g_x^T = \mathrm{diag}(1, \frac{1}{|a|}, |a|)$ and $d(\mathrm{diag}(1, \frac{1}{|a|}, |a|), I) = \sqrt{2}|\log(|a|)| = |\widetilde{g}_x|$, we know that $I \in \mathfrak{min}(\widetilde{g}_x)$. Hence $\widetilde{g}_x$ acts as a translation on the geodesic line through I and $\widetilde{g}_x.I = \mathrm{diag}(1, \frac{1}{|a|}, |a|)$, i.e. on the geodesic $\exp(\mathrm{diag}(0, -t, t))$. The axis of $\widetilde{g}_x$ for $x \neq \pm\sqrt{|a|}$ can be expressed using the conjugacy relation among different solutions g_x from the table above. By Lemma B.6, this single axis forms the whole minimal space. For the fixed point set of $\widetilde{g}_x$ at infinity we again use Lemma B.4 which says that $\mathfrak{fix}_\infty(\widetilde{g}_x) = \partial\mathfrak{min}(\widetilde{g}_x)$, hence the ends of the axis of $\widetilde{g}_x$ are the only fixed points at infinity of $\widetilde{g}_x$. ★

Remark B.10. *An interested reader can verify that in each case where the fixed point set of an elliptic isometry $\widetilde{g} \in SL(3,\mathbb{R})\sigma$ is isometric to a hyperbolic plane, the set $\mathfrak{fix}_\infty(\widetilde{g})$ consists of barycenters of certain chambers.*

B.3 On translation lengths of isometries from $\mathfrak{Iso}(P_1(n,\mathbb{R}))$

In this section we introduce a decomposition of an isometry of $P_1(n,\mathbb{R})$ from $PSL(n,\mathbb{R})$ into three commuting isometries, one (if nontrivial) hyperbolic, one elliptic and the third one (if nontrivial) parabolic with zero translation length. This result gives us a formula to calculate the translation length of any isometry of $P_1(n,\mathbb{R})$ for any $n \in \mathbb{N}$.

In every expression of the form $\sum_{\lambda\in\sigma(X)} \cdots$ below, eigenvalues λ from the spectrum $\sigma(X)$ are counted with multiplicities.

Theorem B.11. *Let $g \in PSL(n,\mathbb{R})$ be an isometry of $P_1(n,\mathbb{R})$. Then g is conjugated to a product HUJ, where all the factors commute, H is a positive diagonal matrix and U is an orthogonal matrix (and hence both are semi-simple isometries), and J is an upper triangular matrix with 1s on the diagonal. Furthermore, g is semi-simple exactly when $J = I$ and the translation length of g equals to the translation length of H and can be expressed as*

$$|g| = 2\sqrt{\sum_{\lambda\in\sigma(g)} \log(|\lambda|)^2}.$$

Proof. Every matrix $g \in SL(n,\mathbb{R})$ can be conjugated by another matrix in $SL(n,\mathbb{R})$ to take on a *modified real Jordan form*, namely a matrix of block diagonal form

$$\mathrm{diag}(D, D_O, J_1, J_2, \dots J_b, J_1^O, J_2^O, \dots J_a^O)$$

where the blocks are as follows. First, D is a pure diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \dots \lambda_d)$. Next, D_O has 2×2 blocks on the diagonal, which are $\mu_i O_i$, $i = 1, 2, \dots c$, for some $\mu_i \in (0,\infty)$ and some $O_i \in O(2)$. Each J_i is a nontrivial modified Jordan block of dimension m_i for real eigenvalues ν_i, $i = 1, 2, \dots b$, which means that it has ν_i on the diagonal and also on the first upper super diagonal (instead of 1s as in the classical Jordan form). Finally, J_i^O is a modified Jordan block of dimension $2k_i$ pertaining to complex eigenvalues, i.e. J_i^O is a block of the form

$$\begin{bmatrix} \kappa_i U_i & \kappa_i U_i & 0 & \dots & 0 & 0 \\ 0 & \kappa_i U_i & \kappa_i U_i & \dots & 0 & 0 \\ \vdots & & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \kappa_i U_i & \kappa_i U_i \\ 0 & 0 & 0 & \dots & 0 & \kappa_i U_i \end{bmatrix},$$

where $U_i \in O(2)$ and κ_i is the absolute value of the corresponding complex eigenvalue.

