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NIL RINGS AND PRIME RINGS

Doctoral thesis

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NILKOLOBARJI IN PRAKOLOBARJI

Doktorska disertacija

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Izjava

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- da sem doktorsko disertacijo z naslovom *Nilkolobarji in prakolobarji* oziroma *Nil rings and prime rings* izdelal samostojno pod mentorstvom prof. dr. Matjaža Omladiča in somentorstvom doc. dr. Primoža Moravca in
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Abstract

The problem of characterizing zero product preserving maps has been studied by several authors in many different settings. Recently such maps have been considered on prime rings with nontrivial idempotents. Most of the known results assume that the map in question is bijective. In the thesis we extend these results by considering non-injective maps. More precisely, we characterize surjective additive zero product preserving maps $\theta : A \to B$, where A is a ring with a nontrivial idempotent and B is a prime ring. We also investigate maps on rings with involution that preserve zeros of xy^*. In particular, we obtain a characterization of surjective additive maps $\theta : A \to B$ such that for all $x, y \in A$ we have $\theta(x)\theta(y)^* = 0$ if and only if $xy^* = 0$. Here A is a unital prime ring with involution that contains a nontrivial idempotent and B is a prime ring with involution.

In the second part of the thesis we devote our attention to nil rings. One of the most important open problems concerning nil rings is the Köthe conjecture, which states that a ring with no nonzero nil ideals should have no nonzero nil one-sided ideals. There are many known statements that are equivalent to the Köthe conjecture and we add one more to the list. It has been proved that, when considering the validity of these statements, we may restrict ourselves to algebras over fields. We observe in the thesis that we may additionally restrict ourselves to finitely generated prime algebras. Furthermore, we investigate the connections between nilpotent, algebraic, and quasi-regular elements. It is well known that an algebraic Jacobson radical algebra over a field is nil. We generalize this result to algebras over certain PIDs and in particular to rings. On the way to this result we introduce the notion of a π-algebraic element, i.e. an element that is a zero of a polynomial with the sum of coefficients equal to one. As a corollary we show that if every element of a ring R is π-algebraic then R is a nil ring, and at the same time obtain a new characterization of the upper nilradical. At the end we investigate the structure of the set of all π-algebraic elements of a ring.

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Keywords: prime ring, zero product preservers, rings of quotients, extended centroid, involution, nil ring, upper nilradical, Jacobson radical, integral ring, π-algebraic element, quasi-regular element, Köthe conjecture

Povzetek

Problem karakterizacije preslikav, ki ohranjajo ničelni produkt, so študirali številni avtorji v mnogih različnih kontekstih. Nedavno so bile take preslikave obravnavane na prakolobarjih z netrivialnimi idenpotenti. Večina znanih rezultatov predpostavlja, da so omenjene preslikave bijektivne. V disertaciji razširimo te rezultate tako, da obravnavamo neinjektivne preslikave. Natančneje, podamo karakterizacijo surjektivnih aditivnih preslikav $\theta : A \to B$, ki ohranjajo ničelni produkt, kjer je A kolobar z netrivialnim idempotentom, B pa prakolobar. Raziščemo tudi preslikave na kolobarjih z involucijo, ki ohranjajo ničle xy^*. V posebnem karakteriziramo surjektivne aditivne preslikave $\theta : A \to B$, za katere za vse $x, y \in A$ velja $\theta(x)\theta(y)^* = 0$ natanko tedaj, ko je $xy^* = 0$. Pri tem je A enotski prakolobar z involucijo, ki vsebuje netrivialen idempotent, B pa prakolobar z involucijo.

V drugem delu disertacije se posvetimo nilkolobarjem. Eden najpomembnejših odprtih problemov s področja nilkolobarjev je Köthejeva domneva, ki pravi, da kolobar brez neničelnih nilidealov nima niti neničelnih nil enostranskih idealov. Znanih je mnogo trditev, ki so ekvivalentne Köthejevi domnevi, in mi dodamo še eno na ta seznam. Dokazano je bilo, da se je za obravnavo veljavnosti teh trditev dovolj omejiti na algebre nad komutativnimi obsegi. V disertaciji opazimo, da se lahko še dodatno omejimo na končno generirane praalgebre. Poleg tega raziščemo povezave med nilpotentnimi, algebraičnimi in kvaziregularnimi elementi. Znano je, da je vsaka algebraična Jacobsonovo radikalna algebra nad komutativnim obsegom nilalgebra. Ta rezultat posplošimo na algebre nad določenimi glavnimi kolobarji in v posebnem na kolobarje. Na poti do tega rezultata vpeljemo pojem π-algebraičnega elementa, tj. elementa, ki je ničla polinoma z vsoto koeficientov ena. Posledično dokažemo, da je kolobar, v katerem je vsak element π-algebraičen, avtomatično nilkolobar, hkrati pa dobimo tudi novo karakterizacijo zgornjega nilradikala. Na koncu raziščemo strukturo množice vseh π-algebraičnih elementov kolobarja.

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Ključne besede: prakolobar, ohranjevalci ničelnega produkta, kolobarji kvocientov, razširjeni centroid, involucija, nilkolobar, zgornji nilradikal, Jacobsonov radikal, celosten kolobar, π-algebraičen element, kvaziregularen element, Köthejeva domneva

Contents

List of notations

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{Z}/p\mathbb{Z}$	the prime field of characteristic p
$[\cdot,\cdot]$	the commutator
$\operatorname{Im}\theta$	the image of θ
$\lvert G\rvert, \operatorname{card} G$	the cardinality of G
$\operatorname{Ann}_R M$	the annihilator of M in R
$I \lhd R$	I is an ideal of R
I_e	the ideal generated by e
R^1	ring R with identity adjoined
$Z(R)$	the center of R
$C(R)$	the extended centroid of R
$Q^r(R), Q^l(R), Q^s(R)$	the Martindale right, left, and symmetric rings of quotients of R
$Q_{mr}(R), Q_{ml}(R)$	the maximal right and left rings of quotients of R
$M_n(R)$	the ring of $n \times n$ matrices over R
$R[x]$	the ring of polynomials over R in one variable
$R[x_1, x_2, x_3, \ldots]$	the ring of polynomials over R in commutative variables $x_1, x_2, x_3, \ldots$
$F\langle X\rangle$	the free F-algebra over the set X
$F\langle x_1, x_2, x_3, \ldots\rangle$	the free F-algebra over the set $\{x_1, x_2, x_3, \ldots\}$
$F[[x]]$	the ring of formal power series in one variable over F
$N(R)$	the set of all nilpotent elements of R
$Nil^*(R)$	the upper nilradical of R
$Nil_*(R)$	the lower nilradical of R
$L(R)$	the Levitzki radical of R
$Q(R)$	the set of all quasi-regular elements of R
$J(R)$	the Jacobson radical of R
$a \circ b$	the quasi-product of a and b
$a^{(-1)}$	the quasi-inverse of a
$A_K(R)$	the set of all algebraic elements of R over K
$I_K(R)$	the set of all integral elements of R over K
$\pi_K(R)$	the set of all π-algebraic elements of R over K

Chapter 1

Prime rings

1.1 Basic definitions and properties

Prime rings are one of the more important types of rings in modern mathematics. They are the most fruitful generalization of integral domains to the setting of non-commutative rings and they come into play in many different fields of mathematics such as algebraic geometry, radical theory, the theory of polynomial and functional identities and many others.

Throughout this section R will denote a ring unless specified otherwise. We begin with the definition of a prime ring.

Definition 1.1.1. A ring R is a *prime ring* if $R \neq 0$ and for any two ideals $I, J \triangleleft R$, $IJ = 0$ implies $I = 0$ or $J = 0$.

We shall also need the notion of a prime ideal.

Definition 1.1.2. An ideal P of a ring R is a *prime ideal* if R/P is a prime ring.

In particular P is a prime ideal if $P \neq R$ and for any two ideals $I, J \triangleleft R$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Note that R is a prime ring if and only if 0 is a prime ideal of R.

Examples of prime rings are the ring of integers, the ring $M_n(F)$ of $n \times n$ matrices over a field F and the ring $F[x]$ of all polynomials over a field F. Also every simple ring is prime.

The above definitions are the standard definitions of prime rings and prime ideals, however, they are usually not the most appropriate to work with. It is often easier to use one of the following equivalent forms.

Proposition 1.1.3. *For an ideal $P \subsetneq R$ the following statements are equivalent:*

(i) *P is a prime ideal,*

(ii) *for any two left ideals $I, J \subseteq R$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$,*

(iii) *for any two right ideals $I, J \subseteq R$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$,*

(iv) *for any two elements* $a, b \in R$, $aRb \subseteq P$ *implies* $a \in P$ *or* $b \in P$.

Note that when verifying (ii) or (iii) we may assume that I and J contain P. The only nontrivial part of Proposition 1.1.3 is the implication (i) implies (iv). Since we will most often make use of (iv), we sketch the proof of this implication. So suppose (i) holds and let $aRb \subseteq P$ for some $a, b \in R$. Then $I = \sum RaR$ and $J = \sum RbR$ are ideals of R with $IJ \subseteq P$. By (i) this implies $RaR \subseteq P$ or $RbR \subseteq P$. Now define $L = \{x \in R\ ;\ RxR \subseteq P\}$. Clearly L is an ideal of R such that $RLR \subseteq P$. Since $P \neq R$ this implies again by (i) that $L \subseteq P$. Consequently $a \in P$ or $b \in P$.

Definition 1.1.4. A ring R is a *semiprime ring* if for any ideal $I \triangleleft R$, $I^2 = 0$ implies $I = 0$. An ideal $P \triangleleft R$ is a *semiprime ideal* if R/P is a semiprime ring.

Clearly every prime ideal is semiprime. The precise relation between these two notions is the following (see [19] for the proof).

Theorem 1.1.5. *An ideal* $P \triangleleft R$ *is a semiprime ideal if and only if* P *is an intersection of prime ideals of* R.

Prime and semiprime ideals and rings have many favourable properties. We mention a few that we shall need. The first one is that any semiprime ring R has zero left and right annihilator. Indeed, if a is an element of the right or left annihilator of a semiprime ring R then $aRa = 0$, which implies $a = 0$.

The second property concerns the relation between ideals of rings and ideals of algebras. For a commutative unital ring K the definition of a prime (semiprime) K-algebra and a prime (semiprime) ideal of a K-algebra is the same as in the ring case. Given a K-algebra R one can view R also solely as a ring. Given an ideal I of the ring R, I need not be an ideal of the algebra R, since it need not be closed for scalar multiplication. So one has to be a little bit careful when passing from rings to algebras and vice versa. However, a semiprime ideal of the ring R is always closed for scalar multiplication, and hence it is also an ideal of the algebra R. Indeed, if P is a prime ideal of R, a an element of P, and $k \in K$ a scalar then $(ka)R(ka) = a(k^2R)a \subseteq aRa \subseteq P$, which implies $ka \in P$.

Lemma 1.1.6. *Let* R *be a* K*-algebra, where* K *is a unital commutative ring. Then any semiprime ideal of the ring* R *is an ideal of the algebra* R.

In an arbitrary ring, or more general an arbitrary algebra over a PID, the set of all torsion elements may be very complicated. This is often an obstacle when generalizing results from algebras over fields to algebras over PIDs. In prime rings and algebras the torsion elements are much more manageable.

Proposition 1.1.7. *Let* K *be a PID and* R *a prime* K*-algebra. If* $ka = 0$ *for some* $k \in K$ *and* $0 \neq a \in R$ *then* $kR = 0$. *In particular, either* R *has no* K*-torsion or there exists an irreducible element* $p \in K$ *such that* $pR = 0$.

Proof. Suppose $ka = 0$ for some $k \in K$ and $0 \neq a \in R$. Then $(kr)Ra = rR(ka) = 0$ for every $r \in R$. Since R is prime, this implies $kr = 0$ for every $r \in R$. If R has K-torsion then $kR = 0$ for some $0 \neq k \in K$ by what we have just proved. Let $k = p_1 p_2 \dots p_n$ be the decomposition of k into irreducible elements and let $I_j = p_j R$ for every $j = 1, 2, \dots, n$. Then each I_j is an ideal of R and $I_1 I_2 \dots I_n \subseteq kR = 0$. Since R is prime, this implies $I_i = 0$ for some $1 \leq i \leq n$. $\square$

The last property that we mention is that the semiprimeness can often be a substitute for the existence of an identity element. To make it more clear what we mean by this, we present a concrete example. Given a unital ring R it is well known that every ideal of the matrix ring $M_n(R)$ is of the form $M_n(J)$ for some ideal $J \lhd R$. It is essential here that R has an identity. If R is not unital this need not be true. For example let I denote the identity matrix in $M_n(\mathbb{Z})$. Then $2I \in M_n(2\mathbb{Z})$ and the ideal $\mathbb{Z} \cdot 2I + M_n(4\mathbb{Z})$ of $M_n(2\mathbb{Z})$ is not of the desired form if $n \geq 2$. However, if P is a prime ideal of $M_n(R)$ then it is always of the desired form, even if R is not a unital ring. The main step in showing this is to prove that (assuming $n = 2$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in P \qquad \text{implies} \qquad \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in P$$

and similarly for other entries. If R is unital, this is clear; all we have to do is multiply the given matrix by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

from both sides. If R is not unital, it is equivalent to showing that P is also an ideal of $M_n(R^1)$, which then reduces the problem to the unital case. The ideal of $M_n(R^1)$ generated by P is equal to $\widehat{P} = \sum M_n(R^1) P M_n(R^1)$, hence we have $(\widehat{P})^3 \subseteq \sum \widehat{P} M_n(R^1) P M_n(R^1) \widehat{P} \subseteq \sum \widehat{P} P \widehat{P}$. Observe that $\widehat{P} \subseteq M_n(R)$ since $P \subseteq M_n(R)$ and $M_n(R) \lhd M_n(R^1)$. Consequently $(\widehat{P})^3 \subseteq P$ and the semiprimeness of P implies $\widehat{P} \subseteq P$, i.e. P in an ideal of $M_n(R^1)$.

For additional properties of prime and semiprime rings we refer the reader to [19].

1.2 Rings of quotients

The method of localization of a commutative ring R at a given multiplicative subset S and in particular the construction of the field of fractions of a given integral domain K is one of the most important tools in commutative algebra. In the noncommutative setting similar constructions are possible, but only when the subset S satisfies certain additional properties. In particular, what we would call 'the (right or left) ring of fractions' of a noncommutative domain may not always exist. In fact, there are noncommutative domains, that cannot be embedded into a division ring (see [18, Theorem 9.11] for a concrete example that was discovered by Mal'cev). Consequently many other rings of quotients have been defined, that seem to be more suitable for the noncommutative setting. The aim of this section is to present some of these rings along with their constructions and a few of their properties that we shall need. For a more detailed discussion of the theory see [4] and [31]. Although several of these constructions work in a wider class of rings, it is not surprising that these rings of quotients have much nicer properties if the starting ring is prime. Some of the constructions in fact require the ring to be prime or at least semiprime. As we are only interested in prime rings, we shall assume throughout that the starting ring is prime.

At the core of the construction of various rings of quotients lie the so called *dense* ideals.

Definition 1.2.1. A right ideal J of a ring R is *dense* if for every $a, b \in R$, $b \neq 0$, there exists $r \in R$ such that $br \neq 0$ and $ar \in J$.

Note that a dense right ideal is always nonzero. Dense left ideals of R are defined in a similar fashion using left multiplication by r.

Remark 1.2.2. Why these ideals are called *dense* is partially explained by the following property. Let K and L be right ideals of R and $f : K \to R$ and $g : L \to R$ homomorphisms of right R-modules. If f and g agree on some dense right ideal $J \subseteq K \cap L$ then they agree on $K \cap L$. In particular, if $f(J) = 0$ then $f = 0$. Indeed, if $f(x) \neq g(x)$ for some $x \in K \cap L$ then there would exist $r \in R$ such that $(f(x) - g(x))r \neq 0$ and $xr \in J$. But this would lead to a contradiction $0 \neq (f(x) - g(x))r = f(xr) - g(xr) = 0$ since f and g agree on J.

We need the following basic property of dense right ideals (see [4, Proposition 2.1.1] for the proof).

Lemma 1.2.3. *Let J and K be dense right ideals of R and $g : K \to R$ a homomorphism of right R-modules. Then $J \cap K$ and $g^{-1}(J)$ are dense right ideals of R.*

We are now ready to construct the first ring of quotients, which is in a sense the largest one. Let R be a prime ring and denote by S the set of all pairs $(f; J)$, where J is a dense right ideal of R and $f : J \to R$ is a homomorphism of right R-modules. Define a relation $\sim$ on S by $(f; J) \sim (g; K)$ if f and g agree on $J \cap K$. Referring to Remark 1.2.2 it is not hard to show that $\sim$ is in fact an equivalence relation on S. We denote the equivalence class of $(f; J) \in S$ by $[f; J]$ and the set of all equivalence classes in S by $Q_{mr}(R)$. We define addition and multiplication on $Q_{mr}(R)$ by

$$\begin{aligned} [f; J] + [g; K] &= [f + g; J \cap K], \\ [f; J] \cdot [g; K] &= [f \circ g; g^{-1}(J)], \end{aligned}$$

where $f \circ g$ denotes the composition of f and g. Observe that by Lemma 1.2.3 the right ideals $J \cap K$ and $g^{-1}(J)$ are indeed dense in R. It is straightforward to show that these operations are well defined and satisfy all ring axioms. We note only that, since R is a prime ring, the left annihilator of R is 0, hence R is a dense right ideal of R. Consequently $[\overline{0}; R]$ is the zero of $Q_{mr}(R)$, where $\overline{0} : R \to R$ is the zero homomorphism. In addition, $[id; R]$ is an identity in $Q_{mr}(R)$.

Definition 1.2.4. The ring $Q_{mr}(R)$ is called the *maximal right ring of quotients* of R.

The ring $Q_{mr}(R)$ was first constructed by Utumi [36] in 1956 in the form presented above. For a different more homological approach to the construction of $Q_{mr}(R)$ we refer the reader to [18, §13], where it is also explained why this ring is called the *maximal* right ring of quotients.

The ring $Q_{mr}(R)$ can be characterized by a certain set of properties ([4, Proposition 2.1.7]).

Theorem 1.2.5. *For a prime ring R the ring $Q_{mr}(R)$ satisfies the following properties:*

(i) *R is a subring of $Q_{mr}(R)$,*

(ii) *for every $q \in Q_{mr}(R)$ there exists a dense right ideal J of R such that $qJ \subseteq R$,*

(iii) *for every $q \in Q_{mr}(R)$ and every dense right ideal J of R, $qJ = 0$ implies $q = 0$,*

(iv) *for every dense right ideal J of R and every homomorphism of right R-modules $f : J \to R$ there exists $q \in Q_{mr}(R)$ such that $f(x) = qx$ for all $x \in J$.*

Furthermore, these properties characterize $Q_{mr}(R)$ up to an isomorphism over R.

Proof. The ring R is a subring of $Q_{mr}(R)$ via the inclusion $F : R \to Q_{mr}(R)$ given by $F(a) = [l_a; R]$, where l_a is the left multiplication by a. We skip the verification that F is indeed an injective ring homomorphism. Henceforth we identify R with its image $F(R)$. Given $[f; J] \in Q_{mr}(R)$ and $a \in J$ we have $[f; J]a = [f; J][l_a; R] = [f \circ l_a; l_a^{-1}(J)]$. Since f is a homomorphism of right R-modules and $a \in J$ it follows that $f \circ l_a = l_{f(a)}$ and $l_a^{-1}(J) = R$, hence $[f; J]a = [l_{f(a)}; R] = f(a)$. Therefore $[f; J]J = f(J) \subseteq R$, which proves (ii). To prove (iii) let $[f; J] \in Q_{mr}(R)$ and let K be a dense right ideal of R such that $[f; J]K = 0$. As above this implies $f(J \cap K) = 0$. By Lemma 1.2.3 $J \cap K$ is a dense right ideal, hence $f(J) = 0$ by Remark 1.2.2, i.e. $[f; J] = 0$. Now let J be a dense right ideal of R and $f : J \to R$ a homomorphism of right R-modules. Then as above $f(x) = [f; J]x$ for all $x \in J$, which proves (iv).