We will now express g as a product of commuting matrices H, U and J and then use the formula $|g| = \lim_{r\to\infty} \frac{1}{r} d(g^r.I, I) = \lim_{r\to\infty} \frac{1}{r}\sqrt{\mathrm{Tr}(\log(g^r g^{rT})^2)}$. The factors H, U, J are as follows. First, the diagonal matrix

$$\begin{aligned} H \;=\; & \operatorname{diag}\Big(|\lambda_1|,|\lambda_2|,\dots|\lambda_d|,\mu_1,\mu_1,\mu_2,\mu_2,\dots\mu_c,\mu_c, \\ & \underbrace{|\nu_1|,|\nu_1|,\dots|\nu_1|}_{m_1-\text{times}},\dots\underbrace{|\nu_b|,|\nu_b|,\dots|\nu_b|}_{m_b-\text{times}},\underbrace{\kappa_1,\kappa_1,\dots\kappa_1}_{2k_1-\text{times}},\dots\underbrace{\kappa_a,\kappa_a,\dots\kappa_a}_{2k_a-\text{times}}\Big). \end{aligned}$$

Next, the orthogonal matrix

$$\begin{aligned} U \;=\; & \operatorname{diag}\Big(\operatorname{sgn}(\lambda_1),\operatorname{sgn}(\lambda_2),\dots\operatorname{sgn}(\lambda_d),O_1,O_2,\dots O_c, \\ & \underbrace{\operatorname{sgn}(\nu_1),\operatorname{sgn}(\nu_1),\dots\operatorname{sgn}(\nu_1)}_{m_1-\text{times}},\dots\underbrace{\operatorname{sgn}(\nu_b),\operatorname{sgn}(\nu_b),\dots\operatorname{sgn}(\nu_b)}_{m_b-\text{times}},\underbrace{U_1,U_1,\dots U_1}_{k_1-\text{times}},\dots\underbrace{U_a,U_a,\dots U_a}_{k_a-\text{times}}\Big) \end{aligned}$$

and finally a Jordan form matrix J with only 1s on the diagonal,

$$J=\operatorname{diag}(\underbrace{1,1,\dots 1}_{(d+2c)-\text{times}},K_{m_1},K_{m_2},\dots K_{m_b},L_{k_1},L_{k_2},\dots L_{k_a}),$$

where K_i is an $i\times i$ Jordan block with 1s on the diagonal and first upper superdiagonal and L_i is a Jordan block with I_2s on the diagonal and first upper superdiagonal, hence a block of dimension $2i\times 2i$.

Note that g is diagonalizable over $\mathbb{C}$ (and hence semi-simple isometry by [BH, Proposition II.10.61]) exactly when there are no nontrivial (i.e. nonidentity) blocks among K_i and no nontrivial blocks among L_i. Therefore g is semi-simple isometry exactly when $J=I$.

Let us now compute the translation length of UJ. Because U and J commute, it means that $(UJ)^r(UJ)^{rT}=J^rU^rU^{rT}J^{rT}=J^rJ^{rT}$ and we get

$$|UJ|=\lim_{r\to\infty}\frac{1}{r}d((UJ)^r(UJ)^{rT},I)=\lim_{r\to\infty}\frac{1}{r}d(J^rJ^{rT},I)=|J|.$$

Take the geodesic ray $\gamma(t):=\exp(t\operatorname{diag}(u_1,u_2,\dots u_n))$, where $u_1>u_2>\dots>u_n$ and calculate

$$|J|\le\lim_{t\to\infty}d(J.\gamma(t),\gamma(t))=\lim_{t\to\infty}d\left(\gamma\left(-\frac{t}{2}\right).J.\gamma(t),I\right)=$$

$$=\lim_{t\to\infty}d\left(\left(\gamma\left(-\frac{t}{2}\right)J\gamma\left(\frac{t}{2}\right)\right).I,I\right).$$