Now suppose Q is another ring that satisfies the properties in the theorem. Define a map $H : Q \to Q_{mr}(R)$ by $H(q) = [l_q; \{x \in R \ ; \ qx \in R\}]$ for all $q \in Q$. The property (ii) ensures that $\{x \in R \ ; \ qx \in R\}$ is a dense right ideal of R, so H is well defined. The property (iii) ensures that H is injective and the property (iv) ensures that H is surjective. Clearly H fixes the elements of R. To verify that H is a ring homomorphism is straightforward. $\square$

Observe that property (iii) in Theorem 1.2.5 implies that the element q in property (iv) is uniquely determined by f.

In a similar fashion we define the *maximal left ring of quotients* $Q_{ml}(R)$. For this purpose we have to use dense left ideals of R and right multiplications by elements of R. It should be noted that multiplication must be defined by $[f; J] \cdot [g; K] = [g \circ f; f^{-1}(K)]$ in this case.

We shall need a few basic properties of the maximal right ring of quotients. Of course the same properties hold for the maximal left ring of quotients as well.

Proposition 1.2.6. *Let R be a prime ring and $p, q \in Q_{mr}(R)$. If $pRq = 0$ then $p = 0$ or $q = 0$.*

Proof. By Theorem 1.2.5 (ii) there exist dense right ideals J and K of R such that $pJ, qK \subseteq R$. Observe that $(pJ)R(qK) \subseteq pRqK = 0$. Since R is prime, this implies $pJ = 0$ or $qK = 0$, hence $p = 0$ or $q = 0$ by Theorem 1.2.5 (iii). $\square$

Let R be a prime ring and S a ring such that $R \subseteq S \subseteq Q_{mr}(R)$. Then Proposition 1.2.6 implies that S is a prime ring as well. In particular, the maximal right ring of quotients of a prime ring is again a prime ring.

We can view Q_{mr} as an operator that assigns to each prime ring R its maximal right ring of quotients $Q_{mr}(R)$, which is again a prime ring. It turns out that this operator is idempotent (see [4, Proposition 2.1.10] for the proof).

Proposition 1.2.7. *Let R be a prime ring and S a subring of $Q_{mr}(R)$ that contains R. Then S is a prime ring and $Q_{mr}(S) \cong Q_{mr}(R)$. In particular, $Q_{mr}(R)$ is a prime ring and $Q_{mr}(Q_{mr}(R)) \cong Q_{mr}(R)$.*

Next we shall define the so called *Martindale rings of quotients* named after W.S. Martindale who introduced them in 1969. In contrast to maximal rings of quotients, these rings are defined using two-sided ideals. Given a prime ring R and a two-sided ideal $I \triangleleft R$ it is not hard to see that I is dense as a right ideal of R if and only if $I \neq 0$. This explains the following definition. For a prime ring R define

$$Q^r(R) = \{q \in Q_{mr}(R) \ ; \ qI \subseteq R \text{ for some } 0 \neq I \triangleleft R\}.$$

Let $p, q \in Q^r(R)$ and choose $0 \neq I, J \triangleleft R$ such that $pI, qJ \subseteq R$. Then $I \cap J \supseteq JI \neq 0$ since R is a prime ring. In addition, $(p-q)I \cap J \subseteq R$ and $(pq)JI \subseteq pRI \subseteq pI \subseteq R$, hence $p - q, pq \in Q^r(R)$. This shows that $Q^r(R)$ is a subring of $Q_{mr}(R)$.

Definition 1.2.8. The ring $Q^r(R)$ is called the *Martindale right ring of quotients* of R.

Since the only difference between the maximal right ring of quotients and the Martindale right ring of quotients is that two-sided ideals are used instead of one-sided ideals, it is not surprising that the latter can be characterized similarly as the former ([4, Proposition 2.2.1]).

Theorem 1.2.9. *For a prime ring R the ring $Q^r(R)$ satisfies the following properties:*

(i) *R is a subring of $Q^r(R)$,*

(ii) *for every* $q \in Q^r(R)$ *there exists* $0 \neq I \triangleleft R$ *such that* $qI \subseteq R$*,*

(iii) *for every* $q \in Q^r(R)$ *and every* $0 \neq I \triangleleft R$*,* $qI = 0$ *implies* $q = 0$*,*

(iv) *for every* $0 \neq I \triangleleft R$ *and every homomorphism of right* R*-modules* $f : I \to R$ *there exists* $q \in Q^r(R)$ *such that* $f(x) = qx$ *for all* $x \in I$*.*

Furthermore, these properties characterize $Q^r(R)$ *up to an isomorphism over* R*.*

In the construction of $Q_{mr}(R)$ as a ring of equivalence classes of partial homomorphisms, the subring $Q^r(R)$ is the subring of those classes, that correspond to homomorphisms of right R-modules $f : I \to R$, where I is a nonzero ideal of R.

In the same manner we define the *Martindale left ring of quotients* $Q^l(R)$ as a subring of $Q_{ml}(R)$.

For a prime ring R define

$$Q^s(R) = \{q \in Q_{mr}(R)\ ;\ qI \cup Iq \subseteq R \text{ for some } 0 \neq I \triangleleft R\}.$$

Similarly as for $Q^r(R)$ we can show that $Q^s(R)$ is a subring of $Q_{mr}(R)$. Clearly $Q^s(R) \subseteq Q^r(R) \subseteq Q_{mr}(R)$.

Definition 1.2.10. The ring $Q^s(R)$ is called the *Martindale symmetric ring of quotients* of R.

Passman has showed that the ring $Q^s(R)$ can also be characterized by four properties analogous to those for $Q^r(R)$ and $Q_{mr}(R)$ ([4, Proposition 2.2.3]).

Theorem 1.2.11. *For a prime ring* R *the ring* $Q^s(R)$ *satisfies the following properties:*

(i) R *is a subring of* $Q^s(R)$*,*

(ii) *for every* $q \in Q^s(R)$ *there exists* $0 \neq I \triangleleft R$ *such that* $qI \cup Iq \subseteq R$*,*

(iii) *for every* $q \in Q^s(R)$ *and every* $0 \neq I \triangleleft R$*,* $qI = 0$ *or* $Iq = 0$ *implies* $q = 0$*,*

(iv) *for every* $0 \neq I \triangleleft R$*, every homomorphism of right* R*-modules* $f : I \to R$*, and every homomorphism of left* R*-modules* $g : I \to R$ *such that* $xf(y) = g(x)y$ *for all* $x, y \in I$*, there exists* $q \in Q^s(R)$ *such that* $f(x) = qx$ *and* $g(x) = xq$ *for all* $x \in I$*.*

Furthermore, these properties characterize $Q^s(R)$ *up to an isomorphism over* R*.*

Observe that, since R is a prime ring, the identity for f and g in property (iv) itself implies, that f and g are homomorphisms of R-modules, so we could omit this in the formulation.

We have defined the ring $Q^s(R)$ as a subring of $Q_{mr}(R)$, however it can also be viewed as a subring of $Q_{ml}(R)$, since it is isomorphic to its analogue

$$\{q \in Q_{ml}(R)\ ;\ qI \cup Iq \subseteq R \text{ for some } 0 \neq I \triangleleft R\} \subseteq Q_{ml}(R).$$

Given $q \in Q^s(R)$ and $0 \neq I \triangleleft R$ such that $qI \cup Iq \subseteq R$, the isomorphism maps $q = [l_q; I]$ to $q' = [r_q; I]$, where l_q and r_q are left and right multiplications by q.

Due to its symmetry the ring $Q^s(R)$ inherits a lot more structure from R then the rings $Q^r(R)$ and $Q_{mr}(R)$. In particular, if R is a prime ring with involution then the involution can be extended to $Q^s(R)$ (see also [4, Proposition 2.5.4]).

Proposition 1.2.12. *Let R be a prime. Any involution on R can be extended uniquely to an involution on $Q^s(R)$.*

Proof. Let $*$ denote the involution on R. Given an element $q \in Q^s(R)$ there is an ideal $0 \neq I \lhd R$ such that $qI \cup Iq \subseteq R$. Hence we can define $f, g : I^* \to R$ by $f(x) = (x^*q)^*$ and $g(x) = (qx^*)^*$ for all $x \in I^*$. Observe that I^* is a nonzero ideal of R and that the maps f and g satisfy $xf(y) = g(x)y$ for all $x, y \in I^*$. Thus by Theorem 1.2.11 there exists a unique element $q^\# \in Q^s(R)$ such that $f(x) = q^\# x$ and $g(x) = xq^\#$ for all $x \in I^*$. It is straightforward to show that the map $q \to q^\#$ is an involution on $Q^s(R)$ that extends $*$. Let us just mention that in the construction of $Q^s(R) \subseteq Q_{mr}(R)$ this involution corresponds to the map $[l_q; I] \to [* \circ r_q \circ *; I^*]$. $\square$

If we attempt to define an involution on $Q_{mr}(R)$ in a similar way, we actually end up with an anti-isomorphism $Q_{mr}(R) \to Q_{ml}(R)$.

Definition 1.2.13. The center of $Q^s(R)$ is denoted by $C(R)$ and called the *extended centroid* of R.

It turns out that $C(R)$ coincides with the center of any other ring of quotients defined above ([4, Remark 2.3.1]).

Proposition 1.2.14. *For every prime ring R,*

$$C(R) = Z(Q^r(R)) = Z(Q_{mr}(R)) = \{q \in Q_{mr}(R) \ ; \ qr = rq \text{ for all } r \in R\}.$$

Proof. Denote $Z := \{q \in Q_{mr}(R) \ ; \ qr = rq \text{ for all } r \in R\}$. It suffices to prove that $Z \subseteq Q^s(R) \cap Z(Q_{mr}(R))$. Take $q \in Z$. By Theorem 1.2.5 there exists a dense right ideal J of R such that $qJ \subseteq R$. Then $\widehat{J} = J + \sum RJ$ is a nonzero ideal of R. Since q commutes with elements of R, we have $\widehat{J}q = q\widehat{J} = qJ + \sum RqJ \subseteq R$, hence $q \in Q^s(R)$. Let x be an arbitrary element of $Q_{mr}(R)$. Then there exists a dense right ideal K of R such that $xK \subseteq R$. This implies $(xq - qx)k = xqk - q(xk) = xkq - (xk)q = 0$ for all $k \in K$, i.e. $(xq - qx)K = 0$. Hence $xq - qx = 0$ by Theorem 1.2.5, which shows that $q \in Z(Q_{mr}(R))$. $\square$

In the construction of $Q_{mr}(R)$ as a ring of equivalence classes of homomorphisms of right R-modules the ring $C(R)$ is the subring of all equivalent classes $[f; I]$, where I is a nonzero ideal of R and $f : I \to R$ is a homomorphism of (R, R)-bimodules.

The next proposition gives the main reason why $C(R)$ plays an important role.

Proposition 1.2.15. *For every prime ring R, $C(R)$ is a field.*

Proof. Let $0 \neq c \in C(R) \subseteq Q^s(R)$ and let I be a nonzero ideal of R such that $cI \cup Ic \subseteq R$. Since c commutes with elements of R, the annihilator $\mathrm{Ann}_R(c) = \{x \in R \ ; \ cx = 0\}$ is a two-sided ideal of R such that $c \,\mathrm{Ann}_R(c) = 0$. By Theorem 1.2.11 this implies $\mathrm{Ann}_R(c) = 0$ since $c \neq 0$. This shows that we have a well defined homomorphism of right R-modules $f : cI \to R$ given by $f(cx) = x$ for all $x \in I$. Observe that cI is a nonzero two-sided ideal of R since c commutes with elements of R. Hence by Theorem 1.2.9 there exists $c' \in Q^r(R)$ such that $f(y) = c'y$ for all $y \in cI$, i.e. $x = c'cx$ for all $x \in I$. This implies $(1-c'c)I = 0$, therefore $c'c = cc' = 1$. Consequently $c' \in C(R)$. □

To summarize, for every prime ring R we have

$$C(R) \subseteq Q^s(R) \subseteq Q^r(R) \subseteq Q_{mr}(R) \quad \text{and} \quad C(R) \subseteq Q^s(R) \subseteq Q^l(R) \subseteq Q_{ml}(R),$$

where all these are prime rings and $C(R)$ is a field.

We conclude with a few standard examples to demonstrate the above definitions. The proofs can mostly be found in [18].

Example 1.2.16. For a commutative domain K we have $Q_{mr}(K) = Q^r(K) = Q^s(K) = C(K) \cong F$, where F denotes the field of fractions of K.

Example 1.2.17. For a simple unital ring R, $Q^r(R) = Q^s(R) \cong R$ and $C(R) \cong Z(R)$.

Example 1.2.18. For the matrix ring $R = M_n(\mathbb{Z})$ we have $Q_{mr}(R) = Q^r(R) = Q^s(R) \cong M_n(\mathbb{Q})$ and $C(R) \cong \mathbb{Q}$.

Example 1.2.19. Let k be a field and σ an automorphism of k of infinite order. Let $R = k[x;\sigma]$ be the skew polynomial ring over k. The elements of R are polynomials with coefficients in k, the addition is the usual addition of polynomials, and the multiplication is defined by $x\lambda = \sigma(\lambda)x$ for all $\lambda \in k$ (and extended appropriately). It turns out that R has a division ring of fractions D. It can be shown that

$$\begin{aligned} Q_{mr}(R) &\cong D, \\ Q^r(R) = Q^s(R) &\cong k[x, x^{-1}; \sigma], \\ C(R) &\cong k^\sigma. \end{aligned}$$

Here k^σ is the field of all fixed points of σ and $k[x, x^{-1}; \sigma]$ is the skew Laurent polynomial ring over k (its elements are Laurent polynomials, i.e. finite Laurent series, addition is the usual addition of Laurent series, and multiplication is defined by $x\lambda = \sigma(\lambda)x$ and $x^{-1}\lambda = \sigma^{-1}(\lambda)x^{-1}$ for all $\lambda \in k$).

Example 1.2.20. Let R be the ring of all $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{C}$ of the form $A + \lambda I$, where A is a finite matrix (i.e. matrix with only finitely many nonzero entries), I is the identity matrix, and $\lambda \in \mathbb{C}$. It is not hard to see that the set of all finite matrices is the only proper nonzero ideal of R. This makes it possible to calculate the Martindale rings of quotients of R. It turns out that

$$\begin{aligned} Q^l(R) &\cong \{\mathbb{N} \times \mathbb{N} \text{ matrices over } \mathbb{C} \text{ with finite rows}\}, \\ Q^r(R) &\cong \{\mathbb{N} \times \mathbb{N} \text{ matrices over } \mathbb{C} \text{ with finite columns}\}, \\ Q^s(R) &\cong \{\mathbb{N} \times \mathbb{N} \text{ matrices over } \mathbb{C} \text{ with finite rows and columns}\}, \\ C(R) &\cong \mathbb{C}. \end{aligned}$$

1.3 Zero product preservers

The theory of preservers is a vast area with applications in many branches of mathematics as well as mathematical physics. It has been an active research area for decades and still is. Loosely speaking a *preserver* is a map between two rings or algebras that preserves certain properties of elements, relations between elements, subsets of elements, or certain operations or identities. The aim of the theory is to characterize all maps that preserve a particular property, i.e. to describe how these maps look like. The most thoroughly studied preservers are preserver on matrix algebras and operator algebras. Classical examples are commutativity preservers, preservers of rank, determinant preservers, adjacency preservers, preservers of spectrum, and preservers of the group of invertible elements to name just a few. In the present section we devote our attention to a specific example of preservers, the zero product preservers.

Definition 1.3.1. Let A and B be two rings. A map $\theta : A \to B$ is said to *preserve zero product* if $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$.

Zero product preservers have been studied by many authors in many different settings. We first mention the result for matrix algebras proved by Chebotar et al. [9, Corollary 2.4], which will also demonstrate what is the expected form of a zero product preserving map.

Theorem 1.3.2. *Let F be an algebraically closed field of characteristic zero and $\theta : M_n(F) \to M_r(F)$ a linear zero product preserving map, where n and r are positive integers with $n \geq 2$ and $n \geq r$. Then either $\operatorname{Im}\theta$ has trivial multiplication or $n = r$ and there exists an invertible matrix $A \in M_n(F)$ and a scalar $\lambda \in F$ such that $\theta(X) = \lambda AXA^{-1}$ for all $X \in M_n(F)$.*

Observe that if $\operatorname{Im}\theta$ has nontrivial multiplication then the map θ is a scalar multiple of an algebra homomorphism. In general the expected form of a zero product preserving map is similar; a homomorphism multiplied by a central element. As we often require the map in question to be surjective, the central element usually turns out to be invertible. Clearly any such map is indeed zero product preserving.

Several other authors have considered zero product preservers in other settings. Wong [41] characterized bijective semilinear zero product preserving maps on simple finite dimensional algebras and more generally on a class of primitive algebras. Ajauro and Jarozs [3] considered zero product preserving maps on subalgebras of the Banach algebra of bounded linear operators and on spaces of continuous operator valued functions. Cui and Hou [10] characterized bounded surjective linear zero product preserving maps on von Neumann algebras. Chebotar et al. [9] characterized surjective bounded linear zero product preserving maps on unital C^*-algebras and also considered such maps on certain standard operator algebras. In 2004 Chebotar et al. [8] generalized some of the above results by considering bijective additive zero product preserving maps on prime rings with nontrivial idempotents. In particular they proved the following theorem ([8, Theorem 1]).

Theorem 1.3.3. *Let A and B be prime rings and $\theta : A \to B$ a bijective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that $Q_{mr}(A)$ contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$.*

(i) *If $1 \in A$, then $\theta(xy) = \lambda\theta(x)\theta(y)$ for all $x, y \in A$, where $\theta(1) \in Z(B)$ and $\lambda = 1/\theta(1) \in C(B)$. In particular, if $\theta(1) = 1$ then θ is a ring isomorphism from A onto B.*

(ii) *If $\deg B \geq 3$, then there exists $\lambda \in C(B)$, the extended centroid of B, such that $\theta(xy) = \lambda\theta(x)\theta(y)$ for all $x, y \in A$.*

Since this theorem describes zero product preservers on general prime rings, the additional condition of existence of a nontrivial idempotent is needed to ensure that we have enough zero products in A. Otherwise A could be a domain, in which case any additive map defined on A would trivially preserve zero product.

Wang [37] has shown that the technical assumption $\deg B \geq 3$ can be removed from this theorem. Moreover Brešar [6] has replaced the assumption that A is a prime ring with a weaker assumption implying that A contains a noncentral idempotent.

In the following theorem we further extend these results by considering surjective (not necessarily injective) additive zero product preserving maps. This result is contained in [33].

Theorem 1.3.4. *Let A be a ring and B a prime ring. Let $\theta : A \to B$ be a surjective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that R is a unital ring that contains A as a subring and let e be an idempotent in R such that $eA \cup Ae \subseteq A$. Denote $f = 1 - e$. If either $e \in A$, $f \in A$, or $A = \sum A^2$ then one of the following holds:*

(i) $\theta(eA + Ae + AeA) = 0$,

(ii) $\theta(fA + Af + AfA) = 0$,

(iii) *there exists $0 \neq \lambda \in C(B)$ such that $\theta(xy) = \lambda\theta(x)\theta(y)$ for all $x, y \in A$.*

We divide the proof into 4 steps. In Steps 1 and 3 we follow the methods used in [8, Theorem 1], slightly modifying the calculations in Step 3. Step 2 is new and is needed to deal with the non-injectivity of θ. In Step 4 we use a similar approach as was used in [37] in order to eliminate the assumption $\deg B \geq 3$. However we avoid the use of the theory of functional identities and present a direct proof instead. We give here the whole proof of the theorem for the sake of completeness.

Proof of Theorem 1.3.4. First notice that $fA \cup Af \subseteq A$. We want to show that the map θ satisfies the identity

$$\theta(xy)\theta(z) = \theta(x)\theta(yz) \qquad \text{for all } x, y, z \in A. \tag{1.1}$$

Since θ is additive and $y = eye + eyf + fye + fyf$ for all $y \in A$ we only need to prove the identity (1.1) for $y \in eAe \cup eAf \cup fAe \cup fAf$.

Step 1. First we prove the identity (1.1) for $y \in eAf \cup fAe$. Let $x, z \in A$. Since $(xe)(z - ez) = 0$, we have $\theta(xe)\theta(z - ez) = 0$ and hence $\theta(xe)\theta(z) = \theta(xe)\theta(ez)$ due to the additivity of θ. Similarly $(x - xe)(ez) = 0$ implies $\theta(x - xe)\theta(ez) = 0$, therefore $\theta(x)\theta(ez) = \theta(xe)\theta(ez)$. Both identities together imply

$$\theta(xe)\theta(z) = \theta(x)\theta(ez) \qquad \text{for all } x, y \in A. \tag{1.2}$$

By symmetry of e and f we also have

$$\theta(xf)\theta(z) = \theta(x)\theta(fz) \qquad \text{for all } x, y \in A. \tag{1.3}$$

Now let $x, y, z \in A$. Since $(xe + xeyf)(eyfz - fz) = 0$, we have

$$\theta(xe + xeyf)\theta(eyfz - fz) = 0.$$

Expanding the left-hand side we get

$$\theta(xe)\theta(eyfz) - \theta(xe)\theta(fz) + \theta(xeyf)\theta(eyfz) - \theta(xeyf)\theta(fz) = 0.$$

Taking into account that $(xe)(fz) = (xeyf)(eyfz) = 0$, which implies $\theta(xe)\theta(fz) = \theta(xeyf)\theta(eyfz) = 0$, this identity reduces to

$$\theta(xeyf)\theta(fz) = \theta(xe)\theta(eyfz).$$

Applying (1.3) on the left and (1.2) on the right we get

$$\theta(xeyf)\theta(z) = \theta(x)\theta(eyfz) \qquad \text{for all } x, y, z \in A. \tag{1.4}$$

Again by symmetry of e and f we also have

$$\theta(xfye)\theta(z) = \theta(x)\theta(fyez) \qquad \text{for all } x, y, z \in A. \tag{1.5}$$

Step 2. In what will follow we will shorten the calculations from [8] a bit. But before we proceed, we prove the following:

(a) $\theta(fAfAeA) = 0$ implies $\theta(AeAf) = 0$ or $\theta(fAeA) = 0$,

(b) the following four conditions are equivalent:
$\theta(AfAe) = 0, \quad \theta(fAeA) = 0, \quad \theta(AeAf) = 0, \quad \theta(eAfA) = 0,$

(c) $\theta(eAfA) = 0$ implies $\theta(eA + Ae + AeA) = 0$ or $\theta(fA + Af + AfA) = 0$.

(a): Let $\theta(fAfAeA) = 0$. Then by (1.5)

$$\theta(AfAfAe)\theta(A) = \theta(A)\theta(fAfAeA) = 0.$$

Since θ is surjective and B is prime this implies $\theta(AfAfAe) = 0$. Hence by (1.4), (1.5) and (1.2)

$$\theta(AeAf)\theta(A)\theta(fAeA) = \theta(A)\theta(eAfAfAe)\theta(A) = \theta(Ae)\theta(AfAfAe)\theta(A) = 0.$$

Since θ is surjective and B is prime this implies $\theta(AeAf) = 0$ or $\theta(fAeA) = 0$.

(b): By (1.5) we have $\theta(AfAe)\theta(A) = \theta(A)\theta(fAeA)$. Since θ is surjective and B prime this means that the first two conditions in (b) are equivalent. By symmetry of e and f the last two conditions in (b) are also equivalent. In addition by (1.5) and (1.3) we have

$$\theta(AfAe)\theta(A)\theta(fAeA) = \theta(A)\theta(fAeAf)\theta(AeA) = \theta(Af)\theta(AeAf)\theta(AeA),$$

so the third condition in (b) implies the first two. By symmetry of e and f the first condition in (b) implies the last two. So all the conditions are equivalent.

(c): Let $\theta(eAfA) = 0$ and let $x, y \in A$. Then by (b) we have $\theta(xfye) = \theta(fxey) = \theta(xeyf) = \theta(exfy) = 0$. Since $f = 1 - e$ this implies $\theta(xye) = \theta(xeye)$, $\theta(xey) = \theta(exey)$, $\theta(xey) = \theta(xeye)$ and $\theta(exy) = \theta(exey)$. Hence

$$\theta(exy) = \theta(xey) = \theta(xye) \qquad \text{for all } x, y \in A. \tag{1.6}$$

Since $e = 1 - f$ this also implies

$$\theta(fxy) = \theta(xfy) = \theta(xyf) \qquad \text{for all } x, y \in A. \tag{1.7}$$

If $A = \sum A^2$ then (1.6) implies $\theta(ex) = \theta(xe)$ for all $x \in A$ since θ is additive. If $e \in A$ then (1.6) with $x = e$ implies $\theta(ey) = \theta(eye)$ for all $y \in A$ and (1.6) with $y = e$ implies $\theta(exe) = \theta(xe)$ for all $x \in A$. These two identities together then imply $\theta(ex) = \theta(xe)$ for all $x \in A$. Similarly if $f \in A$ then (1.7) implies $\theta(fx) = \theta(xf)$ for all $x \in A$. So in any case we have

$$\theta(ex) = \theta(xe) \qquad \text{for all } x \in A \tag{1.8}$$

and

$$\theta(fx) = \theta(xf) \qquad \text{for all } x \in A, \tag{1.9}$$

since these two conditions are in fact equivalent. By (1.2), (1.3) and (1.8) we have

$$\theta(Af)\theta(A)\theta(eA) = \theta(A)\theta(fAe)\theta(A) = \theta(A)\theta(efA)\theta(A) = \theta(A)\theta(0)\theta(A) = 0,$$

where $\theta(0) = 0$ since θ is additive. Hence either $\theta(Af) = 0$ or $\theta(eA) = 0$. In the first case (1.9) and (1.7) imply $\theta(Af) = \theta(fA) = \theta(AfA) = 0$ and in the later case (1.8) and (1.6) imply $\theta(eA) = \theta(Ae) = \theta(AeA) = 0$. This completes the proof of (c).

Step 3. Now we prove the identity (1.1) for $y \in eAe \cup fAf$. In view of (a), (b) and (c), if either $\theta(fAfAeA) = 0$ or $\theta(eAfA) = 0$ then one of the conditions (i) and (ii) from the theorem holds. So we may assume that $\theta(fAfAeA) \neq 0$ and $\theta(eAfA) \neq 0$. Let $x, y, z, w, u, v \in A$ be arbitrary. Then

$$\begin{aligned}
&\theta(xeye)\theta(zf)\theta(w)\theta(eufv) \\
=\ &\theta(xeye)\theta(z)\theta(fweuf)\theta(v) && \text{by (1.3) and (1.4)} \\
=\ &\theta(xey)\theta(ezfwe)\theta(uf)\theta(v) && \text{by (1.2) and (1.5)} \\
=\ &\theta(xeyezf)\theta(we)\theta(uf)\theta(v) && \text{by (1.4)} \\
=\ &\theta(x)\theta(eyezfwe)\theta(uf)\theta(v) && \text{by (1.4)} \\
=\ &\theta(x)\theta(eyez)\theta(fweuf)\theta(v) && \text{by (1.5)} \\
=\ &\theta(x)\theta(eyezf)\theta(w)\theta(eufv) && \text{by (1.3) and (1.4)}\ ,
\end{aligned}$$

which implies $(\theta(xeye)\theta(zf) - \theta(x)\theta(eyezf))\theta(w)\theta(eufv) = 0$. Since θ is surjective, B is prime and $\theta(eAfA) \neq 0$ this implies

$$\theta(xeye)\theta(zf) = \theta(x)\theta(eyezf) \qquad \text{for all } x, y, z \in A. \tag{1.10}$$

Let $x, y, z, w, t, u, v \in A$ be arbitrary. Then

$$\begin{aligned}
& \theta(xeye)\theta(ze)\theta(w)\theta(ftfuev) \\
= \ & \theta(xeye)\theta(z)\theta(ewftfue)\theta(v) && \text{by (1.2) and (1.5)} \\
= \ & \theta(xey)\theta(ezewftf)\theta(ue)\theta(v) && \text{by (1.2) and (1.4)} \\
= \ & \theta(xeyezewf)\theta(tf)\theta(ue)\theta(v) && \text{by (1.4)} \\
= \ & \theta(x)\theta(eyezewftf)\theta(ue)\theta(v) && \text{by (1.4)} \\
= \ & \theta(x)\theta(eyez)\theta(ewftfue)\theta(v) && \text{by (1.4)} \\
= \ & \theta(x)\theta(eyeze)\theta(w)\theta(ftfuev) && \text{by (1.2) and (1.5)}\ ,
\end{aligned}$$

which implies $(\theta(xeye)\theta(ze) - \theta(x)\theta(eyeze))\theta(w)\theta(ftfuev) = 0$. Again since θ is surjective, B is prime and $\theta(fAfAeA) \neq 0$ this implies

$$\theta(xeye)\theta(ze) = \theta(x)\theta(eyeze) \qquad \text{for all } x, y, z \in A. \tag{1.11}$$

Summing (1.10) and (1.11) gives

$$\theta(xeye)\theta(z) = \theta(x)\theta(eyez) \qquad \text{for all } x, y, z \in A. \tag{1.12}$$

By symmetry of e and f we also have

$$\theta(xfyf)\theta(z) = \theta(x)\theta(fyfz) \qquad \text{for all } x, y, z \in A. \tag{1.13}$$

Finally summing (1.4), (1.5), (1.12) and (1.13) gives the identity (1.1).

Step 4. Now we treat the identity (1.1). Since $\theta : A \to B$ is surjective, let $\psi : B \to A$ be any set-theoretic right inverse of θ, so that $\theta\psi = id_B$. Let $y \in A$ be a fixed element and define

$$f(X) = \theta(\psi(X)y) \qquad \text{and} \qquad g(Z) = \theta(y\psi(Z))$$

for all $X, Z \in B$. Then by (1.1)

$$f(X)Z = \theta(\psi(X)y)\theta(\psi(Z)) = \theta(\psi(X))\theta(y\psi(Z)) = Xg(Z)$$

for all $X, Z \in B$. By Theorem 1.2.11 there exists a unique $q \in Q^s(B) \subseteq Q_{mr}(B)$ such that

$$f(X) = Xq \qquad \text{and} \qquad g(Z) = qZ$$

for all $X, Z \in B$ (the above identity itself implies that $f : B \to B$ is a homomorphism of left B-modules and $g : B \to B$ is a homomorphism of right B-modules). Let $x, z \in A$ be arbitrary. Since $\theta\psi = id_B$ we have

$$\theta(xy)\theta(z) = \theta(x)\theta(yz) = \theta(\psi(\theta(x)))\theta(yz) = \theta(\psi(\theta(x))y)\theta(z),$$

hence $(\theta(xy)-\theta(\psi(\theta(x))y))\theta(z)=0$. Since θ is surjective and B is prime this implies

$$\theta(xy)=\theta(\psi(\theta(x))y)=f(\theta(x))=\theta(x)q \qquad \text{for all } x\in A.$$

Similarly we get

$$\theta(yz)=\theta(y\psi(\theta(z)))=g(\theta(z))=q\theta(z) \qquad \text{for all } z\in A.$$

Since q depends on y we shall write $q=h(y)$. Then $h:A\to Q_{mr}(B)$ is a map that satisfies

$$\theta(x)h(y)=\theta(xy)=h(x)\theta(y) \qquad \text{for all } x,y\in A. \tag{1.14}$$

Since θ is surjective this identity implies $h(A)\subseteq I_{Q_{mr}(B)}(B)$, where $I_{Q_{mr}(B)}(B)=\{r\in Q_{mr}(B)\ ;\ rB\subseteq B,\ Br\subseteq B\}$ is the idealizer of B in $Q_{mr}(B)$. Now define

$$H(X)=h(\psi(X))$$

for all $X\in B$. Then $H:B\to I_{Q_{mr}(B)}(B)$ and

$$H(X)Y=h(\psi(X))\theta(\psi(Y))=\theta(\psi(X))h(\psi(Y))=XH(Y)$$

for all $X,Y\in B$. Since $I_{Q_{mr}(B)}(B)$ is a ring such that $B\subseteq I_{Q_{mr}(B)}(B)\subseteq Q_{mr}(B)$ and B is a prime ring, Theorem 1.2.7 implies that $I_{Q_{mr}(B)}(B)$ is a prime ring as well and $Q_{mr}(I_{Q_{mr}(B)}(B))=Q_{mr}(B)$. Observe that B is a nonzero ideal of $I_{Q_{mr}(B)}(B)$, hence Theorem 1.2.11 together with the above identity implies that there exists a unique $\lambda\in Q^s(I_{Q_{mr}(B)}(B))\subseteq Q_{mr}(I_{Q_{mr}(B)}(B))=Q_{mr}(B)$ such that

$$H(X)=X\lambda \qquad \text{and} \qquad H(Y)=\lambda Y$$

for all $X,Y\in B$. This implies $X\lambda=\lambda X$ for all $X\in B$, hence $\lambda\in C(B)$. Let $x,y\in A$ be arbitrary. Then by (1.14)

$$h(x)\theta(y)=\theta(x)h(y)=\theta(\psi(\theta(x)))h(y)=h(\psi(\theta(x)))\theta(y),$$

hence $(h(x)-h(\psi(\theta(x))))\theta(y)=0$. As before this implies

$$h(x)=h(\psi(\theta(x)))=H(\theta(x))=\lambda\theta(x) \qquad \text{for all } x\in A.$$

By (1.14) we have $\theta(xy)=\lambda\theta(x)\theta(y)$ for all $x,y\in A$ as required. Suppose $\lambda=0$. Then

$$\theta(xy)=0 \qquad \text{for all } x,y\in A. \tag{1.15}$$

If $A=\sum A^2$ then (1.15) implies $\theta(A)=0$, which is impossible since θ is surjective. If $e\in A$ then (1.15) implies $\theta(eA+Ae+AeA)=0$ and θ satisfies condition (i) of the theorem. Similarly if $f\in A$ then θ satisfies condition (ii) of the theorem. Therefore we may assume that $\lambda\neq 0$. $\square$

We should remark that in order to get any information from Theorem 1.3.4, the idempotent e must be nontrivial, otherwise one of the conditions (i) and (ii) will be satisfied trivially. Even more, e must act nontrivially on A, i.e. not as 0 or 1. This in a way means that e must have some meaningful connection to A. In Theorem 1.3.3 this connection was that e lies in the maximal right ring of quotients of A.

In certain situations we can guarantee that condition (iii) of Theorem 1.3.4 will hold. Suppose that e acts nontrivially on A. Then $eA + Ae + \sum AeA$ and $fA + Af + \sum AfA$ are nonzero ideals of A. So if A is a simple ring or if θ is injective then the condition (iii) must hold. This shows in particular that in [6, Corollary 4.3] no additional assumptions on the ring A are needed besides the existence of a nontrivial idempotent.

Observe that if a map θ satisfies condition (iii) of Theorem 1.3.4 then it indeed preserves zero product. On the other hand conditions (i) or (ii) are not sufficient for θ to preserve zero product. So one would think that perhaps more information about θ could be extracted. However, a simple example shows that this is not the case. In other words the conclusions of (i) and (ii) are optimal.

Example 1.3.5. Let T be a ring, S a domain, B a prime ring, and $A = S \oplus T$. Embed S and T into unital rings S^1 and T^1. Then A is embedded into $R = S^1 \oplus T^1$ and $e = (1, 0) \in R$ is an idempotent such that $eA + Ae + AeA = S$ and $fA + Af + AfA = T$. For an arbitrary (additive, surjective) map $\phi : S \to B$ with $\phi(0) = 0$ the map $\theta : A \to B$ defined by $\theta(s, t) = \phi(s)$ for all $s \in S$, $t \in T$ preserves zero product and satisfies condition (ii) of Theorem 1.3.4. This shows that the information in (i) or (ii) of Theorem 1.3.4 is the most information we can extract from the existence of a single idempotent.

Notice that for example condition (ii) of Theorem 1.3.4 implies $\theta(x) = \theta(exe)$ for all $x \in A$. Of course the restriction of θ to a subring $eAe \subseteq A$ again preserves zero product. However this does not reduce the problem to the ring eAe since the map $\theta(x) = \psi(exe)$ might not preserve zero product even if $\psi : eAe \to B$ does.

1.4 Maps preserving zeros of xy^*

In this section we consider a variant of the zero product preservers for rings with involution. By *involution* we mean an anti-automorphism of order ≤ 2.

Definition 1.4.1. Let A and B be two rings with involution. A map $\theta : A \to B$ is said to *preserve zeros of xy^** if $\theta(x)\theta(y)^* = 0$ for all $x, y \in A$ with $xy^* = 0$.

The preservers of zeros of xy^* have not been studied as extensively as the zero product preservers, as they have appeared more recently. Nevertheless there are a few known results on this subject and on related questions. Again we start with a basic result for matrices, that was proved by Swain [34, Corollary 5].

Theorem 1.4.2. *Let F be a field, $*$ an involution on $M_n(F)$, where $n \geq 2$, and $\theta : M_n(F) \to M_n(F)$ a bijective linear map that preserves zeros of xy^*. Then there* *exist invertible matrices $B, U \in M_n(F)$, with $U^* = U^{-1}$, such that $\theta(X) = BXU$ for all $X \in M_n(F)$.*

Observe that the map θ can also be written in the form $\theta(X) = CU^{-1}XU$ for all $X \in M_n(F)$, where $C = BU \in M_n(F)$. In particular, θ is a composition of a $*$-homomorphism of $M_n(F)$ and a left multiplication by some element of $M_n(F)$. This is in general the expected form of a map that preserves zeros of xy^*; a $*$-homomorphism multiplied from the left by some element of the ring. As we often require the map to be surjective (or even bijective), the corresponding element usually turns out to be invertible.

Swain [34] has also considered maps that preserve zeros of xy^* on prime rings with involution. He has obtained a characterization of bijective additive maps $\theta : A \to A$ that preserve zeros of xy^* in the case when A is a unital prime ring with involution that is generated by idempotents. Examples of rings generated by idempotents are simple rings with nontrivial idempotents and the rings of $n \times n$ matrices over any unital ring, where $n \geq 2$ (see [6] for details). The assumption that the ring is generated by idempotents is quite strong, however as Swain has pointed out, it might be difficult to obtain such a characterization for general prime rings with involution that contain a nontrivial idempotent.

In the last decade several related problems have appeared in the literature. For example Wong [40] considered linear maps θ on C^*-algebras such that $\theta(x)\theta(y)^* = \theta(x)^*\theta(y) = 0$ for all x, y with $xy^* = x^*y = 0$. He called such maps *disjointness preserving* maps. Some work has also been done on maps preserving zeros of other $*$-polynomials, such as $xy - yx^*$ (see [7]).

There is one crucial difference between the setting with involution and the setting without involution. The condition for a zero product preserver is completely symmetric, while the condition for a map that preserves zeros of xy^* is not symmetric, as the $*$ only appears on the right-hand side. This loss of symmetry in the case with involution has certain consequences. Firstly, the class of expected solutions is somewhat larger. In both settings the expected solutions are morphisms multiplied by some element, however, in the case without involution this element has to be central, while in the case with involution it may be general. And secondly, the results are usually less general and often some additional assumptions are needed to be able to describe maps preserving zeros of xy^*. For example, in the aforementioned result of Swain [34] on prime rings with involution, the additional assumption was that the ring is generated by idempotents. The aim of the present section is to prove some results in which we avoid this strong additional assumption and rather impose additional assumptions on the map itself. Most of these results are contained in [33].

First we demonstrate how the problem of maps preserving zeros of xy^* can in fact be viewed as a generalization of the problem of zero product preservers. Suppose A and B are rings with involution and $\theta : A \to B$ is an arbitrary map. Define a map $\phi : A \to B$ by setting $\phi(x) = \theta(x^*)^*$ for all $x \in A$. Then the map θ will preserve zeros of xy^* if and only if $\theta(x)\phi(y) = 0$ for all $x, y \in A$ with $xy = 0$. This condition is a generalization of the condition for a zero product preserver in the sense that it involves two maps instead of only one. The fact that ϕ is closely related to θ plays no significant role in our proofs, hence most results could be formulated for arbitrary pairs of maps θ and ϕ. We do this explicitly only in the most interesting case.

Our first result shows that, similar to the case with no involution, the injectivity of the map in [34, Theorem 4] can be omitted. The proof is similar to that of Theorem 1.3.4 so we leave the details out.