Because J is an upper triangular matrix with 1s on the diagonal and the eigenvalues of the generator of the geodesic line γ are decreasing, the matrix $\gamma\left(-\frac{t}{2}\right)J\gamma\left(\frac{t}{2}\right)$ tends to the identity as t tends to infinity, see [BH, Proposition 10.64]. Hence the above limit equals 0 and also $|J|=|UJ|=0$.

Recall from the definition of H that it is a diagonal matrix with positive diagonal entries. Such a matrix acts as an elliptic isometry exactly when $H=I$, otherwise it acts as a translation on the geodesic line through I and H. It moves I to H^2 and one can easily compute

$$|H|=d(I,H.I)=d(I,H^2)=\|\log(H^2)\|_2=\sqrt{\sum_{\lambda\in\sigma(H)}\log(\lambda^2)^2}=2\sqrt{\sum_{\lambda\in\sigma(H)}\log(\lambda)^2}.$$

Compute further

$$2\sqrt{\sum_{\lambda\in\sigma(H)}\log(\lambda)^2}=|H|=$$

$$=\lim_{r\to\infty}\frac{1}{r}d(H^rH^{rT},I)=\lim_{r\to\infty}\frac{1}{r}d(H^rH^{rT},I)+\lim_{r\to\infty}\frac{1}{r}d((UJ)^r(UJ)^{rT},I)=$$

$$=\lim_{r\to\infty}\frac{1}{r}d((UJ)^rH^rH^{rT}(UJ)^{rT},(UJ)^r(UJ)^{rT})+\lim_{r\to\infty}\frac{1}{r}d((UJ)^r(UJ)^{rT},I)\ge$$

$$\ge\lim_{r\to\infty}\frac{1}{r}d((UJ)^rH^rH^{rT}(UJ)^{rT},I)=\lim_{r\to\infty}\frac{1}{r}d((HUJ)^r(HUJ)^{rT},I)=|g|\ge$$

$$\geq \lim_{r\to\infty} \frac{1}{r} d((UJ)^r H^r H^{rT} (UJ)^{rT}, (UJ)^r (UJ)^{rT}) - \lim_{r\to\infty} \frac{1}{r} d((UJ)^r (UJ)^{rT}, I) =$$

$$= \lim_{r\to\infty} \frac{1}{r} d(H^r H^{rT}, I) = |H|.$$

Because the absolute values of eigenvalues of g and their multiplicities are exactly the same as those of H we infer

$$|g| = 2\sqrt{\sum_{\lambda\in\sigma(g)} \log(|\lambda|)^2}.$$

★

Theorem B.11 together with Corollary B.3 yields the following corollary.

Corollary B.12. *Given $\widetilde{g} \in PSL(n,\mathbb{R})\sigma$, its translation length is*

$$|\widetilde{g}| = \sqrt{\sum_{\lambda\in\sigma(gg^{-T})} \log(|\lambda|)^2}.$$

Razširjeni povzetek

Uvod

V zgodnjih osemdesetih letih prejšnjega stoletja je Mikhail Gromov dokazal številne rezultate iz teorije Riemannovih mnogoterosti nepozitivnih prereznih ukrivljenosti brez uporabe gladke (Riemannove) strukture, temveč le preko lastnosti inducirane funkcije razdalje. Natančneje, njegov pristop se sklicuje na CAT(0) neenakost, ki ji inducirana funkcija razdalje na Riemannovi mnogoterosti nepozitivnih prereznih ukrivljenosti zadošča. Revolucionarni seminar, na katerem je predstavil svoje rezultate, je motiviral številne matematike, da so pričeli s študijem splošnih CAT(0) prostorov, tj. geodezičnih metričnih prostorov, katerih metrika zadošča CAT(0) neenakosti.