Proposition 1.4.3. *Let A be a unital ring with involution generated by idempotents and B a prime ring with involution. Let $\theta : A \to B$ be a surjective additive map that preserves zeros of xy^*. Then there exists a $*$-homomorphism $h : A \to Q^s(B)$ such that $\theta(x) = \theta(1)h(x)$ for all $x \in A$.*

Proof. As in [34, Theorem 4] we have

$$\theta(xy)\theta(z)^* = \theta(x)\theta(zy^*)^* \qquad \text{for all } x, y, z \in A.$$

Let $\psi : B \to A$ be any set-theoretic right inverse of θ and let $y \in A$ be a fixed element. Now define

$$f(X) = \theta(\psi(X)y) \qquad \text{and} \qquad g(Z) = \theta(\psi(Z^*)y^*)^*$$

for all $X, Z \in B$. Then

$$f(X)Z = Xg(Z) \qquad \text{for all } X, Z \in B.$$

By Theorem 1.2.11 there exists a unique $q \in Q^s(B)$ such that

$$f(X) = Xq \qquad \text{and} \qquad g(Z) = qZ \qquad \text{for all } X, Z \in B.$$

Similarly as in the proof of Theorem 1.3.4 this implies

$$\theta(xy) = \theta(x)q \qquad \text{and} \qquad \theta(zy^*) = \theta(z)q^* \qquad \text{for all } x, z \in A.$$

Since q depends on y we write $q = h(y)$. Then $h : A \to Q^s(B)$ is a map, such that

$$\theta(xy) = \theta(x)h(y) \qquad \text{and} \qquad \theta(zy^*) = \theta(z)h(y)^* \qquad \text{for all } x, y, z \in A.$$

As in [34, Theorem 4] this implies that h is a $*$-homomorphism of rings and $\theta(x) = \theta(1)h(x)$ for all $x \in A$. $\square$

Let A be a unital ring and let $E(A)$ denote the set of all idempotents of A. For $e \in E(A)$ denote by I_e the ideal of A generated by e, that is $I_e = \sum AeA$. Observe that I_e is in fact the ideal generated by all left and right annihilators of $1 - e$. As in [6] let $I(A)$ denote the ideal of A generated by $[E(A), A]$, where $[\cdot, \cdot]$ denotes the commutator in A.

We claim that

$$I(A) = \sum_{e \in E(A)} I_e \cap I_{1-e}.$$

For an arbitrary $e \in E(A)$ we have $[e, x] = -[1-e, x] \in I_e \cap I_{1-e}$ for all $x \in A$, thus $I(A) \subseteq \sum_{e \in E(A)} I_e \cap I_{1-e}$. Conversely, if a is an element of $I_e \cap I_{1-e}$ then $a = \sum_i x_i e y_i = \sum_j z_j (1-e) w_j$ for some $x_i, y_i, z_j, w_j \in A$. Thus

$$(1-e)a = \sum_i [1-e, x_i] e y_i \qquad \text{and} \qquad ea = \sum_j [e, z_j](1-e) w_j.$$

Summing the last two equalities gives

$$a = \sum_i [1-e, x_i] e y_i + \sum_j [e, z_j](1-e) w_j,$$

which lies in $I(A)$. Note that if A is a ring with involution then $I(A)^* = I(A)$.

Lemma 1.4.4. *Let A be a unital ring and B an arbitrary ring. Let $\theta, \phi : A \to B$ be additive maps such that $\theta(x)\phi(y) = 0$ for all $x, y \in A$ with $xy = 0$. Then*

$$\theta(xy)\phi(z) = \theta(x)\phi(yz) \qquad \textit{for all } y, z \in A,\ x \in I(A).$$

Proof. Let e be an idempotent in A and let $f = 1 - e$. In the same way as in the proof of Theorem 1.3.4 we can prove that the following identities hold for all $x, y, z \in A$:

$$\begin{aligned}
\theta(xe)\phi(z) &= \theta(x)\phi(ez), && (1.16)\\
\theta(xf)\phi(z) &= \theta(x)\phi(fz), && (1.17)\\
\theta(xeyf)\phi(z) &= \theta(x)\phi(eyfz), && (1.18)\\
\theta(xfye)\phi(z) &= \theta(x)\phi(fyez). && (1.19)
\end{aligned}$$

Now let $y, z, u, v \in A$ be arbitrary. Then we have

$$\begin{aligned}
&\theta(ufveye)\phi(z) \\
&= \theta(u)\phi(fveyez) \quad \text{by (1.19)} \\
&= \theta(ufve)\phi(yez) \quad \text{by (1.19)} \\
&= \theta(ufv)\phi(eyez) \quad \text{by (1.16)}\ .
\end{aligned}$$

Since θ is additive, this implies

$$\theta(xeye)\phi(z) = \theta(x)\phi(eyez) \qquad \text{for all } y, z \in A,\ x \in I_f. \tag{1.20}$$

By symmetry of e and f we also have

$$\theta(xfyf)\phi(z) = \theta(x)\phi(fyfz) \qquad \text{for all } y, z \in A,\ x \in I_e. \tag{1.21}$$

Summing (1.18), (1.19), (1.20) and (1.21), we obtain

$$\theta(xy)\phi(z) = \theta(x)\phi(yz) \qquad \text{for all } y, z \in A,\ x \in I_e \cap I_f. \tag{1.22}$$

Since θ is additive and $I(A) = \sum_{e \in E(A)} I_e \cap I_{1-e}$, this implies

$$\theta(xy)\phi(z) = \theta(x)\phi(yz) \qquad \text{for all } y, z \in A,\ x \in I(A).$$

□

Clearly we could show that $\theta(xy)\phi(z) = \theta(x)\phi(yz)$ holds also when x is arbitrary and $z \in I(A)$.

With the above lemma we are ready to prove the main result of this section, which describes pairs of maps θ and ϕ that satisfy the condition

$$\theta(x)\phi(y) = 0 \quad \text{iff} \quad xy = 0. \tag{1.23}$$

This will then help us characterize maps that preserve zeros of xy^* in both directions.

Theorem 1.4.5. *Let A be a unital prime ring with a nontrivial idempotent and B a prime ring. Let $\theta, \phi : A \to B$ be surjective additive maps such that for all $x, y \in A$ we have $\theta(x)\phi(y) = 0$ if and only if $xy = 0$. Then $\theta(1)$ is invertible in $Q^r(B)$, $\phi(1)$ is invertible in $Q^l(B)$, and there exists an injective homomorphism $h : A \to Q^s(B)$ such that $\theta(x) = \theta(1)h(x)$ and $\phi(x) = h(x)\phi(1)$ for all $x \in A$.*

Proof. Let $I = I(A)$. Since A is a prime ring with a nontrivial idempotent, I is a nonzero ideal of A. By Lemma 1.4.4 we have

$$\theta(xy)\phi(z) = \theta(x)\phi(yz) \qquad \text{for all } y, z \in A,\ x \in I. \tag{1.24}$$

Since θ and ϕ are additive, we have $\theta(0) = \phi(0) = 0$. If $\theta(x) = 0$ for some $x \in A$, then $\theta(x)\phi(1) = 0$ implies $x1 = 0$, hence $x = 0$. Since θ is additive, this shows that it is injective. In particular $\theta(I) \neq 0$. Similarly ϕ is injective.

Since θ is surjective, the same argument as above shows that the left annihilator of $\phi(1)$ in B is zero. Now let $L = B\phi(1)$, which is a left ideal of B. We want to show that L is dense in B. Let $b_1, b_2 \in B$ with $b_2 \neq 0$. Since ϕ is surjective, we may write $b_1 = \phi(a_1)$ and $b_2 = \phi(a_2)$. The injectivity of ϕ implies $a_2 \neq 0$. Since I is a nonzero ideal of a prime ring A, it has zero right annihilator. Hence there exists $i \in I$ such that $ia_2 \neq 0$, which implies $\theta(i)b_2 = \theta(i)\phi(a_2) \neq 0$. On the other hand the identity (1.24) implies

$$\theta(i)b_1 = \theta(i)\phi(a_1) = \theta(ia_1)\phi(1) \in L.$$

This shows that L is dense in B. Now define $g : L \to B$ by $g(b\phi(1)) = b$ for all $b \in B$. Since the left annihilator of $\phi(1)$ in B is zero, g is a well defined homomorphism of left B-modules. By the left version of Theorem 1.2.5 there exists $r \in Q_{ml}(B)$ such that $g(x) = xr$ for all $x \in L$, i.e. $b = b\phi(1)r$ for all $b \in B$. This implies $B(1 - \theta(1)r) = 0$, therefore $\theta(1)r = 1$ by the left version of Theorem 1.2.5. Hence r

is a right inverse of $\phi(1)$ in $Q_{ml}(B)$. If we insert $z = 1$ into (1.24) and multiply it by r from the right, we get

$$\theta(xy) = \theta(x)\phi(y)r \qquad \text{for all } y \in A,\ x \in I. \tag{1.25}$$

The right annihilator of r in $Q_{ml}(B)$ is zero since r has left inverse $\theta(1)$. Next we want to show that the left annihilator of r in $Q_{ml}(B)$ is zero as well. Suppose that $qr = 0$ for some $0 \neq q \in Q_{ml}(B)$. By the left version of Theorem 1.2.5 there exists a dense left ideal J of B such that $Jq \subseteq B$. Since q is nonzero, Jq is also nonzero. Take any nonzero $c \in Jq \subseteq B$. Then $cr = 0$. Since ϕ is bijective, there exists a nonzero $d \in A$ such that $c = \phi(d)$. By (1.25)

$$\theta(xd) = \theta(x)\phi(d)r = \theta(x)cr = 0 \qquad \text{for all } x \in I.$$

Since θ is injective, this implies $Id = 0$. Hence $d = 0$, which is a contradiction.

Let $x \in I$ and $y, z, w \in A$ be arbitrary. Then by (1.25) on one hand

$$\theta(xyzw) = \theta(xy)\phi(zw)r = \theta(x)\phi(y)r\phi(zw)r,$$

on the other hand

$$\theta(xyzw) = \theta(xyz)\phi(w)r = \theta(xy)\phi(z)r\phi(w)r = \theta(x)\phi(y)r\phi(z)r\phi(w)r.$$

Since ϕ is surjective, this implies $\theta(I)B(r\phi(zw)r - r\phi(z)r\phi(w)r) = 0$. By the left version of Proposition 1.2.6 it follows that $r(\phi(zw) - \phi(z)r\phi(w))r = 0$. Taking into account that both left and right annihilator of r in $Q_{ml}(B)$ are zero we get

$$\phi(zw) = \phi(z)r\phi(w) \qquad \text{for all } z, w \in A. \tag{1.26}$$

If we insert $w = 1$ into (1.26), we get $\phi(z) = \phi(z)r\phi(1)$. Since ϕ is surjective, this implies $B(1 - r\phi(1)) = 0$, hence $r\phi(1) = 1$ by the left version of Theorem 1.2.5. So $\phi(1)$ is invertible in $Q_{ml}(B)$ and r is its inverse. Notice that (1.25) implies $\theta(I)Br \subseteq B$, therefore $(\sum B\theta(I)B)r \subseteq B$. This means that $r \in Q^l(B)$ since $\sum B\theta(I)B$ is a nonzero ideal of B.

In the same way we can show that $\theta(1)$ is invertible in $Q^r(B)$ with inverse s and the maps θ and ϕ satisfy

$$\phi(xy) = s\theta(x)\phi(y) \qquad \text{for all } x \in A,\ y \in I. \tag{1.27}$$

and

$$\theta(zw) = \theta(z)s\theta(w) \qquad \text{for all } z, w \in A. \tag{1.28}$$

We would now like to combine identities (1.25), (1.26), (1.27), and (1.28), however we have to be a little bit careful, because the first two are identities in $Q_{ml}(B)$ while the last two are identities in $Q_{mr}(B)$. Nevertheless, we may interpret all these identities in $Q^s(B)$ once we observe that $Br, sB \subseteq Q^s(R)$. To verify this notice that (1.25) implies $\theta(I)Br \subseteq B$, hence $(\sum B\theta(I)B)Br \subseteq B$. In addition (1.26) implies

$BrB \subseteq B$, hence $Br(\sum B\theta(I)B) \subseteq B$. Since $\sum B\theta(I)B$ is a nonzero ideal of B, this means that $Br \subseteq Q^s(R)$. Similarly (1.27) and (1.28) imply $sB \subseteq Q^s(R)$. Now let $x \in I$ and $y, z \in A$ be arbitrary. Then on one hand by (1.25)

$$\theta(xyz) = \theta(xy)(\phi(z)r) = \theta(x)(\phi(y)r)(\phi(z)r),$$

on the other hand by (1.28) and (1.25)

$$\theta(xyz) = \theta(xy)(s\theta(z)) = \theta(x)(\phi(y)r)(s\theta(z)).$$

Since all the terms in brackets lie in $Q^s(B)$, we can combine these two identities to get

$$\theta(x)(\phi(y)r)(\phi(z)r) = \theta(x)(\phi(y)r)(s\theta(z)).$$

If we view this in $Q_{ml}(B)$ we may rewrite it as

$$\theta(x)\phi(y)r((\phi(z)r) - (s\theta(z))) = 0.$$

Since ϕ is surjective, this implies $\theta(I)Br((\phi(z)r) - (s\theta(z))) = 0$. By the left version of Proposition 1.2.6 it follows that $r((\phi(z)r) - (s\theta(z))) = 0$. Hence

$$\phi(z)r = s\theta(z) \quad \text{for all } z \in A$$

since r is invertible. Now define $h : A \to Q^s(B)$ by $h(x) = \phi(x)r = s\theta(x)$ for all $x \in A$. Since r is the inverse of $\phi(1)$ and s is the inverse of $\theta(1)$, we have $\phi(x) = h(x)\phi(1)$ and $\theta(x) = \theta(1)h(x)$ for all $x \in A$. Clearly h is additive. Since r is invertible and ϕ is injective, h is injective as well. By (1.26) we have

$$h(xy) = \phi(xy)r = \phi(x)r\phi(y)r = h(x)h(y)$$

for all $x, y \in A$, hence h is a homomorphism of rings as needed. □

Observe that the conclusions of Theorem 1.4.5 are also sufficient for θ and ϕ to satisfy the condition (1.23). A careful inspection of the proof shows that instead of A being a prime ring with a nontrivial idempotent, it would be enough to assume that the ideal $I(A)$ has zero left and right annihilator. As a corollary of Theorem 1.4.5 we have the following.

Theorem 1.4.6. *Let A be a unital prime ring with involution that contains a nontrivial idempotent and B a prime ring with involution. Let $\theta : A \to B$ be a surjective additive map such that for all $x, y \in A$ we have $\theta(x)\theta(y)^* = 0$ if and only if $xy^* = 0$. Then $\theta(1)$ is invertible in $Q^r(B)$ and there exists an injective $*$-homomorphism $h : A \to Q^s(B)$ such that $\theta(x) = \theta(1)h(x)$ for all $x \in A$.*

Proof. Let $\phi(x) = \theta(x^*)^*$ for all $x \in A$. Then θ and ϕ satisfy condition (1.23). By Theorem 1.4.5 $\theta(1)$ is invertible in $Q^r(B)$ and there exists an injective homomorphism $h : A \to Q^s(B)$ such that $\theta(x) = \theta(1)h(x)$ and $\phi(x) = h(x)\phi(1)$ for all $x \in A$.

By Proposition 1.2.12 the involution on B can be extended uniquely to an involution on $Q^s(B)$, which we denote by $*$ as well. Hence we have

$$\theta(1)h(x^*) = \theta(x^*) = \phi(x)^* = \phi(1)^* h(x)^* = \theta(1)h(x)^*$$

for all $x \in A$. Since $\theta(1)$ is invertible in $Q^r(B)$, this implies $h(x^*) = h(x)^*$ for all $x \in A$. $\square$

Remark 1.4.7. By the proof of Theorem 1.4.5 the map θ from Theorem 1.4.6 is injective and satisfies the identity

$$\theta(xy^*) = \theta(x)\theta(y)^* r \qquad \text{for all } x, y \in A,$$

where r is the inverse of $\theta(1)^*$ in $Q^l(B)$.

Theorem 1.4.6 characterizes maps that preserve zeros of xy^* in both directions. The question remains what can be said about maps on rings with nontrivial idempotents that preserve zeros of xy^*. The next result describes such maps $\theta : A \to B$ in the special case when $\theta(1)$ is a central element in B. In particular, this is the case if we assume that B is unital and $\theta(1) = 1$.

Proposition 1.4.8. *Let A be a unital ring with involution that contains a nontrivial idempotent e and B a prime ring with involution. Let $\theta : A \to B$ be a surjective additive map that preserves zeros of xy^* and suppose $\theta(1) \in Z(B)$. Then one of the following holds:*

(i) $\theta(I_e \cap I_{1-e} + I_{e^*} \cap I_{1-e^*}) = 0$,

(ii) $\theta(xy^*) = \lambda\theta(x)\theta(y)^*$ *for all $x, y \in A$, where $\lambda = 1/\theta(1)^* \in C(B)$.*

Proof. Let $\phi(x) = \theta(x^*)^*$ for all $x \in A$. The conclusions follow from the proof of Theorem 1.4.5 upon noticing that what was needed in the proof is satisfied automatically here. More precisely, unless $\theta(1) = 0$, both $\theta(1)$ and $\phi(1)$ are invertible in $C(B)$ since they are central in B, so we obtain (1.25) and (1.27) immediately where $r = 1/\theta(1)^* = \lambda$ and $s = 1/\theta(1) = \lambda^*$. In view of condition (i) we may assume that $\theta(I) \neq 0$, where $I = I(A)$. Since r and s are invertible, their left and right annihilators are zero, which gives (1.26) and (1.28). These facts are sufficient for the rest of the proof as well. By Remark 1.4.7 we get the desired conclusion. The only remaining case we need to settle is the case when $\theta(1) = 0$. If this is the case then (1.24) with $z = 1$ implies $\theta(I)B = 0$, hence $\theta(I) = 0$. $\square$

Observe that if we additionally assume in Theorem 1.4.8 that θ is injective then condition (i) implies $I_e \cap I_{1-e} = 0$. Since $I_e + I_{1-e} = A$ this means that $A = I_e \oplus I_{1-e}$. In particular, if A is a prime ring, this cannot happen.

Chapter 2

Nil rings

2.1 Definitions and properties

An element a of a ring R is *nilpotent* if $a^n = 0$ for some positive integer n. The smallest integer n for which $a^n = 0$ is called the *index of nilpotency* of a or just the *index* of a. The set of all nilpotent elements of a ring R will be denoted by $N(R)$.

Definition 2.1.1. A ring R is *nil* if every element in R is nilpotent.

Nil rings are a generalization of nilpotent rings, which are historically one of the most important class of rings. A ring R is *nilpotent* if there exists a positive integer n such that $R^n = 0$, i.e. any product of n elements of R is zero. Every nilpotent ring is nil, but a nil ring need not be nilpotent.

Example 2.1.2. Let $S_n(\mathbb{C})$ denote the ring of strictly upper triangular $n \times n$ matrices over $\mathbb{C}$. Then the ring $R = \bigoplus_{n\in\mathbb{N}} S_n(\mathbb{C})$ is nil but not nilpotent.

Let R be an arbitrary ring. An ideal $I \lhd R$ is *nil* if it is nil as a ring. If I and J are two nil ideals of R then $I + J$ is again a nil ideal of R. Indeed, if $a \in I + J$ then $a + J$ is nilpotent element of $(I + J)/J$, since $(I + J)/J \cong I/(I \cap J)$ is nil. Thus $a^n \in J$ for some positive integer n and hence a is nilpotent, because J is nil. This implies that in every ring R the sum of all nil ideals of R is again a nil ideal and is thus the largest nil ideal of R.

Definition 2.1.3. The *upper nilradical* of R is the largest nil ideal of R and is denoted by $Nil^*(R)$.

It is well known that in a commutative ring R the set of all nilpotent elements is in fact an ideal of R, so $N(R) = Nil^*(R)$ for any commutative ring R. In noncommutative rings the inclusion $Nil^*(R) \subseteq N(R)$ is usually strict. Rings R for which $N(R) = Nil^*(R)$ are called *NI rings*.

One of the shortcomings of the notion of nilpotent rings is that the sum of all nilpotent ideals of a ring need not be a nilpotent ideal. There are different ways to overcome this shortcoming. One possibility is to work with locally nilpotent rings instead. A ring R is *locally nilpotent* if every finitely generated subring of R is nilpotent. Clearly a nilpotent ring is locally nilpotent. On the other hand the ring R from Example 2.1.2 is locally nilpotent but not nilpotent. It is easy to prove that the sum of all the locally nilpotent ideals of a ring is again a locally nilpotent ideal.

Definition 2.1.4. The *Levitzki radical* of R (also called the *locally nilpotent radical*) is the largest locally nilpotent ideal of R and is denoted by $L(R)$.

Clearly $L(R) \subseteq Nil^*(R)$. The first example of a nil ring that is not locally nilpotent was given by Golod [14] as a consequence of joint work of Golod and Shafarevich [15] (the example can also be found in [35, §20]). The ring in this example is a nil algebra generated by three elements, which has infinite dimension over the corresponding field, hence it cannot be nilpotent.

Definition 2.1.5. The *lower nilradical* of R (also called the *prime radical*) is the intersection of all prime ideals of R and is denoted by $Nil_*(R)$. In particular, $Nil_*(R) = R$ if R has no prime ideals.

The following classical result shows that $Nil_*(R)$ is a nil ideal, which justifies the naming (see [19]).

Theorem 2.1.6. *For every ring R we have $Nil_*(R) \subseteq L(R)$, hence $Nil_*(R)$ is a locally nilpotent ideal of R.*

It is well known that in a commutative ring R we always have $Nil_*(R) = L(R) = Nil^*(R)$. In particular a commutative prime ring cannot be nil. In general, however this is not the case. In fact, below we give an example of a nonzero prime ring R, which is locally nilpotent. For such a ring R we have in particular $0 = Nil_*(R) \neq L(R) = R$.

Example 2.1.7. Let F be a field and let $F\langle x_1, x_2, x_3, \ldots\rangle$ denote the free F-algebra over the infinite set of indeterminates $\{x_1, x_2, x_3, \ldots\}$. Moreover let A denote the subalgebra of $F\langle x_1, x_2, x_3, \ldots\rangle$ consisting of polynomials with constant term 0. For any integer $n \geq 2$ let M_n denote the set of all monomials of degree n containing only indeterminates $x_1, x_2, \ldots, x_n$ and let I be the ideal of the algebra A generated by $M = \bigcup_{n=2}^{\infty} M_n$. Clearly A/I is a locally nilpotent ring, because every finitely generated subring of A contains polynomials in only finitely many indeterminates. In addition A/I is nonzero since I does not contain any monomial of degree 1. We claim that A/I is a prime ring.

Suppose p and q are two polynomials in A such that $pAq \subseteq I$. Choose k big enough such that polynomials p and q contain only indeterminates $x_1, x_2, \ldots, x_k$ and $\deg p, \deg q \leq k$. By assumption $px_{2k+2}q$ is an element of I. Since I is generated by monomials, every term of $px_{2k+2}q$ must lie in I. But the terms of $px_{2k+2}q$ are all of degree $\leq 2k+1$, so they must lie even in the ideal I' of A generated by $M' = M_2 \cup M_3 \cup \ldots \cup M_{2k+1}$. Let u be a monomial appearing in p with coefficient $\alpha \neq 0$ and v be a monomial appearing in q with coefficient $\beta \neq 0$. If u' is a monomial appearing in p and v' is a monomial appearing in q such that $ux_{2k+2}v = u'x_{2k+2}v'$ then $u = u'$ and $v = v'$, since the indeterminate x_{2k+2} does not appear in any of the monomials u, u', v, v'. This implies that the monomial $ux_{2k+2}v$ appears in $px_{2k+2}q$

with coefficient $\alpha\beta \neq 0$, hence $ux_{2k+2}v \in I'$. So there exists a monomial $m \in M'$ such that m is a subword of $ux_{2k+2}v$. But m does not contain the indeterminate x_{2k+1}, thus m must be a subword of u or v. Hence $u \in I' \subseteq I$ or $v \in I' \subseteq I$. Since u and v were arbitrary monomials appearing in p and q with nonzero coefficients, we conclude that $p \in I$ or $q \in I$, which shows that I is a prime ideal of A, i.e. A/I is a prime ring.

The lower nilradical is closely related to nilpotent ideals since it can also be characterized in the following way. Let R be a ring. For all ordinal numbers α we define ideals $\mathcal{N}_\alpha(R) \lhd R$ inductively as follows. For $\alpha = 1$ let

$$\mathcal{N}_1(R) = \sum\{I \lhd R\ ;\ I \text{ is nilpotent}\}.$$

For an ordinal number α that is a successor of β the ideal $\mathcal{N}_\alpha(R) \lhd R$ is uniquely defined by $\mathcal{N}_\alpha(R) \supseteq \mathcal{N}_\beta(R)$ and by the relation

$$\mathcal{N}_\alpha(R)/\mathcal{N}_\beta(R) = \mathcal{N}_1(R/\mathcal{N}_\beta(R)).$$

For a limit ordinal number α let

$$\mathcal{N}_\alpha(R) = \bigcup_{\beta<\alpha} \mathcal{N}_\beta(R).$$

As proved by Levitzki [21], $Nil_*(R) = \mathcal{N}_\alpha(R)$ for any ordinal α with $\operatorname{card}\alpha > \operatorname{card} R$.

Next we shall define the *Jacobson radical* of a ring. Given a ring $(R, +, \cdot)$, define an operation $\circ$ on R, called *quasi-multiplication*, by

$$a \circ b = a + b - ab.$$

It is easy to see that $(R, \circ)$ is a monoid with identity element 0. An element $a \in R$ is called *left quasi-regular* if it is left invertible in $(R, \circ)$, i.e. if there exists $a' \in R$ such that $a' \circ a = 0$. In this case we say that a' is the *left quasi-inverse* of a. If R is unital then this is equivalent to $1 - a$ being left invertible in $(R, \cdot)$ with left inverse $1 - a'$. In fact the map $f : (R, \circ) \to (R, \cdot)$ given by $x \mapsto 1 - x$ is a monoid homomorphism, since $1 - a \circ b = (1 - a)(1 - b)$. Similarly we define *right quasi-regular* elements and *right quasi-inverses*. An element $a \in R$ is called *quasi-regular* if it is both left and right quasi-regular. In this case a has a unique inverse in $(R, \circ)$ which we call the *quasi-inverse* of a. The set of all quasi-regular elements of R will be denoted by $Q(R)$. Clearly $(Q(R), \circ)$ is a group, since this is just the group of invertible elements of the monoid $(R, \circ)$. For every $a \in Q(R)$ and every $n \in \mathbb{Z}$ the n-th power of a in $(Q(R), \circ)$ will be denoted by $a^{(n)}$ to distinguish it from a^n, the n-th power of a in $(R, \cdot)$. In particular $a^{(0)} = 0$ and $a^{(-1)}$ is the quasi-inverse of a. If R is unital then $1 - a^{(-1)} = (1 - a)^{-1}$. An ideal $I \lhd R$ is called *quasi-regular* if $I \subseteq Q(R)$. The quasi-inverse of an element of I is again an element of I, thus I is quasi-regular iff $I \subseteq Q(I)$.

Definition 2.1.8. The *Jacobson radical* of R is the largest quasi-regular ideal of R and is denoted by $J(R)$.

Every nilpotent element is quasi-regular, in fact if $x^n = 0$ then $-x-x^2-\ldots-x^{n-1}$ is the quasi-inverse of x. Hence $N(R) \subseteq Q(R)$ and $Nil^*(R) \subseteq J(R)$.

Definition 2.1.9. A ring R is called *Jacobson radical* if $J(R) = R$.

There are many examples of rings which are Jacobson radical but not nil. We mention one here.

Example 2.1.10. Consider $R = \{\frac{2m}{2n-1} \ ; \ m,n \in \mathbb{Z}\}$ as a subring of the field of rational numbers. The quasi-inverse of $\frac{2m}{2n-1}$ is easily seen to be $\frac{2m}{2m-2n+1}$, which is again an element of R. So R is a Jacobson radical ring which is not nil.

It turns out that the Jacobson radical of a ring R contains even all one-sided quasi-regular ideals of R. This is a consequence of the fact that ab is quasi-regular if and only if ba is quasi-regular. In fact, if ab is quasi-regular then

$$(ba)^{(-1)} = b(ab)^{(-1)}a - ba.$$

The Jacobson radical has many other characterizations which are very useful in various situations. To present them we need a few more definitions.

Definition 2.1.11. A ring R is called *left primitive* if there exists a simple faithful left R-module. An ideal $I \lhd R$ is called *left primitive* if R/I is a left primitive ring.

By definition a left R-module M is *simple* if $RM \neq 0$ and M has only trivial R-submodules. It is well known that a left R-module M is simple if and only if $M \neq 0$ and $Rm = M$ for all $0 \neq m \in M$. In particular a primitive ring is nonzero.

Similarly one can define *right primitive* rings. The notion of primitivity is not left-right symmetric, because there exist rings which are left primitive but not right primitive and vice versa (see [35, §27] for an explicit example). Nevertheless we will simply speak of *primitive* rings instead of *left primitive* rings, since we will only work with left modules.

Suppose M is a simple left R-module and let P be the annihilator of M, i.e. $P = \operatorname{Ann} M = \{r \in R \ ; \ rM = 0\}$. Then P is an ideal of R and M can be made into an R/P-module with scalar multiplication defined by $(r+P)m = rm$. In fact M is a simple faithful R/P-module, so P is a primitive ideal of R. Conversely, if P is a primitive ideal of R and M is a simple faithful R/P-module, then M can be made into a simple R-module with scalar multiplication defined by $rm = (r+P)m$. Then the annihilator of M as an R-module is just P. This shows that the primitive ideals of R are precisely the annihilators of simple R-modules.

Now choose a nonzero m in a simple R-module M. Since M is simple we have $Rm = M$. Hence the map $f : R \to M$ defined by $f(r) = rm$ is a surjective homomorphism of left R-modules. Let L denote the kernel of f, which is a left ideal of R. Then $R/L \cong M$ and since M is simple, L is a maximal left ideal of R. In addition there exists $e \in R$ such that $em = m$. This implies that $(re - r)m = 0$ for every $r \in R$, i.e. $re - r \in L$ for every $r \in R$.

Definition 2.1.12. A left ideal L of R is *modular* if there exists $e \in R$ such that $re - r \in L$ for every $r \in R$.

By the above every simple R-module is isomorphic to R/L for some modular maximal left ideal L of R. It is not hard to see that the converse is also true. The annihilator of the R-module R/L is equal to $\{r \in R \ ; \ rR \subseteq L\}$, which is the largest ideal of R contained in L. This gives the following classical characterization of primitive ideals of R.

Proposition 2.1.13. *An ideal P of R is primitive if and only if there exists a modular maximal left ideal L of R such that P is the largest ideal of R contained in L, i.e. $P = \{r \in R \ ; \ rR \subseteq L\}$.*

This characterization is important because it does not make any use of modules, instead, it characterizes primitive ideals within the ring itself.

We mention two more classical results about primitive rings (see [19]). The first one gives a relation between primitive rings and prime rings.

Proposition 2.1.14. *Every primitive ring is a prime ring.*

The second one shows that in the category of commutative rings the notion of primitive rings is equivalent to the notion of fields.

Proposition 2.1.15. *A commutative ring is primitive if and only if it is a field.*

Now we can state the theorem that gives different characterizations of the Jacobson radical (see [35] and [19] for the full proof).

Theorem 2.1.16. *For every ring R the following hold:*

(i) *$J(R)$ is the intersection of all primitive ideals of R,*

(ii) *$J(R)$ is the intersection of annihilators of all simple left R-modules,*

(iii) *$J(R)$ is the intersection of all modular maximal left ideals of R,*

(iv) *$J(R) = \{r \in R \ ; \ Rr$ is left quasi-regular$\}$.*

To summarize, for every ring R we have a chain of inclusions

$$Nil_*(R) \subseteq L(R) \subseteq Nil^*(R) \subseteq J(R), \tag{2.1}$$

and in general, each of these inclusions may be strict. It turns out that all these radicals are semiprime ideals. For additional properties of the radicals defined above we refer the reader to [13] or [19].

Another class of rings that are closely connected to nilpotent rings is the class of nil rings of bounded index.

Definition 2.1.17. A ring R is *nil of bounded index* $\leq n$ if every element in R is nilpotent of index $\leq n$. A ring R is *nil of bounded index* if there exists a positive integer n such that R is nil of bounded index $\leq n$.

Clearly every nilpotent ring is nil of bounded index. The converse is not true in general as we will see in the example below. However there is a partial converse for algebras over fields that was proved by Nagata [22].

Theorem 2.1.18. *Let A be a nil algebra of bounded index $\leq n$ over a field of characteristic 0 or $p > n$. Then A is nilpotent. In fact, $A^{2^n-1} = 0$.*

Theorem 2.1.18 also gives a good idea of what can go wrong in general and how to construct a counterexample.

Example 2.1.19. Let $R = (\mathbb{Z}/2\mathbb{Z})[x_1, x_2, x_3, \ldots]$ denote the algebra of all polynomials in commutative indeterminates $x_1, x_2, x_3, \ldots$ over the field $\mathbb{Z}/2\mathbb{Z}$. Moreover let I denote the ideal of R generated by the set $\{x_1, x_2, x_3, \ldots\}$ and J denote the ideal of R generated by the set $\{x_1^2, x_2^2, x_3^2, \ldots\}$. Clearly $u^2 \in J$ for all monomials $u \in I$, since R is commutative. In addition $(u+v)^2 = u^2 + v^2$ for any monomials in $u, v \in R$, since the characteristic of R is 2. Hence $p^2 \in J$ for every polynomial $p \in I$. This shows that I/J is nil of bounded index ≤ 2. However I/J is not nilpotent, since the elements $x_1, x_1x_2, x_1x_2x_3, \ldots$ are not contained in J.

Nevertheless the following result of Levitzki [20, Theorem 4] shows that we have a good approximation of the converse of Theorem 2.1.18.

Theorem 2.1.20. *If R is a nil ring of bounded index then $Nil_*(R) = R$. In particular R is locally nilpotent.*

For the rest of this section let K be a commutative unital ring and R a K-algebra, possibly noncommutative and nonunital.

An element $a \in R$ is *algebraic* over K if there exists a nonzero polynomial $p \in K[x]$ such that $p(0) = 0$ and $p(a) = 0$. If in addition p can be chosen monic (i.e. the leading coefficient is equal to 1) then a is called *integral* over K. The condition $p(0) = 0$ is necessary only because R may be nonunital, in which case only polynomials with zero constant term can be evaluated at elements of R. The set of all algebraic elements of R will be denoted by $A_K(R)$, the set of all integral elements of R will be denoted by $I_K(R)$. A K-algebra R is *algebraic (integral)* over K if every element in R is algebraic (integral) over K. Note the special case of the above definitions when R is just a ring and $K = \mathbb{Z}$. In this case we will also write $A(R) = A_{\mathbb{Z}}(R)$ and $I(R) = I_{\mathbb{Z}}(R)$. Of course every nilpotent element of R is integral, so $N(R) \subseteq I_K(R) \subseteq A_K(R)$.

The following proposition is a classical result from commutative algebra.

Proposition 2.1.21. *If R is a commutative K-algebra then $I_K(R)$ is a subalgebra of R.*

Let a be an integral element of R. The smallest integer n, for which there exists a monic polynomial p of degree n such that $p(0) = 0$ and $p(a) = 0$, is called the *integral degree* of a or just the *degree* of a. Similarly one defines the *algebraic degree* of elements. Note that the algebraic degree and the integral degree of an integral element need not be equal.

Definition 2.1.22. An algebra R is *integral of bounded degree* $\leq n$ if every element in R is integral of degree $\leq n$. An algebra R is *integral of bounded degree* if there exists a positive integer n such that R is integral of bounded degree $\leq n$.

Recall that an ideal of the K-algebra R is an ideal of the ring R which is also closed for scalar multiplication. So when R is a K-algebra the radicals in (2.1) can a priori be defined in two ways; via ring ideals of R or via algebra ideals of R. However, since all these radicals are semiprime ideals, these two definitions coincide by Lemma 1.1.6.

2.2 The Köthe conjecture

One of the most important problems concerning nil rings is the Köthe conjecture. In 1930 Köthe [16] conjectured

Köthe conjecture 2.2.1. *If a ring has no nonzero nil ideals then it has no nonzero nil one-sided ideals.*

The importance of the Köthe conjecture lies in the fact that it would imply that the upper nilradical $Nil^*(R)$ of a ring R would contain not only all nil ideals but also all nil one-sided ideals of R. Although the question whether the conjecture is true is still open, considerable progress has been made since 1930's on this subject. There are many well known classes of rings that satisfy the conjecture. These classes include the class of all commutative rings, the class of all noetherian rings, and the class of all algebras over uncountable fields. These examples, especially the last one, suggest that the conjecture might be true. However there are also a few more recent results which indicate that perhaps a counterexample to the conjecture could be found. In this section we will present some of these results and examples, along with some known statements that are equivalent to the Köthe conjecture.

2.2.1 Equivalent statements

There are many known statements that are equivalent to the Köthe conjecture. We will state these statements as conjectures. So every conjecture stated in this subsection will be equivalent to the Köthe conjecture. We start with two basic ones.

Conjecture 2.2.2. *Every nil one-sided ideal of a ring R is contained in $Nil^*(R)$.*

Conjecture 2.2.3. *The sum of two nil left ideals of a ring is a nil.*

Of course we also have the right-handed version of Conjecture 2.2.3.

Proof of equivalence. Suppose that the Köthe conjecture holds. The ring $R/Nil^*(R)$ has no nonzero nil ideals, so by Köthe conjecture it has no nonzero nil one-sided ideals. Hence every nil one-sided ideal of R is contained in $Nil^*(R)$, which is precisely Conjecture 2.2.2.

Suppose Conjecture 2.2.2 holds. If L_1 and L_2 are two nil left ideals of R then they are contained in $Nil_*(R)$ by Conjecture 2.2.2. So their sum $L_1 + L_2$ is also contained in $Nil^*(R)$, hence it is nil. This implies Conjecture 2.2.3.

Now suppose Conjecture 2.2.3 holds. Let R be a ring. We show that for any $a \in R$ the principal left ideal R^1a is nil if and only if the principal right ideal aR^1 is nil. Suppose R^1a is nil and take $r \in R^1$. Then ra is nilpotent, say $(ra)^n = 0$. Since $(ar)^{n+1} = a(ra)^n r = 0$, ar is nilpotent as well. This shows that aR^1 is nil. Similarly the converse holds. Now let N denote the sum of all nil principal left ideals of R. By the above N is also the sum of all nil principal right ideals of R. In particular N is a two-sided ideal of R. Every element of N is contained in a finite sum of nil left ideals of R, which is a nil left ideal of R by Conjecture 2.2.3. Hence N is a nil ideal of R. This implies the Köthe conjecture. □

It turns out that the Köthe conjecture has a lot to do with the problem of describing the Jacobson radical of polynomial rings. One of the most important results concerning this problem is the following theorem of Amitsur [2, Theorem 1].

Theorem 2.2.4. *If R is a ring then $J(R[x]) = N[x]$ where $N = J(R[x]) \cap R$ is a nil ideal of R.*

Proof. First we show that $J(R[x]) \cap R = 0$ implies $J(R[x]) = 0$.

Suppose $J(R[x]) \cap R = 0$ but $J(R[x]) \neq 0$. We reduce the general case to the case when R is unital. Since $R[x] \lhd R^1[x]$, we have $J(R[x]) = J(R^1[x]) \cap R[x]$. Since the only invertible elements in $\mathbb{Z}[x]$ are 1 and -1, the only quasi-regular elements are 0 and 2. Hence $J(R^1[x]/R[x]) \cong J((R^1/R)[x]) \cong J(\mathbb{Z}[x]) = 0$, which implies $J(R^1[x]) \subseteq R[x]$ and consequently $J(R[x]) = J(R^1[x])$. In addition $J(R^1[x]) \cap R^1 = J(R[x]) \cap R^1 = J(R[x]) \cap R$. So it suffices to show that $J(R^1[x]) \cap R^1 = 0$ implies $J(R^1[x]) = 0$, i.e. we may assume that R is unital.