Za številne lastnosti Riemannovih mnogoterosti nepozitivnih prereznih ukrivljenosti je še danes neznano, ali se posplošijo na širši razred CAT(0) prostorov oziroma pod kakšnimi dodatnimi predpostavkami se to zgodi. Pomemben vir vprašanj v CAT(0) geometriji so tudi fundamentalne grupe kompaktnih Riemannovih mnogoterosti nepozitivnih prereznih ukrivljenosti. Vemo, da le-te delujejo kokompaktno diskretno z izometrijami na univerzalnem krovu, ki je CAT(0). Naravno vprašanje je torej, katere lastnosti teh fundamentalnih grup se posplošijo na grupe, ki delujejo kokompaktno diskretno z izometrijami na kakšnem CAT(0) prostoru. Kot kažejo številni primeri iz geometrične teorije grup, se geometrija prostora močno prepleta z algebraičnimi lastnostmi grupe, ki deluje na prostoru (z izometrijami, kokompaktno, diskretno ...). V tej disertaciji se ukvarjamo s konkretnim primerom problema v tem duhu. V nadaljevanju bo Γ vedno označevala diskretno grupo, ki deluje kokompaktno diskretno z izometrijami na nekem CAT(0) prostoru X. **Kokompaktnost** pomeni, da obstaja kompaktna podmnožica $K \subseteq X$, da je $\Gamma K = \bigcup_{\gamma \in \Gamma} \gamma K = X$, **diskretnost** pa, da je za vsako omejeno množico $B \subseteq X$ število elementov $\gamma \in \Gamma$, za ka tere $\gamma B \cap B \neq \emptyset$, končno. Kokompaktno diskretno delovanje z izometrijami bomo označevali tudi $\Gamma \overset{geo}{\curvearrowright} X$ ali da Γ deluje na X geometrično ali da je Γ CAT(0) grupa.

Problem periodičnih ravnin

Ena izmed trditev, ki so že dolga leta znane v kontekstu mnogoterosti nepozitivnih prereznih ukrivljenosti [LY], je izrek o ravnem torusu.

Izrek 1 (**Lawson in Yau, ravni torus**). *Naj bo M kompaktna Riemannova mnogoterost nepozitivnih prereznih ukrivljenosti. Če je $\mathbb{Z}^n$ podgrupa $\pi_1(M)$, tedaj je $\mathbb{R}^n$ vsebovan kot izometrično vložen podprostor v univerzalnem krovnem prostoru mnogoterosti M in na njem $\mathbb{Z}^n$ deluje s translacijami.*

Gromov je izrek posplošil v naslednjo obliko.

Izrek 2 (**Ravni torus, CAT(0)**). *Naj bo X CAT(0) prostor in $\Gamma \overset{geo}{\curvearrowright} X$. Naj $\mathbb{Z}^n \cong A \leq \Gamma$. Tedaj X vsebuje izometrično vloženo kopijo $\mathbb{R}^n$, na kateri A deluje kokompaktno diskretno s translacijami.*

Zgornja izreka govorita le o eni smeri – če Γ ($\pi_1(M)$) vsebuje podgrupo $\mathbb{Z}^n$, tedaj X ($\widetilde{M}$) vsebuje podprostor $\mathbb{R}^n$. Naravno je torej vprašanje obrata zgornjih izrekov.

Problem 3 (**Problem periodičnih ravnin**). *Naj bo X CAT(0) prostor ($\widetilde{M}$ univerzalni krov kompaktne Riemannove mnogoterosti M nepozitivnih prereznih ukrivljenosti), ki vsebuje podprostor $\mathbb{R}^n$. Ali Γ, ki deluje geometrično na X (ali $\pi_1(M)$) vsebuje podgrupo $\mathbb{Z}^n$?*

V začetku devedesetih let prejšnjega stoletja sta Bangert in Schröder v [BS] pritrdilno odgovorila na zgornje vprašanje za realno analitično mnogoterost M. Še več, dokazala sta, da za vsako kopijo maksimalnega evklidskega prostora $\mathbb{R}^n$ (tj. prostora, ki ni vsebovan v večjem evklidskem), obstaja podgrupa

$\mathbb{Z}^n$ v $\pi_1(M)$, ki na dani kopiji $\mathbb{R}^n$ deluje s translacijami. Tej situaciji pravimo, da je $\mathbb{R}^n$ **periodični evklidski podprostor**. Za ta strožji rezultat je že znano, da v kontekstu CAT(0) prostorov ne drži. V delih [W1, W2] sta primera $\Gamma \overset{geo}{\curvearrowright} X$, kjer X vsebuje evklidsko ravnino $\mathbb{R}^2$, ki ni periodična, in celo evklidsko ravnino, ki ni limita periodičnih ravnin. A v obeh primerih se kljub temu nekje v Γ nahaja podgrupa $\mathbb{Z}^2$ (ki po izreku 2 deluje kokompaktno s translacijami na nekem $\mathbb{R}^2 \subseteq X$).

Pomembnost ravnin v CAT(0) prostorih, ki dopuščajo geometrično delovanje kakšne grupe, se skriva v naslednjem razdelku.

Negativna ukrivljenost, hiperboličnost

Poddružina CAT(0) prostorov so CAT(κ) prostori za negativen parameter κ. Tako kot so CAT(0) prostori posplošitev Riemannovih mnogoterosti prereznih ukrivljenosti največ 0, so CAT(κ) prostori posplošitev Riemannovih mnogoterosti prereznih ukrivljenosti največ κ. O grupah, ki delujejo na CAT(κ) prostorih geometrično, je znano mnogo več kot o CAT(0) grupah. Znane so tudi kot *hiperbolične grupe*, vendar ta izraz originalno prihaja iz drugačne definicije negativne ukrivljenosti, prav tako Gromove.

Definicija 4. *Geodezični metrični prostor X je δ-hiperboličen za nenegativen parameter δ, če za poljubno trojico točk $x, y, z \in X$ velja, da poljubna geodetka med x in y leži v δ-okolici unije poljubnih geodetk med x in z ter med z in y. Če vrednost parametra δ ni pomembna, pravimo, da je X hiperbolični prostor. Grupa je hiperbolična, če deluje geometrično na kakšnem hiperboličnem prostoru.*

Po obnašanju "velikih" trikotnikov v hiperboličnih prostorih so le-ti podobni CAT(κ) prostorom za negativne κ (oziroma hiperbolični ravnini $\mathbb{H}^2$). Iz definicije CAT(κ) neenakosti je z nekaj računanja v $\mathbb{H}^2$ lahko preveriti, da so CAT(κ) prostori hiperbolični za $\kappa < 0$. Obrat seveda ne drži, saj definicija δ-hiperboličnosti ne zahteva ničesar od trikotnikov obsega manj kot δ, definicija CAT(κ) pa. Toda v grobi geometriji, s pogledom "od daleč", pa hiperbolični prostori seveda imajo duh negativne ukrivljenosti.

Čeprav ni znano, ali je vsaka hiperbolična grupa CAT(0), pa imajo hiperbolične grupe številne lastnosti, ki jih CAT(0) grupe v splošnem nimajo oz. za njih niso znane. Zato je z vidika CAT(0) in hiperbolične geometrije pomemben naslednji izrek.

Izrek 5 (**[BH, Theorem II.9.33]**)**.** *Naj bo X pravi kokompakten CAT(0) prostor (npr. $\Gamma \overset{geo}{\curvearrowright} X$). Tedaj je X hiperboličen natanko tedaj, ko ne vsebuje podprostora $\mathbb{R}^2$.*

Izrek podaja jasno oviro, ki CAT(0) prostoru preprečuje biti hiperboličen. Problem periodičnih ravnin pa osvetljuje to oviro še z algebraičnega vidika.