Now let $f(x)$ be a nonzero polynomial in $J(R[x])$ of minimal degree. The map $g(x) \mapsto g(x+1)$ is an automorphism of $R[x]$. Since the Jacobson radical is invariant under automorphisms, it follows that $f(x+1) \in J(R[x])$. Now $f(x+1) - f(x)$ is a polynomial in $J(R[x])$ of smaller degree than $f(x)$, hence it must be 0. Writing $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_n \neq 0$, we have

$$0 = f(x+1) - f(x) = na_n x^{n-1} + \ldots,$$

hence $na_n = 0$. This implies that $nf(x)$ is a polynomial in $J(R[x])$ of smaller degree than $f(x)$, so it must be 0. Let m be the smallest positive integer such that $mf(x) = 0$ and let p be a prime divisor of m. Define $h(x) = \frac{m}{p} f(x)$. Then $h(x)$ is a nonzero polynomial in $J(R[x])$ with $ph(x) = 0$ and $h(x+1) = h(x)$. Denote $R_p = \{r \in R \; ; \; pr = 0\} \lhd R$, so $h(x) \in R_p[x] \cap J(R[x]) = J(R_p[x])$.

We now show that if a polynomial $g(x) \in R_p[x]$ satisfies $g(x+1) = g(x)$ then $g(x) \in R_p[x^p - x]$, i.e. $g(x) = \widehat{g}(x^p - x)$ for some polynomial $\widehat{g}(x) \in R_p[x]$. We do this by induction on the degree of $g(x)$. The ring R_p is an algebra over the field $\mathbb{Z}/p\mathbb{Z}$. The equality $g(x+1) = g(x)$ implies that every element of $\mathbb{Z}/p\mathbb{Z}$ is a zero of the polynomial $g(x) - g(0)$. So if the degree of $g(x)$ is less than p then $g(x) - g(0)$

must be zero. In this case $g(x)$ is a constant polynomial, hence it lies in $R_p[x^p - x]$. Suppose the degree of $g(x)$ is greater or equal to p. By the division algorithm we have

$$g(x) = s(x)(x^p - x) + t(x)$$

for some polynomials $s(x), t(x) \in R_p[x]$, where the degree of $s(x)$ is less than the degree of $g(x)$ and the degree of $t(x)$ is less than p. Clearly

$$g(x+1) = s(x+1)(x^p - x) + t(x+1),$$

thus $g(x+1) = g(x)$ implies

$$(s(x+1) - s(x))(x^p - x) = -(t(x+1) - t(x)).$$

Since the degree of the right-hand side is less than p, we conclude that $s(x+1) = s(x)$ and $t(x+1) = t(x)$. By induction $s(x), t(x) \in R_p[x^p - x]$, which implies $g(x) \in R_p[x^p - x]$. In particular, $h(x) = \widehat{h}(x^p - x)$ for some polynomial $\widehat{h}(x) \in R_p[x]$.

Next we show that if a polynomial $g(x^p - x)$ belongs to $J(R_p[x])$ then it belongs to $J(R_p[x^p - x])$. Indeed, let $r(x)$ be an element of the ideal of $R_p[x^p - x]$ generated by $g(x^p - x)$. Clearly $r(x+1) = r(x)$. Since $g(x^p - x) \in J(R_p[x])$, it follows that $r(x) \in J(R_p[x])$. Let $r'(x)$ be the quasi-inverse of $r(x)$ in $R_p[x]$. Obviously $r'(x+1)$ is the quasi-inverse of $r(x+1) = r(x)$, hence $r'(x+1) = r'(x)$. By the above $r'(x) \in R_p[x^p - x]$, so $r(x)$ is quasi-regular in $R_p[x^p - x]$. This implies that $g(x^p - x)$ belongs to $J(R_p[x^p - x])$.

The map $g(x) \mapsto g(x^p - x)$ is an isomorphism between $R_p[x]$ and $R_p[x^p - x]$, thus it maps $J(R_p[x])$ to $J(R_p[x^p - x])$. By what we have just proved, $\widehat{h}(x^p - x) \in J(R_p[x^p - x])$, hence $\widehat{h}(x) \in J(R_p[x]) \subseteq J(R[x])$. However, unless $h(x)$ is constant, the degree of $\widehat{h}(x)$ is less than the degree of $h(x)$ and $f(x)$, which is a contradiction with the choice of $f(x)$. Hence $h(x)$ is constant and so is $f(x)$. But then $0 \neq f(x) \in J(R[x]) \cap R$ which is a contradiction.

Now we treat the general case. Since $N \subseteq J(R[x])$ we have $N[x]R[x] \subset NR[x] \subseteq J(R[x])$. As an intersection of prime ideals $J(R[x])$ is a semiprime ideal. Hence it follows that $N[x] \subseteq J(R[x])$. The natural isomorphism $(R/N)[x] \cong R[x]/N[x]$ maps R/N to $(R + N[x])/N[x]$, hence

$$\begin{aligned} J((R/N)[x]) \cap R/N &\cong J(R[x]/N[x]) \cap (R + N[x])/N[x] = \\ &= J(R[x])/N[x] \cap (R + N[x])/N[x] = \\ &= (J(R[x]) \cap (R + N[x]))/N[x] = \\ &= (J(R[x]) \cap R + N[x])/N[x] = \\ &= (N + N[x])/N[x] = 0. \end{aligned}$$

It follows by the above that $J(R[x]/N[x]) \cong J((R/N)[x]) = 0$, thus $J(R[x]) \subseteq N[x]$. We conclude that $J(R[x]) = N[x]$.

To prove that N is a nil ideal, take $a \in N$ and consider the element ax. Since it lies in $J(R[x])$, it is quasi-regular. Hence there is a polynomial $p(x) \in R[x]$, say of degree n, such that $ax \circ p(x) = 0$, i.e. $p(x) = -ax + axp(x)$. Using this equality on its right-hand side repeatedly we get

$$p(x) = -ax - a^2x^2 - a^3x^3 - \ldots - a^{n+1}x^{n+1} + a^{n+2}x^{n+2}p(x).$$

Since the degree of $p(x)$ is n and every term in $a^{n+2}x^{n+2}p(x)$ is of degree $\geq n+2$ we must have $a^{n+1} = 0$. $\square$

Krempa [17, Theorem 1] characterized via matrix rings when the polynomial ring $R[x]$ is Jacobson radical.

Theorem 2.2.5. *For a ring R the polynomial ring $R[x]$ is Jacobson radical if and only if the ring $M_n(R)$ is nil for every positive integer n.*

With the help of Theorem 2.2.5 Krempa [17] proved that the following statements are equivalent to the Köthe conjecture. The first two were independently discovered also by Sands [25].

Conjecture 2.2.6. *If R is a nil ring then $M_2(R)$ is a nil ring.*

Conjecture 2.2.7. *If R is a nil ring then $M_n(R)$ is a nil ring for every positive integer n.*

Conjecture 2.2.8. *If R is a nil ring then $R[x]$ is Jacobson radical.*

Proof of equivalence. Suppose the Köthe conjecture holds and let R be a nil ring. Let L_1 denote the set of all matrices in $M_2(R)$ with zero second column and L_2 the set of all matrices in $M_2(R)$ with zero first column. Then L_1 and L_2 are left ideals of $M_2(R)$. Let $A = [a_{ij}]_{i,j}$ be an element of L_1. Since R is nil, $a_{11}^n = 0$ for some positive integer n. Hence A^n is strictly lower triangular and so $A^{2n} = 0$. This shows that L_1 is nil. Similarly L_2 is nil. Hence by Conjecture 2.2.3 $M_2(R) = L_1 + L_2$ is also nil.

Suppose Conjecture 2.2.6 holds and let R be a nil ring. Then by an induction argument $M_{2^n}(R)$ is nil for every positive integer n. Since $M_n(R)$ is embedded in $M_{2^n}(R)$, $M_n(R)$ is nil for every positive integer n.

By Theorem 2.2.5 Conjecture 2.2.7 is equivalent to Conjecture 2.2.8, so all that is left to show is that the Conjecture 2.2.8 implies Conjecture 2.2.2. Suppose Conjecture 2.2.8 holds and let L be nil one-sided ideal of a ring R. Then $L[x]$ is a Jacobson radical ring, hence it is a quasi-regular one-sided ideal of $R[x]$ and consequently contained in $J(R[x])$. Let I denote the ideal of R generated by L. Then $I[x] \subseteq J(R[x])$, hence $I[x]$ is Jacobson radical. By Theorem 2.2.4 I is a nil ring. This implies $L \subseteq I \subseteq Nil^*(R)$ as required. $\square$

Conjecture 2.2.8 connects the Köthe conjecture to polynomial rings. In particular, if the conjecture was true, it would imply $J(R[x]) = Nil^*(R)[x]$ for any ring R, i.e. the ideal N in Theorem 2.2.4 would be equal to the upper nilradical of R. It would thus provide a complete description of the Jacobson radical of polynomial ring in terms of the base ring. In addition, it follows from [13, Proposition 4.9.1] that the upper nilradical of $M_n(R)$ is of the form $M_n(I)$ for some ideal $I \lhd R$. Hence Conjecture 2.2.7 would imply $Nil^*(M_n(R)) = M_n(Nil^*(R))$ for every ring R and every positive integer n.

More recently a much weaker statement than that of Conjecture 2.2.8 has been proved to be equivalent to the Köthe conjecture by Smoktunowicz [27]. She proved that for a nil ring R every primitive ideal in $R[x]$ is of the form $I[x]$ for some ideal $I \lhd R$ and consequently gave the following.

Conjecture 2.2.9. *If R is a nil ring then $R[x]$ is not primitive.*

Proof of equivalence. Clearly Conjecture 2.2.8 implies Conjecture 2.2.9. To show the converse, suppose that R is a nil ring such that $R[x]$ is not Jacobson radical. Then there exists a primitive ideal in $R[x]$ and by [27, Theorem 1] it is of the form $I[x]$ for some ideal $I \lhd R$. Hence $(R/I)[x] \cong R[x]/I[x]$ is a primitive ring. This contradicts Conjecture 2.2.9, since R/I is a nil ring. $\square$

Ferrero and Puczyłowski [11] investigated when a ring can be a sum of two subrings of a certain kind and gave the following statement equivalent to the Köthe conjecture. See [11] for the proof of equivalence.

Conjecture 2.2.10. *Let R be a ring such that $R = R_1 + R_2$, where R_1 and R_2 are subrings of R. If R_1 is nilpotent and R_2 is nil then R is nil.*

Definition 2.2.11. We say that a ring or an algebra R *satisfies the Köthe conjecture* if $Nil^*(R)$ contains every nil one-sided ideal of R.

The Köthe conjecture simply states that every ring should satisfy the Köthe conjecture. Yonghua [42, Corollary 1.1] gave a necessary and sufficient condition for a ring to satisfy the Köthe conjecture by introducing the concept of Köthe subsets.

Definition 2.2.12. A subset M of a ring R is a *Köthe subset* if there exists a maximal nil left ideal L of R such that $M = (L + LR + Nil^*(R)) \backslash Nil^*(R)$.

Proposition 2.2.13. *A ring R satisfies the Köthe conjecture if and only if for every Köthe subset $M \subseteq R$, R satisfies the ACC for left annihilators of elements of M.*

This equivalent condition itself is interesting, however, note that if R satisfies the Köthe conjecture then only the empty set is a Köthe subset of R.

The next statement has a more group theoretic flavour. Its equivalence with the Köthe conjecture was established by Fisher and Krempa [12]. For a subgroup G of the group of automorphisms of a ring R denote by R^G the *subring of fixed points*, i.e. $R^G = \{r \in R\ ;\ g(r) = r \text{ for all } g \in G\}$. An element $r \in R$ is an *additive $|G|$-torsion element* if $r \neq 0$ and $|G|r = 0$.

Conjecture 2.2.14. *Let R be a ring and G a finite subgroup of the group of automorphisms of R such that R has no additive $|G|$-torsion. If R^G is nil then R is nil.*

More information on the background of this statement along with the proof of equivalence can be found in [12]. We give one more statement equivalent to the Köthe conjecture that is a combination of the previous ones.

Conjecture 2.2.15. *Let R be a ring such that $R[x]$ is primitive and let $G \neq 1$ be a finite subgroup of the group of automorphisms of R. If R^G is nil then R has additive $|G|$-torsion.*

Proof of equivalence. Suppose the Köthe conjecture holds. Let R and G be as in Conjecture 2.2.15 and suppose R^G is nil but R has no additive $|G|$-torsion. Then by Conjecture 2.2.14 R is a nil ring, hence by Conjecture 2.2.9 $R[x]$ is not primitive, which is a contradiction.

Now we show that Conjecture 2.2.15 implies Conjecture 2.2.9. Suppose R is a nil ring such that $R[x]$ is primitive. Then $R[x]$ is prime by Proposition 2.1.14, hence R is prime as well. By Proposition 1.1.7 either R has no additive 2-torsion or $2R = 0$.

Suppose R has no additive 2-torsion. Let G' be the group all diagonal matrices in $M_2(\mathbb{Z})$ with ± 1 on the diagonal. Then G' acts on $M_2(R)$ by conjugation. Since R has no 2-torsion, the kernel of this action is $\{I, -I\}$, where I is the identity matrix. Since the order of G' is 4, the image G of this action is a group of automorphisms of $M_2(R)$ of order 2. An easy calculation shows that $M_2(R)^G$ is the set of all diagonal matrices in $M_2(R)$, hence isomorphic to $R \oplus R$, which is a nil ring. Since $R[x]$ is primitive, $M_2(R)[x] \cong M_2(R[x])$ is primitive as well. By Conjecture 2.2.15 $M_2(R)$ has additive 2-torsion, hence so does R, which is a contradiction.

So $2R = 0$ and R is an algebra over the field $\mathbb{Z}/2\mathbb{Z}$. Let G be the group of matrices in $M_3(\mathbb{Z}/2\mathbb{Z})$ generated by

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It can easily be calculated that $P^3 = I$, $Q^7 = I$ and $QP = PQ^2$. Hence $G = \{P^iQ^j \ ; \ 0 \leq i \leq 2, 0 \leq j \leq 6\}$ has order 21. None of the matrices in G are diagonal except I, thus G acts on $M_3(R)$ faithfully by conjugation. Hence we can identify G with its image under this action. To calculate $M_3(R)^G$ let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

be an element of $M_3(R)^G$. Then

$$0 = PA - AP = \begin{bmatrix} g - c & h - b & i - a - c \\ d - f & 0 & -d \\ a + g - i & b & c - g \end{bmatrix}$$

and

$$0 = QA - AQ = \begin{bmatrix} d - c & b + e - a & f - b \\ g - d - f & h - d & i - e - f \\ a - i & b + h - g & c - h \end{bmatrix}.$$

It follows from the first equation that $b = d = h = f = 0$. Consequently it follows from the second equation that $c = g = 0$ and $a = e = i$. This shows that A is a scalar matrix. So $M_3(R)^G$ is isomorphic to R, which is a nil ring. As above $M_3(R)[x]$ is a primitive ring, thus Conjecture 2.2.15 implies that $M_3(R)$ has additive 21-torsion, and hence so does R. But this is impossible since $2R = 0$. □

Krempa has proved that it would be enough to consider the Köthe conjecture in the class of algebras over fields. More precisely, he proved that the following statement is equivalent to the Köthe conjecture (see [17] for the proof).

Conjecture 2.2.16. *If F is a field and R is an F-algebra then every nil one-sided ideal of R is contained in $Nil^*(R)$.*

We can even restrict ourselves to prime algebras.

Conjecture 2.2.17. *If F is a field and R is a prime F-algebra then every nil one-sided ideal of R is contained in $Nil^*(R)$.*

Proof of equivalence. Suppose Conjecture 2.2.17 holds. Let F be a field and R an arbitrary F-algebra. Suppose there exists a nil one-sided ideal L of R which is not contained in $Nil^*(R)$. Choose $a \in L \backslash Nil^*(R)$. By [17, Lemma 2] there exists a prime ideal P of R such that $a \notin P$ and $Nil^*(R/P) = 0$ (all the ideals mentioned here are ideals in the ring sense as well as in the algebra sense, because they are all semiprime). Since R/P is a prime algebra and $(L + P)/P$ is a nil one-sided ideal in R/P it follows by Conjecture 2.2.17 that $(L+P)/P$ is contained in $Nil^*(R/P) = 0$. Hence $L \subseteq P$, but this is a contradiction, since $a \in L \backslash P$. So Conjecture 2.2.17 implies Conjecture 2.2.16. The converse is obvious. □

Remark 2.2.18. Conjecture 2.2.17 is a version of Conjecture 2.2.2 for prime algebras over fields. Consequently we can show that the versions of Conjectures 2.2.3, 2.2.6, 2.2.7 and 2.2.8 for prime algebras over fields are also equivalent to the Köthe conjecture. To see this we just have to follow the proof of the implications Conjecture 2.2.2 ⇒ Conjecture 2.2.3 ⇒ Conjecture 2.2.6 ⇒ Conjecture 2.2.7 ⇒ Conjecture 2.2.8, referring to the fact that R is prime iff $R[x]$ is prime iff $M_n(R)$ is prime for some (or for all) n. Since '$R[x]$ is primitive' implies 'R is prime', the version of Conjecture 2.2.8 for prime algebras over fields clearly implies the version of Conjecture 2.2.9 for arbitrary algebras over fields. To finish the proof we have to show that the version of Conjecture 2.2.9 for arbitrary algebras over fields implies Conjecture 2.2.16. To do this we just have to follow the proof of the implications Conjecture 2.2.9 ⇒ Conjecture 2.2.8 ⇒ Conjecture 2.2.2, referring to Lemma 1.1.6 when needed.

The version of Conjecture 2.2.6 for prime algebras states that in order to prove the Köthe conjecture it would be enough to prove that for a prime nil algebra R the algebra $M_2(R)$ is nil as well. Although the notions of prime algebras and nil algebras are in a way opposite notions in the commutative setting, it is not hard to find examples of noncommutative prime nil algebras. In fact, the ring in Example 2.1.7

is a prime nil algebra, which is even locally nilpotent. In 2002 Smoktunowicz [26] has even constructed a simple nil algebra.

More information on the Köthe conjecture and related problems can be found in [29] and [30].

2.2.2 Rings that satisfy the Köthe conjecture

Recall that a ring R is said to satisfy the Köthe conjecture if $Nil^*(R)$ contains all nil one-sided ideals of R. In this subsection we present some classes of rings that satisfy the Köthe conjecture. It should be pointed out that if a certain class of rings satisfies the Köthe conjecture it does not necessarily mean that this class will satisfy any of the other statements equivalent to the Köthe conjecture, since these statements need not be equivalent within this class. For example the statements 'for every commutative ring R, R is nil implies $M_2(R)$ is nil' and 'for every commutative ring R, R is nil implies $M_n(R)$ is nil for all n' need not be a priori equivalent. So for a certain class $\mathcal{C}$ of rings it is reasonable to ask not only if $\mathcal{C}$ satisfies the Köthe conjecture but also whether $\mathcal{C}$ satisfies any of the other statements equivalent to Köthe conjecture. We shall present some known results on this subject, mainly focusing on Conjectures 2.2.7 and 2.2.8.

We begin with a few simple observations. It is clear that the class of all commutative rings satisfies the Köthe conjecture. If for a given ring R we have $J(R) = Nil^*(R)$ then the ring R satisfies the Köthe conjecture. This is because $J(R)$ contains every quasi-regular one-sided ideal of R and any nil one-sided ideal is quasi-regular. In particular this means that the class of all left (resp. right) artinian rings satisfies the Köthe conjecture, since for any left artinian ring R the Jacobson radical $J(R)$ is a nilpotent ideal of R.

A theorem of Levitzki gives an even bigger class of rings that satisfy the Köthe conjecture, namely the class of left (resp. right) noetherian rings.

Theorem 2.2.19. *Let R be a left noetherian ring. Then every nil one-sided ideal of R is nilpotent. In particular $Nil_*(R) = L(R) = Nil^*(R)$ is the largest nilpotent ideal of R.*

It is well known (see [35, p. 144]) that the Jacobson radical of an algebraic algebra R over a field F is nil, so $J(R) = Nil^*(R)$ for such an algebra. Hence any algebraic algebra over a field and in particular any finite dimensional algebra satisfies the Köthe conjecture.

It is not hard to prove that if R is a locally nilpotent ring then $R[x]$ as well as $M_n(R)$ are locally nilpotent rings. This implies that locally nilpotent rings satisfy Conjectures 2.2.7 and 2.2.8. In particular this means that every commutative ring satisfies Conjectures 2.2.7 and 2.2.8, since every commutative nil ring is locally nilpotent. Consequently we have the following.