Trditev 6. *Če je odgovor na problem periodičnih ravnin pritrdilen, tedaj je CAT(0) grupa hiperbolična natanko tedaj, ko ne vsebuje kopije $\mathbb{Z}^2$.*

Naravni primer nehiperboličnega CAT(0) prostora je produkt vsaj dveh geodezično polnih faktorjev. Očitne ravnine v takšnem produktu so produkti geodezičnih premic v posameznih faktorjih. Natančneje, če je $X \cong X_1 \times \cdots \times X_m$ in so $\ell_1, \ldots \ell_m$ geodezične premice v ustreznih faktorjih, tedaj je $\ell_1 \times \cdots \times \ell_m$ izometrično $\mathbb{R}^m$. V disertaciji pritrdilno odgovorimo na problem periodičnih ravnin za evklidske podprostore takšne oblike.

Izrek 7 (**[CZ, Corollary 1]**)**.** *Naj bo X geodezično poln CAT(0) prostor, ki je produkt m faktorjev. Naj $\Gamma \overset{geo}{\curvearrowright} X$. Tedaj Γ vsebuje podgrupo $\mathbb{Z}^m$, ki deluje kokompakto diskretno s translacijami na $\ell_1 \times \cdots \times \ell_m$ za neke geodezične premice $\ell_i \subseteq X_i$.*

Čeprav obravnavamo zgolj te posebne oblike CAT(0) prostorov in evklidskih podprostorov v njih, pa dokaz zgornjega izreka zahteva uporabo globokih rezultatov strukturne teorije za grupe izometrij CAT(0) prostorov. Glavna vira uporabljene teorije sta [CM-ST, CM-DS].

Strukturna teorija za grupo izometrij CAT(0) prostora

Kombinacija rešitve Hilbertovega petega problema [MZ1, MZ2, T1] ter izrekov o kanoničnem razcepu [FL, Theorem 1.1] ter o podgrupah edinkah grupe izometrij geodezično polnega CAT(0) prostora, ki dopušča geometrično delovanje kakšne grupe [CM-ST, Theorem 1.10], pripelje do naslednjega izreka.

Izrek 8. *Naj bo X geodezično poln CAT(0) prostor in $\Gamma \overset{geo}{\curvearrowright} X$. Tedaj je X izometričen produktu (z ℓ^2-metriko)*

$$\mathbb{R}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q, \quad n, p, q \geq 0,$$

kjer so $X_1, \ldots X_p, Y_1, \ldots Y_q$ nerazcepni, neizometrični $\mathbb{R}$ in je

(i) grupa izometrij X_i je enostavna Liejeva grupa s končno povezanimi komponentami za vse i in

(ii) grupa izometrij Y_j je popolnoma nepovezana lokalno kompaktna topološka grupa za vse j.

Poleg tega velja še, da je grupa izometrij X končna razširitev direktnega produkta grup izometrij $\mathbb{R}^n$, X_i in Y_j za vse i, j.

Izrek je ključen za dokaz izreka 7 v primeru, da je evklidski faktor trivialen. Če je n iz zgornjega izreka pozitiven, se najprej skličemo na [CM-ST, Theorem 3.8], ki je nekakšna posplošitev Bieberbachovega izreka o mrežah (diskretnih podgrupah končnega kovolumna) v $\mathbb{R}^n \rtimes O(n)$.

Izrek 9. *Naj $\Gamma \overset{geo}{\curvearrowright} X$, kjer je X geodezično poln CAT(0) prostor, ki razpade kot produkt $\mathbb{R}^n \times X'$. Tedaj ima Γ podgrupo končnega indeksa, ki razpade kot direktni produkt $\mathbb{Z}^n \times \Gamma'$, kjer $\mathbb{Z}^n$-faktor deluje trivialno na X' in Γ' deluje kokompaktno diskretno z izometrijami na X'.*

Ta trditev nam omogoča, da se že na začetku znebimo evklidskega faktorja. Z drugimi besedami, Γ (virtualno) vsebuje faktor $\mathbb{Z}^n$ in njena projekcija na grupo izometrij prostora X' deluje geometrično na X'. Vendar pa faktorjev z Liejevo grupo izometrij ter faktorjev s popolnoma nepovezano grupo izometrij z vidika grupe Γ ne ločimo tako zlahka. Projekcija Γ na grupo izometrij X_i ali Y_i je kaj lahko nediskretna oziroma gosta. Da obidemo ta problem, moramo kombinirati rezultate iz teorije Liejevih grup [Pra, PR, Sel] ter dejstva o CAT(0) prostorih s popolnoma nepovezano grupo izometrij, [CM-ST, §6].

Skica dokaza

Po razpravi iz prejšnjega razdelka smo v situaciji, ko $\Gamma \overset{geo}{\curvearrowright} X \cong X' \times Y_1 \times \cdots \times Y_q$, kjer je grupa izometrij X' polenostavna Liejeva grupa in so grupe izometrij faktorjev Y_j popolnoma nepovezane lokalno kompaktne grupe. Vsi faktorji so geodezično polni. Poiskali bomo element $\gamma \in \Gamma$, ki deluje kot hiperbolična izometrija na vseh faktorjih prostora X. Še več, minimalni prostor X'-komponente izometrije γ bo maksimalen evklidski podprostor v X'. Takšen γ ima minimalni prostor $\mathfrak{min}(\gamma)$ izometričen $\mathbb{R}^{r+q}$, kjer je r rang grupe izometrij X'. Po naslednjem izreku centralizator $\mathfrak{Z}_\Gamma(\gamma)$ deluje geometrično na $\mathfrak{min}(\gamma)$, torej lahko ponovno uporabimo izrek 9, da zagotovimo $\mathbb{Z}^{r+q} \leq \Gamma$, saj

$$\mathfrak{min}(\gamma) = \mathfrak{min}(\gamma_{X'}) \times \mathfrak{min}(\gamma_{Y_1}) \times \cdots \times \mathfrak{min}(\gamma_{Y_q}) \cong \mathbb{R}^r \times (\mathbb{R} \times C_1) \times \cdots \times (\mathbb{R} \times C_q).$$

Izrek 10 (**[Rua, Theorem 3.2]**). *Naj $\Gamma \overset{geo}{\curvearrowright} X$. Tedaj za poljuben $\gamma \in \Gamma$ velja, da $\mathfrak{Z}_\Gamma(\gamma) \overset{geo}{\curvearrowright} \mathfrak{min}(\gamma)$.*

Pri iskanju kandidata za γ najprej poiščemo ustrezen element v grupi Γ, ki deluje kot hiperbolična izometrija na vseh faktorjih Y_j. Za to uporabimo argument, ki ga Swenson uporabi pri dokazu obstoja hiperbolične izometrije v poljubni (neskončni) CAT(0) grupi, [Swe, Theorem 11], v kombinaciji z rigidnostjo kota Aleksandrova, ki zagotavlja "diskretno" obnašanje poljubnih eliptičnih izometrij CAT(0) prostora s popolnoma nepovezano grupo izometrij.

Izrek 11 (**Rigidnost kota Aleksandrova, [CM-ST, Proposition 6.8]**). *Naj bo X pravi kokompakten geodezično poln CAT(0) prostor s popolnoma nepovezano grupo izometrij. Tedaj obstaja $\varepsilon > 0$, tako da za poljubno eliptično izometrijo g prostora X, za poljubno točko x, za katero $gx \neq x$, velja $\sphericalangle_c(x, gx) \geq \varepsilon$, kjer je c projekcija x na množico g-fiksnih točk.*

To pa je natanko lastnost, ki jo Swensonov argument uporabi za dokaz hiperboličnosti ustreznega elementa. V našem primeru jo uporabimo q-krat, po enkrat na vsakem faktorju Y_j, kar nam da element γ, ki deluje hiperbolično na vsakem od Y_j-faktorjev. Naj bo $\gamma = (\gamma_{X'}, \gamma_Y)$.