Proposition 2.2.20. *If R is a commutative ring then $J(R[x]) = Nil^*(R)[x]$ and $Nil^*(M_n(R)) = M_n(Nil^*(R))$ for every positive integer n.*

A ring R is said to satisfy a noncommutative polynomial $p \in \mathbb{Z}\langle x_1, x_2, \ldots\rangle$ if $p(a_1, a_2, \ldots) = 0$ for all $a_1, a_2, \ldots \in R$. In this case p is called a *polynomial identity* in R. A polynomial identity p is called *proper* if for every $r \in R$ there exists a coefficient n of p such that $nr \neq 0$ (see [23, Chapter 1] for details). A ring R is called a *PI-ring* if it satisfies a proper polynomial identity. Equivalently, R satisfies a homogeneous multilinear polynomial identity with all coefficients equal to ± 1 (see [23, Theorem 4.1]). In particular every commutative ring is a PI-ring since it satisfies the identity $x_1x_2 - x_2x_1 = 0$. PI-rings are an important generalization of commutative rings. In a way they are close to being commutative and have many similar properties as commutative rings. In particular Levitzki [20, Corollary] proved the following.

Theorem 2.2.21. *If R is a PI-ring then $Nil_*(R) = L(R) = Nil^*(R)$ and R satisfies the Köthe conjecture.*

So if a PI-ring is nil then it is locally nilpotent. Hence PI-rings also satisfy Conjectures 2.2.7 and 2.2.8.

Another class of rings that satisfy the Köthe conjecture is the class of monomial algebras. The usual definition of a monomial algebra requires that this algebra is unital, however when speaking about nil algebras one would like to exclude the unit, since it is not a nilpotent element. To this end we use the following definition. Let F be a field, X a nonempty set, and $F\langle X\rangle$ the free unital F-algebra over X. Let $F_0\langle X\rangle$ denote the ideal of $F\langle X\rangle$ generated by X, i.e. the ideal of all polynomials with zero constant term. Let I be an ideal of $F_0\langle X\rangle$ generated by some set of monomials in $F_0\langle X\rangle$. The factor algebra $F_0\langle X\rangle/I$ is called a *monomial algebra*. Observe that I is also an ideal of $F\langle X\rangle$. The factor algebra $F\langle X\rangle/I$ will be called a *unital monomial algebra*. Clearly $(F_0\langle X\rangle/I)^1 \cong F\langle X\rangle/I$, in particular all the radicals of $F_0\langle X\rangle/I$ and $F\langle X\rangle/I$ defined in section 2.1 coincide. The fact that monomial algebras satisfy the Köthe conjecture was proved by Beidar and Fong [5] in the most general case. In the case of characteristic 0 the result is due to Jaspers and Puczyłowski.

Theorem 2.2.22. *If R is a monomial algebra then $J(R) = Nil^*(R) = L(R)$.*

The original result is stated for unital monomial algebras, however it is clear from the above that the same holds for arbitrary monomial algebras. Theorem 2.2.22 implies that monomial algebras satisfy the Köthe conjecture as well as Conjectures 2.2.7 and 2.2.8.

Perhaps the most fascinating results concerning the truth of Köthe conjecture are the results of Amitsur [1], [2] on algebras over uncountable fields.

Theorem 2.2.23. *For an algebra R over an uncountable field the following hold:*

(i) *$Nil^*(R)$ contains all nil one-sided ideals of R,*

(ii) *if R is nil then $M_n(R)$ is nil for every positive integer n,*

(iii) *if R is nil then $R[x]$ is nil.*

Note that the conclusion of Theorem 2.2.23 (iii) is much stronger than that of Conjecture 2.2.8. In view of Conjecture 2.2.17, Theorem 2.2.23 implies that the only remaining source of possible counterexamples to the Köthe conjecture are effectively prime algebras over countable fields. For algebras over countable fields the situation is much different. In particular the stronger conclusion of Theorem 2.2.23 (iii) is not true for algebras over countable fields as proved by Smoktunowicz [28].

Theorem 2.2.24. *For every countable field F there exists a nil F-algebra R such that $R[x]$ is not nil.*

A little later an even stronger example was found by Puczyłowski and Smoktunowicz [24].

Theorem 2.2.25. *For every countable field F there exists an F-algebra R such that $R[x]$ is Jacobson radical but not nil.*

The examples of Theorems 2.2.24 and 2.2.25 and the methods behind their constructions suggest that a counterexample to the Köthe conjecture could perhaps be constructed using similar methods. The idea behind the construction is as follows. Let F be a countable field and A the algebra of all polynomials in $F\langle x, y, z\rangle$ with zero constant term. Then the elements of A can be enumerated, say $A = \{p_1, p_2, p_3, \ldots\}$. Let I be the ideal of A generated by $\{p_1^{m_1}, p_2^{m_2}, p_3^{m_3}, \ldots\}$ for some positive integers $m_1, m_2, m_3, \ldots$. Clearly the algebra A/I is nil (and this is essentially the only way to construct a nil algebra if the field is countable). In [28] it was essentially proved that if the sequence $m_1, m_2, m_3, \ldots$ increases rapidly enough then the algebra $(A/I)[u, v]$ is not nil. Consequently either A/I or $(A/I)[u]$ is the example for Theorem 2.2.24.

2.2.3 The Jacobson radical of polynomial rings

Recall that if the Köthe conjecture was true, it would imply $J(R[x]) = Nil^*(R)[x]$ for every ring R. Since we do not know if this is the case it is natural to try to find alternative descriptions of the Jacobson radical of polynomial rings. By Theorem 2.2.4 the radical $J(R[x])$ is of the form $N[x]$ where $N = J(R[x]) \cap R$ is a nil ideal of R. It is clear that N contains the Levitzki radical of R, since the ring $L(R)[x]$ is locally nilpotent. Hence $L(R) \subseteq N \subseteq Nil^*(R)$. The next theorem gives additional information about the ideal N.

Theorem 2.2.26. *For any ring R we have $J(R[x]) = N[x]$, where*

$$N = \bigcap \{P \lhd R\ ;\ P \text{ prime ideal and } J((R/P)[x]) = 0\}.$$

This result follows from a more general theory of radicals of polynomial rings (see [13, §4.9]). Since we are only interested in a special case of this theory, namely the Jacobson radical, we give a more direct proof.

Lemma 2.2.27. *If $J(R[x]) = N[x]$ and $g \notin N[x]$ then there exists a prime ideal $P \lhd R$ such that $J(R[x]/P[x]) = 0$ and $g \notin P[x]$.*

Proof. Since $g \notin N[x] = J(R[x])$ there exists $h \in R[x]g$ which is not left quasi-regular. For this h we must have $\{f \circ h \ ; \ f \in R[x]\} \cap N[x] = \emptyset$, since $f \circ h \in N[x]$ for some $f \in R[x]$ would imply that $f \circ h$ is left quasi-regular and consequently h would be left quasi-regular as well. By Zorn's lemma there exists an ideal $P \triangleleft R$ which is maximal with respect to the property $\{f \circ h \ ; \ f \in R[x]\} \cap P[x] = \emptyset$.

Suppose I and J are ideals of R such that $P \subsetneq I$, $P \subsetneq J$ and $IJ \subseteq P$. By the maximality of P there exist $f_1, f_2 \in R[x]$ such that $f_1 \circ h \in I[x]$ and $f_2 \circ h \in J[x]$. Define $q = f_1 + f_2 - f_1 \circ h \circ f_2 \in R[x]$. Then

$$\begin{aligned} q \circ h &= q + h - qh = (f_1 + f_2 - f_1 \circ h \circ f_2) + h - (f_1 + f_2 - f_1 \circ h \circ f_2)h = \\ &= (f_1 + h - f_1 h) + (f_2 + h - f_2 h) - (f_1 \circ h \circ f_2 + h - (f_1 \circ h \circ f_2)h) = \\ &= f_1 \circ h + f_2 \circ h - (f_1 \circ h \circ f_2) \circ h = f_1 \circ h + f_2 \circ h - (f_1 \circ h) \circ (f_2 \circ h) = \\ &= (f_1 \circ h)(f_2 \circ h), \end{aligned}$$

so $q \circ h \in I[x]J[x] \subseteq IJ[x] \subseteq P[x]$, which is a contradiction. This shows that P is a prime ideal of R.

By [2, Theorem 1] we have $J((R/P)[x]) = (M/P)[x]$ for some ideal $M \triangleleft R$ with $P \subseteq M$. Suppose $M \neq P$. Then by the maximality of P there exists $f \in R[x]$ such that $f \circ h \in M[x]$. This implies that $f \circ h + P[x]$ is left quasi-regular in $R[x]/P[x]$, hence there exists $t \in R[x]$ such that $t \circ f \circ h \in P[x]$, which is a contradiction. So $M = P$ and $J(R[x]/P[x]) \cong J((R/P)[x]) = 0$.

If $g \in P[x]$ then $h \in P[x]$ and hence $0 \circ h = h \in P[x]$ which is a contradiction. So $g \notin P[x]$. $\square$

Proof of Theorem 2.2.26. Denote $S = \bigcap\{P \triangleleft R \ ; \ P \text{ prime ideal and } J((R/P)[x]) = 0\} \triangleleft R$. By [2, Theorem 1] we have $J(R[x]) = N[x]$ for some ideal $N \triangleleft R$. If P is an ideal of R with $J(R[x]/P[x]) \cong J((R/P)[x]) = 0$ then $N[x] = J(R[x]) \subseteq P[x]$, hence $N \subseteq P$. This implies $N \subseteq S$. If $g \notin N[x]$ then by Lemma 2.2.27 there exists a prime ideal $P \triangleleft R$ such that $J(R[x]/P[x]) = 0$ and $g \notin P[x]$, which implies $g \notin S[x]$. Hence $S[x] \subseteq N[x]$ and so $S \subseteq N$. $\square$

Recall that the Köthe conjecture claims that if R is a nil ring then $R[x]$ should be Jacobson radical. In particular the ring $R[x]$ should have no primitive ideals. Smoktunowicz [27, Theorem 1] has proved that if R is a nil ring then any primitive ideal in $R[x]$ (if it exists) is of the form $I[x]$ for some ideal $I \triangleleft R$. A natural question arises whether this can be generalized to arbitrary rings. Of course it is not true that for every ring R every primitive ideal of $R[x]$ is of the form $I[x]$, however, we can ask whether there exists at least one primitive ideal of this form (provided that $R[x]$ is not Jacobson radical). Unfortunately the answer is still negative.

Example 2.2.28. Let R be a commutative ring which is not nil. Then by Proposition 2.2.20 the ring $R[x]$ is not Jacobson radical. However, $R[x]$ contains no primitive ideals of the form $I[x]$ with $I \triangleleft R$. Indeed, if $I[x]$ was a primitive ideal in $R[x]$ then $(R/I)[x] \cong R[x]/I[x]$ would be a field by Proposition 2.1.15. But this is impossible unless $I = R$, which is not the case.

Nevertheless Lemma 2.2.27 shows that slightly less is true, namely for every ring R, if $R[x]$ is not Jacobson radical then there exists a ideal $I \lhd R$ such that I is prime and $(R/I)[x]$ has zero Jacobson radical.

The following proposition gives another interesting property of the Jacobson radical of polynomial rings.

Proposition 2.2.29. *If R is a ring and $S \subseteq R$ a subring then $J(R[x]) \cap S[x] \subseteq J(S[x])$.*

Proof. Suppose $p(x) \in J(R[x]) \cap S[x]$ and let $u(x)$ be an element of the ideal of $S[x]$ generated by $p(x)$. We have to show that $u(x)$ is quasi-regular in $S[x]$. Since $p(x) \in J(R[x])$ and S is a subring of R we have $u(x) \in J(R[x])$. Let $v(x) \in R[x]$ be the quasi-inverse of $u(x)$, so $u(x) + v(x) - u(x)v(x) = u(x) + v(x) - v(x)u(x) = 0$. Denote $u(x) = u_0 + u_1x + u_2x^2 + \ldots$ and $v(x) = v_0 + v_1x + v_2x^2 + \ldots$, where $u_j = 0$ and $v_j = 0$ for j big enough. Then the equalities above imply

$$u_i + v_i - \sum_{k+l=i} u_k v_l = u_i + v_i - \sum_{k+l=i} v_l u_k = 0$$

for all $i \geq 0$. In particular $u_0 + v_0 - u_0v_0 = u_0 + v_0 - v_0u_0 = 0$, i.e. v_0 is the quasi-inverse of u_0. In addition

$$v_i - u_0v_i = -u_i + \sum_{\substack{k+l=i \\ l\neq i}} u_k v_l$$

for all $i \geq 1$, hence

$$u_0 \circ v_i = u_0 + v_i - u_0v_i = u_0 - u_i + \sum_{\substack{k+l=i \\ l\neq i}} u_k v_l$$

for all $i \geq 1$. Quasi-multiplying by v_0 from the left we get

$$v_i = v_0 \circ (u_0 - u_i + \sum_{\substack{k+l=i \\ l\neq i}} u_k v_l)$$

for all $i \geq 1$. This means that for $i \geq 1$ the element v_i lies in the subring generated by $\{v_0, v_1, v_2, \ldots, v_{i-1}, u_0, u_1, u_2, \ldots\}$. Inductively it follows that all v_i lie in the subring generated by $\{v_0, u_0, u_1, u_2, \ldots\}$. By Theorem 2.2.4 the element u_0 is nilpotent, hence its quasi-inverse v_0 is a polynomial in u_0. Therefore all v_i lie in the subring generated by $\{u_0, u_1, u_2, \ldots\}$. Since $u_i \in S$ for all i, $v_i \in S$ for all i as well. This shows that $v(x) \in S[x]$, hence $u(x)$ is quasi-regular in $S[x]$. Consequently $p(x) \in J(S[x])$. □

Corollary 2.2.30. *The Jacobson radical of a polynomial ring over any ring is a union of Jacobson radical polynomial rings over finitely generated rings.*

Proof. Let R be a ring. By Theorem 2.2.4 we have $J(R[x]) = N[x]$ for some ideal $N \lhd R$. If S is a finitely generated subring of N then $S[x]$ is Jacobson radical by Proposition 2.2.29. Clearly $J(R[x])$ is a union of $S[x]$ when S runs through all finitely generated subrings of N. $\square$

When considering the truth of Conjecture 2.2.8 we can in fact restrict ourselves to finitely generated rings and algebras.

Proposition 2.2.31. *The Köthe conjecture is equivalent to the statement 'if F is a countable field and R is a finitely generated prime nil F-algebra then $R[x]$ is Jacobson radical'.*

Proof. Consider the following statements:

(A) If R is a nil algebra then $R[x]$ is Jacobson radical.

(B) If R is a finitely generated nil algebra then $R[x]$ is Jacobson radical.

(C) If R is a finitely generated prime nil algebra then $R[x]$ is Jacobson radical.

By Remark 2.2.18 statement (A) is clearly equivalent to the Köthe conjecture. Also (A) obviously implies (B) and (B) implies (C). Hence it is enough to prove the implications (C) $\Rightarrow$ (B) $\Rightarrow$ (A), since by Theorem 2.2.23 the statement (C) is true, if the field is uncountable.

Suppose (C) holds. Let R be a finitely generated nil algebra such that $R[x]$ is not Jacobson radical. Then there exists a primitive algebra ideal $P \lhd R[x]$. If M is a simple faithful $R[x]/P$-module then $(R[x]/P)m = M$ for every $0 \neq m \in M$. Hence M is simple and faithful also as a module over the ring $R[x]/P$. This shows that P is primitive also as a ring ideal. By Theorem [27, Theorem 1] we have $P = I[x]$ for some ring ideal $I \lhd R$. Clearly I is also an algebra ideal. Hence $(R/I)[x] \cong R[x]/P$ is a primitive algebra. In particular by Proposition 2.1.14 this implies that R/I is a prime algebra. In addition it is finitely generated and nil. Hence $(R/I)[x]$ is Jacobson radical, which is in contradiction with $(R/I)[x]$ being primitive unless $I = R$, which is not the case. This proves (B).

Suppose (B) holds. Let R be a nil algebra such that $R[x]$ is not Jacobson radical. Then there exists $f(x) \in R[x]$ which is not quasi-regular. Let S be the subalgebra of R generated by the coefficients of $f(x)$. Then S is a finitely generated nil F-algebra, hence $S[x]$ is Jacobson radical by (B). This is a contradiction since $f(x) \in S[x]$ is not quasi-regular even in $R[x]$, let alone in $S[x]$. So (B) implies (A). $\square$

2.3 π-algebraicity

Let R be a ring or an algebra. Every nilpotent element of R is quasi-regular and of course algebraic. In addition the quasi-inverse of a nilpotent element is a polynomial in this element. In the present section we will be interested in the connections between these three notions; nilpotency, algebraicity, and quasi-regularity. In particular we will investigate how close algebraic elements are to being nilpotent and how close quasi-regular elements are to being nilpotent. We are motivated by the following two questions:

Q1. Algebraic rings and algebras are usually thought of as nice and well behaved. For example an algebraic algebra over a field with no zero divisors is a division algebra. On the other hand nil rings and algebras, which are of course algebraic, are bad and hard to deal with. It is thus natural to ask what makes the nil rings and algebras bad among all the algebraic ones.

The answer for algebras over fields is well known, namely they are Jacobson radical. We generalize this to algebras over certain principal ideal domains and in particular to rings.

Q2. Can nilpotent elements among all quasi-regular elements be characterized by the property "quasi-inverse of a is a polynomial in a"?

It is somewhat obvious that element-by-element this will not be possible, however we will be able to characterize the upper nilradical in this way.

Most of the results of this section are contained in [32].

2.3.1 π-algebraic elements

In this section K will always denote a commutative unital ring, F a field, and R an algebra over K or F. The two questions from the introduction of this section motivate the following definition, which will play a crucial role in our considerations.

Definition 2.3.1. An element a of a K-algebra R is *π-algebraic* (over K) if there exists a polynomial $p \in K[x]$ such that $p(0) = 0$, $p(1) = 1$ and $p(a) = 0$. In this case we will also say that a is *π-algebraic with polynomial p*. A subset $S \subseteq R$ is *π-algebraic* if every element in S is π-algebraic. The set of all π-algebraic elements of a K-algebra R will be denoted by $\pi_K(R)$.

Note the special case of the above definition when R is just a ring and $K = \mathbb{Z}$. In this case we will also write $\pi(R) = \pi_{\mathbb{Z}}(R)$. The crucial condition in this definition is the condition $p(1) = 1$. The condition $p(0) = 0$ is there simply because R may not be unital, in which case only polynomials with zero constant term can be evaluated at an element of R.

We first present some basic properties of π-algebraic elements along with some examples. For the definitions and notations see section 2.1.

Lemma 2.3.2. *If R is a K-algebra then $N(R) \subseteq \pi_K(R) \subseteq A_K(R) \cap Q(R)$. If R is an F-algebra then $N(R) \subseteq \pi_F(R) = A_F(R) \cap Q(R)$. The quasi-inverse of a π-algebraic element is a polynomial in this element.*

Proof. Obviously every nilpotent element is π-algebraic and every π-algebraic element is algebraic. Suppose $a \in R$ is π-algebraic with polynomial p. Then $P(x) = 1-(1-p(x))/(1-x)$ is again a polynomial with $P(0) = 0$ (and proper coefficients). Hence we may define $a' = P(a)$. Since $x \circ P(x) = x + P(x) - xP(x) = p(x)$, we have $a \circ a' = 0$. Similarly we get $a' \circ a = 0$. Hence a' in the quasi-inverse of a and it is a polynomial in a. Now suppose R is an F-algebra and a is an element of $A_F(R) \cap Q(R)$. Let $r \in F[x]$ be the minimal polynomial of a (if R is not unital then $r(0)$ must be zero) and let a' be the quasi-inverse of a. Suppose $r(1) = 0$. Then $r(x) = (1-x)q(x) = q(x) - xq(x)$ for some polynomial $q \in F[x]$ of degree less then that of r. If R is not unital then $q(0) = 0$, so we may evaluate q at a in any case. Hence $0 = r(a) - a'r(a) = q(a) - aq(a) - a'q(a) + a'aq(a) = q(a) - (a' \circ a)q(a) = q(a)$, which is a contradiction since r was the minimal polynomial for a. Thus $r(1)$ is an invertible element of F and hence the element a is π-algebraic with polynomial $p(x) = r(1)^{-1}r(x)x$. $\square$

We shall see in the examples that the inclusion $\pi_K(R) \subseteq A_K(R) \cap Q(R)$ may be strict.