V drugem koraku uporabimo lastnost odprtih stabilizatorjev točk v $Y_1 \times \cdots \times Y_q$.

Izrek 12 (**[CM-ST, Theorem 6.1]**). *Naj bo X pravi kokompakten geodezično poln CAT(0) prostor s popolnoma nepovezano grupo izometrij. Tedaj je stabilizator poljubne omejene podmnožice X odprt v grupi izometrij prostora X.*

Ker je izometrija g, za katero je $[g^{-1}x, x] \cup [x, gx] = [g^{-1}x, gx]$ za kakšen x, hiperbolična z $x \in \mathfrak{min}(g)$, so vsi elementi iz $U\gamma$ hiperbolični na vseh faktorjih Y_i, kjer je U stabilizator množice $\{y, \gamma y\}$ za kakšen $y \in \mathfrak{min}(\gamma)$. Po izreku 12 pa je $U\gamma$ odprta, kar nam daje dovolj prostora, da zagotovimo maksimalno regularnost tudi na X'-komponenti nekega elementa, ki deluje kot hiperbolična izometrija na vseh faktorjih Y_i. Velja namreč naslednja trditev.

Trditev 13 ([CM-DS, **Lemma 3.2**]). *Naj bo Γ kokompaktna mreža v $G \times H$, kjer je G polenostavna Liejeva grupa in H popolnoma nepovezana lokalno kompaktna grupa. Naj bo U poljubna odprta kompaktna podgrupa v H. Tedaj je $\Gamma \cap (G \times U)$ kokompaktna mreža v $G \times U$, njena projekcija na G pa kokompaktna mreža v G.*

V $\Gamma_U := \mathrm{pr}_G(\Gamma \cap (G \times U))$, kjer je G grupa izometrij X' in U stabilizator $\{y, \gamma y\}$ za neki $y \in \mathfrak{min}(\gamma)$, po klasičnih izrekih iz teorije Liejevih grup poiščemo $\mathbb{R}$-regularni element γ' (tj. element z maksimalnim centralizatorjem, torej element, ki ima minimalni prostor izometričen $\mathbb{R}^r$). Slednji obstaja, saj so $\mathbb{R}$-regularni elementi Zariski odprti [Pra, Theorem], mreža pa je Zariski gosta po Borelovem izreku o gostoti, [Bor, Trditev (ii)]. Po [PR, Lemma 3.5] obstaja še Zariski odprta podmnožica $V = V(\gamma') \subseteq G$ z naslednjo lastnostjo. Za vsak $v \in V$ obstaja $n_v \in \mathbb{N}$, tako da $\gamma'^n v$ deluje kot $\mathbb{R}$-regularen element na X' za vsak $n \geq n_v$. Torej je $\Gamma_U \cap (V\gamma_{X'})$ neprazna (pomnimo, $\gamma_{X'}$ je X'-komponenta izometrije $\gamma \in \Gamma$, ki deluje hiperbolino na vseh Y_i-faktorjih). Izberimo element $w \in \Gamma_U \cap (V\gamma_{X'}^{-1})$. Torej je $w\gamma_{X'} \in V$, kar po definiciji pomeni, da je za dovolj velik $n \in \mathbb{N}$ element $\gamma'^n w\gamma_{X'} \in \Gamma_U$ deluje kot $\mathbb{R}$-regularen element na X'. Naj bo $(\gamma'', u) \in \Gamma \cap (G \times U)$ iz pr_G-praslike $\gamma'^n w \in \Gamma_U$. Tedaj je $(\gamma'', u)\gamma = (\gamma'^n w\gamma_{X'}, u\gamma_Y)$ ustrezen element, ki deluje $\mathbb{R}$-regularno na X' in kot hiperbolična izometrija na vseh Y_j-faktorjih.

Izjava

Podpisani Gašper Zadnik izjavljam, da je disertacija z naslovom *Pomen ravnin v CAT(0) geometriji* oziroma *Significance of flats in CAT(0) geometry* plod lastnega raziskovalnega dela.

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