Lemma 2.3.3. *If R is a unital K-algebra then $2 - \pi_K(R) \subseteq \pi_K(R)$. In particular $0, 2 \in \pi_K(R)$ and $1 \notin \pi_K(R)$. If R is a unital F-algebra then $2 - \pi_F(R) \subseteq \pi_F(R)$. In addition $F \backslash \{1\} \subseteq \pi_F(R)$ and $1 \notin \pi_F(R)$.*

Proof. If a is π-algebraic with polynomial p then $2-a$ is π-algebraic with polynomial $q(x) = p(2-x)x$. We always have $0 \in \pi_K(R)$, hence $2 \in \pi_K(R)$. The identity element is never π-algebraic since it is not quasi-regular. If R is a unital F-algebra and $\lambda \neq 1$ is a scalar then λ is π-algebraic with polynomial $p(x) = (1-\lambda)^{-1}(x-\lambda)x$. $\square$

Next we give a few examples.

Example 2.3.4. For a finite ring R, $\pi(R) = Q(R)$ and $J(R) = Nil^*(R)$. To verify the first part observe that $(Q(R), \circ)$ is a finite group, say of order n. So for every $a \in Q(R)$ we have $a^{(n)} = 0$, hence every $a \in Q(R)$ is π-algebraic with polynomial $p(x) = x^{(n)} = 1 - (1-x)^n$. The second part is well known and it also follows from the first part and Theorem 2.3.16.

Example 2.3.5. For any field F, $\pi_F(F) = F \backslash \{1\} = Q(F)$ by Lemma 2.3.3. In particular $\pi_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q} \backslash \{1\} = Q(\mathbb{Q})$. On the other hand we will show that $\pi(\mathbb{Q}) = \{1 + \frac{1}{n}\ ;\ n \in \mathbb{Z} \backslash \{0\}\}$. Indeed, if n is a nonzero integer then $1 + \frac{1}{n}$ is π-algebraic over $\mathbb{Z}$ with polynomial $p(x) = (1 - n(x-1))x$. Conversely, suppose $\frac{a}{b} \in \mathbb{Q}$ (a and b coprime) is π-algebraic with polynomial $P \in \mathbb{Z}[x]$ of degree d. Then $Q(x) = b^d P(\frac{x}{b})$ is a polynomial with integer coefficients. Hence $a-b$ divides $Q(a) - Q(b) = b^d P(\frac{a}{b}) - b^d P(1) = -b^d$. Since a and b are coprime, this is only possible if $a - b = \pm 1$ (any prime that would divide $a-b$ would divide b and hence a). Thus $\frac{a}{b} = 1 \pm \frac{1}{b}$ as needed. Obviously $A(\mathbb{Q}) = \mathbb{Q}$, so the inclusion $\pi(\mathbb{Q}) \subseteq A(\mathbb{Q}) \cap Q(\mathbb{Q})$ from Lemma 2.3.2 is strict here.

Example 2.3.6. Let $F \subseteq E$ be fields and $M_n(E)$ the ring of $n \times n$ matrices over E. Then

$$\begin{aligned} N(M_n(E)) &= \text{matrices with eigenvalues } 0, \\ \pi_F(M_n(E)) &= \text{matrices with eigenvalues in } \overline{F} \backslash \{1\}, \\ Q(M_n(E)) &= \text{matrices with eigenvalues in } \overline{E} \backslash \{1\}, \end{aligned}$$

where $\overline{F} \subseteq \overline{E}$ are algebraic closures of F and E. A matrix is quasi-regular iff it has no eigenvalue equal to 1. So in view of Lemma 2.3.2, to verify the above, we only need to prove that

$$A_F(M_n(E)) = \text{ matrices with eigenvalues in } \overline{F}.$$

If $A \in M_n(E)$ is algebraic over F, it clearly has eigenvalues in $\overline{F}$. So suppose $A \in M_n(E)$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \overline{F}$. For every $i = 1, 2, \dots, n$, let p_i be the minimal polynomial of λ_i over F. Then the minimal polynomial m_A of A over E divides $P(x) = \prod_{i=1}^n p_i(x)$, hence $P(A) = 0$. Since P has coefficients in F, A is algebraic over F.

The following proposition gives a connection between π-algebraic and integral elements.

Proposition 2.3.7. *Let R be a K-algebra and a an element of R. The following are equivalent:*

(i) *a is π-algebraic,*

(ii) *a is quasi-regular and $a^{(-1)}$ is integral,*

(iii) *a is quasi-regular and $a^{(-1)}$ is a polynomial in a.*

Proof. By Lemma 2.3.2 (i) implies (iii). On the other hand, if $a^{(-1)} = P(a)$ where P is a polynomial in $K[x]$, then $a + P(a) - aP(a) = 0$, so a is π-algebraic with polynomial $(x + P(x) - xP(x))x$. It remains to prove the equivalence of (i) and (ii).

For a polynomial $p \in K[x]$ define $\widehat{p}(x) = (x-1)^{\deg p} p(\frac{x}{x-1})$, which is again a polynomial in $K[x]$. Notice that $\widehat{p}(1)$ equals the leading coefficient of p and the leading coefficient of $\widehat{p}$ equals $p(1)$ if $p(1) \neq 0$. In addition $\widehat{p}(0) = 0$ iff $p(0) = 0$.

We may assume that R is unital, otherwise we just adjoin a unit to R. Let a be a quasi-regular element. Then the inverse of $1-a$ is $1-a^{(-1)}$, so the term $\frac{x}{x-1}$ evaluated at a equals $-a(1-a^{(-1)}) = -a+aa^{(-1)} = a^{(-1)}$. Thus $\widehat{p}(a) = (a-1)^{\deg p} p(a^{(-1)})$. This shows that $\widehat{p}(a) = 0$ iff $p(a^{(-1)}) = 0$, since $1-a$ is invertible. Similarly $\widehat{p}(a^{(-1)}) = 0$ iff $p(a) = 0$.

If p is a monic polynomial such that $p(0) = 0$ and $p(a^{(-1)}) = 0$ then a is π-algebraic with polynomial $\widehat{p}$. If a is π-algebraic with polynomial p then $\widehat{p}$ is a monic polynomial such that $\widehat{p}(0) = 0$ and $\widehat{p}(a^{(-1)}) = 0$, so $a^{(-1)}$ is integral. $\square$

In particular Proposition 2.3.7 states that $\pi_K(R) = (Q(R) \cap I_K(R))^{(-1)}$ (compare with Lemma 2.3.2). In what follows we will see that there is a strong connection between π-algebraic elements and nilpotent elements, in case K satisfies certain properties given by the following definition.

Definition 2.3.8. We shall say that a principal ideal domain K is *special* if there is no nonconstant polynomial $p \in K[x]$ with $p(0) \neq 0$ such that $p(k)$ would be invertible in K for all $k \in K$ coprime to $p(0)$.

There is a simple condition that a special PID always satisfies.

Proposition 2.3.9. *If K is a special PID then $J(K) = 0$. In particular, if K is not a field then K has infinitely many nonassociated irreducible elements.*

Proof. Let K be a special PID. Suppose $J(K) \neq 0$ and take $0 \neq a \in J(K)$. Since K is commutative and unital, this implies that $1-ak$ is invertible in K for every $k \in K$. But then the polynomial $p(x) = 1 - ax$ contradicts the definition of a special PID. Hence $J(K) = 0$. Since K is commutative and unital, $J(K)$ is just the intersection of all maximal ideals of K by Theorem 2.1.16. If K is not a field then the maximal ideals of K are the principal ideals generated by the irreducible elements. If there are only finitely many such ideals then their intersection is nonzero. □

Proposition 2.3.9 has a partial converse.

Proposition 2.3.10. *If K is a PID that has finite group of units and is not a field then the following are equivalent:*

(i) *K has infinitely many nonassociated irreducible elements,*

(ii) $J(K) = 0$,

(iii) *K is special.*

Proof. It follows by Proposition 2.3.9 and its proof that (i) and (ii) are equivalent and (iii) implies (ii).

(i) $\Rightarrow$ (iii): Let p be a polynomial in $K[x]$ with $p(0) \neq 0$ such that $p(k)$ is invertible for all $k \in K$ coprime to $p(0)$. Since $p(0) \neq 0$ and K has infinitely many nonassociated irreducible elements, there are infinitely many elements in K that are coprime to $p(0)$. Since there are only finitely many invertible elements in K, there exist infinitely many elements $k \in K$ coprime to $p(0)$ such that $p(k)$ is the same for all these k, say $p(k) = u$ where u is invertible. But then the polynomial $p(x) - u$ has infinitely many zeros, so it must be zero. Hence p is a constant polynomial. □

For general PIDs the condition $J(K) = 0$ is not sufficient for K to be special. The simplest counterexample is given by any field that is not algebraically closed (see Proposition 2.3.12), however fields are rather extremal among all PID, since they have no irreducible elements. Hence we give another counterexample which is not a field.

Example 2.3.11. Let $S \subseteq \mathbb{Z}$ be a multiplicatively closed subset generated by all primes p with $p = 2$ or $p \equiv 1 \pmod 4$ and let $K = S^{-1}\mathbb{Z}$ be the localization of $\mathbb{Z}$ at S. Then K has infinitely many nonassociated irreducible elements, represented by the primes p with $p \equiv 3 \pmod 4$, hence $J(K) = 0$. Now let $p(x) = x^2 + 1$. To see that K is not special, we will show that $p(k)$ is invertible in K for all $k \in K$. For

$k = \frac{m}{n} \in K$ we have $p(k) = \frac{m^2+n^2}{n^2}$. To see that this is invertible in K we need to show that any prime dividing $m^2 + n^2$ is contained in S. Suppose p is a prime with $p \equiv 3 \pmod 4$ that divides $m^2 + n^2$. Then $m^2 \equiv -n^2 \pmod p$. Since $n \in S$, this implies that both m and n are coprime to p. Hence we have

$$1 \equiv m^{p-1} \equiv (m^2)^{\frac{p-1}{2}} \equiv (-n^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} n^{p-1} \equiv (-1)^{\frac{p-1}{2}} \equiv -1 \pmod p.$$

This is a contradiction since $p \neq 2$, which finishes the proof.

Here are some examples of special PIDs.

Proposition 2.3.12.

(i) *A field is a special PID if and only if it is algebraically closed.*

(ii) *The ring of integers $\mathbb{Z}$ and the ring of Gaussian integers $\mathbb{Z}[i]$ are special PIDs.*

(iii) *For any field F the polynomial ring $F[x]$ is a special PID.*

Proof. (i): Let F be a field that is a special PID. If $p \in F[x]$ is a nonconstant polynomial then either $p(0) = 0$ or there exists $\lambda \in F$ such that $p(\lambda)$ is not invertible, i.e. $p(\lambda) = 0$. So F is algebraically closed. Clearly any algebraically closed field is a special PID.

(ii): This follows from Proposition 2.3.10.

(iii): Let F be a field and $P(y)$ a nonconstant polynomial in $(F[x])[y]$ with $P(0) \neq 0$. Let $P(y) = p_0(x) + p_1(x)y + \ldots + p_n(x)y^n$ where $p_0(x) \neq 0$, $p_n(x) \neq 0$, and $n \geq 1$. Denote $d_i = \deg p_i$ and $d = \max\{d_i\ ;\ i = 0, 1, 2, \ldots, n\}$, where the degree of the zero polynomial is equal to $-\infty$. Now let $p(x) = p_0(x)x^{d+1} + 1$. Then $p(x)$ is coprime to $p_0(x) = P(0)$. The degree of $p_i(x)p(x)^i$ is equal to $d_i + i(d_0 + d + 1)$. Since $d_0, d_n, d \neq -\infty$ we have $d_n + n(d_0 + d + 1) \geq n(d_0 + d + 1) > d + (n-1)(d_0 + d + 1) \geq d_i + i(d_0 + d + 1)$ for all $i < n$. This implies that the degree of $P(p(x))$ is equal to $d_n + n(d_0 + d + 1) \geq 1$, hence $P(p(x))$ is not invertible in $F[x]$. $\square$

The next proposition was our main motivation for the introduction of special PIDs.

Proposition 2.3.13. *Let K be a special PID and R a K-algebra. If a is an element of R such that $Ka \subseteq \pi_K(R)$ then there exists $0 \neq k \in K$ such that ka is nilpotent. In particular, if R has no K-torsion then a is nilpotent.*

Proof. For a nonzero polynomial $f \in K[x]$, let $\delta(f)$ denote the greatest common divisor of all coefficients of f. First we show that for any π-algebraic element r there exists a nonzero polynomial $f \in K[x]$ and a nonzero element $c \in K$ such that $f(1) = 1$, $cf(r) = 0$, and f divides (within $K[x]$) any polynomial that annihilates r. So let $r \in R$ be π-algebraic with polynomial $h \in K[x]$. Choose a nonzero polynomial $p \in K[x]$ of minimal degree such that $p(r) = 0$ and let $c = \delta(p)$ and $f(x) = \frac{p(x)}{c} \in K[x]$. So $cf(r) = 0$ and $\delta(f) = 1$. Suppose $P \in K[x]$ is a polynomial with $P(r) = 0$. By the division algorithm there exists $0 \neq k \in K$ and polynomials $s, t \in K[x]$ with

$\deg t < \deg f = \deg p$ such that $kP(x) = s(x)f(x) + t(x)$ (divide in $D[x]$, where D is the field of fractions of K, and multiply by a common denominator of all fractions). Multiplying by c we get $ckP(x) = cs(x)f(x) + ct(x) = s(x)p(x) + ct(x)$. The minimality of p now implies $ct(x) = 0$, hence $t(x) = 0$ and so $kP(x) = s(x)f(x)$. By Gauss's lemma this implies $\delta(s) = k\delta(P)$ up to association, so k divides $\delta(s)$. Thus the polynomial $\frac{s(x)}{k}$ has integer coefficients and $P(x) = \frac{s(x)}{k}f(x)$, i.e. f divides P. In particular f divides h, so there is a polynomial S such that $h(x) = S(x)f(x)$. Evaluating at 1 we get $1 = S(1)f(1)$, so $f(1)$ is invertible in K. We may assume that $f(1) = 1$, otherwise we just multiply f by $f(1)^{-1} = S(1)$.

By the above for any $k \in K$ there exists $0 \neq c_k \in K$ and $0 \neq f_k \in K[x]$ such that $f_k(1) = 1$, $c_k f_k(ka) = 0$, and f_k divides any polynomial that annihilates ka. Let $k \neq 0$. Then f_1 divides $c_k f_k(kx)$, since $c_k f_k(kx)$ annihilates a. Similarly $c_1 k^{\deg f_1} f_1(\frac{x}{k})$ is a polynomial in $K[x]$ that annihilates ka, so f_k divides $c_1 k^{\deg f_1} f_1(\frac{x}{k})$. This in particular implies that all these polynomials have the same degree, so there exists $d_k \in K$ such that $c_1 k^{\deg f_1} f_1(\frac{x}{k}) = d_k f_k(x)$. We have $f_k(1) = 1$, hence $\delta(f_k) = 1$. Consequently $c_1\delta(k^{\deg f_1} f_1(\frac{x}{k})) = d_k$ up to association. If k is coprime to the leading coefficient of f_1 then $\delta(k^{\deg f_1} f_1(\frac{x}{k})) = 1$ since $\delta(f_1) = 1$. For such k we have $c_1 = d_k$ up to association, hence c_1 divides d_k and $u_k = \frac{d_k}{c_1}$ is invertible. In addition $k^{\deg f_1} f_1(\frac{x}{k}) = u_k f_k(x)$. Evaluating at 1 we get $k^{\deg f_1} f_1(\frac{1}{k}) = u_k$. Now $p(x) = x^{\deg f_1} f_1(\frac{1}{x})$ is a polynomial in $K[x]$ with $p(0)$ equal to the leading coefficient of f_1. Hence we have proved above that $p(k)$ is invertible for every $k \neq 0$ coprime to $p(0)$. If 0 is coprime to $p(0)$ then $p(0)$ is automatically invertible. Since K is a special PID, it follows that p is a constant polynomial. Hence there is a constant $c \in K$ such that $f_1(\frac{1}{x}) = \frac{c}{x^{\deg f_1}}$ and therefore $f_1(x) = cx^{\deg f_1}$. Consequently $c_1 ca^{\deg f_1} = 0$, thus $c_1 ca$ is nilpotent. Clearly $c_1 c \neq 0$. $\square$

The conclusions of Proposition 2.3.13 are not necessarily true if K is not a special PID. In particular, they are not true if K is a field which is not algebraically closed.

Example 2.3.14. Let F be a field which is not algebraically closed and p a non-constant polynomial over F which has no zeros in F. Let a be a zero of p in the algebraic closure of F. Then for every nonzero $\lambda \in F$ the element λa is π-algebraic with polynomial $p(\lambda^{-1})^{-1}p(\lambda^{-1}x)x$, however there is no $0 \neq \lambda \in F$ such that λa would be nilpotent.

2.3.2 π-algebraic rings and algebras

The aim of this subsection is to answer questions Q1 and Q2. By Lemma 2.3.2 every π-algebraic algebra is algebraic and Jacobson radical so the following proposition is a restatement of the well known fact that any algebraic Jacobson radical algebra over a field is nil (see for example [35, p. 144]). In fact, every algebraic element in the Jacobson radical of an algebra over a field is nilpotent and its index is equal to its degree.

Proposition 2.3.15. *Every π-algebraic F-algebra is nil.*

Next we extend this result to algebras over special PIDs and in particular to rings.

Theorem 2.3.16. *If K is a special PID then every π-algebraic K-algebra is nil.*

Proof. We shall first prove this for the particular case of rings, since we would like to point out that the result for rings can be derived from the results of Watters [38]. So suppose R is a π-algebraic ring. Let a be an element of R, S the subring of R generated by a, and $s \in S$. Since s is π-algebraic, $s^{(-1)}$ is a polynomial in s by Proposition 2.3.7. Hence $s^{(-1)} \in S$. This implies that S is Jacobson radical. Since $a^{(-1)}$ is π-algebraic, a is integral by Proposition 2.3.7. Thus the additive group $(S, +)$ is finitely generated (by the powers of a). By [38] the ring S is nilpotent, hence the element a is nilpotent. This shows that R is nil.

The above proof cannot be imitated in the general case since one of the tools used in [38] was Dirichlet's theorem on primes in arithmetic progressions.

Now suppose R is a π-algebraic K-algebra where K is a special PID. Suppose the element $a \in R$ is not nilpotent. A standard application of Zorn's lemma shows that there exists a prime ideal $P \triangleleft R$ that does not contain any power of a. Since R/P is a prime K-algebra, it follows by Proposition 1.1.7 that either R/P has no K-torsion or there exists an irreducible element $k \in K$, such that $k(R/P) = 0$. In the first case Proposition 2.3.13 implies that a is nilpotent modulo P, which is a contradiction. In the second case R/P is an algebra over the field K/kK, which is π-algebraic even over K/kK, so Proposition 2.3.15 implies that a is nilpotent modulo P, which is again a contradiction. Hence R must be nil. $\square$

Theorem 2.3.17. *If K is a special PID then every integral Jacobson radical K-algebra is nil.*

Proof. This follows directly from Proposition 2.3.7 and Theorem 2.3.16. $\square$

An algebraic Jacobson radical K-algebra need not be nil. In fact the ring in Example 2.1.10 is an algebraic Jacobson radical ring which is not nil. So the integral condition in Theorem 2.3.17 is crucial.

This answers question Q1 in two ways: the fact that distinguishes nil rings and algebras from all other algebraic ones is firstly that they are integral and Jacobson radical and secondly that the polynomials ensuring algebraicity in the nil case have the sum of their coefficients equal to 1. It is perhaps interesting that this rather large family of polynomials with the sum of coefficients equal to 1 produces the same effect as the rather restrictive family $\{x, x^2, x^3, x^4, \ldots\}$.

Observe that by Lemma 2.3.2 in an algebraic division algebra over a field only the identity is not π-algebraic. So if only one element in an algebra is not π-algebraic then the algebra may be very nice instead of nil.

Definition 2.3.18. A K-algebra R is *π-algebraic of bounded degree* $\leq n$ if every element of R is π-algebraic with some polynomial of degree $\leq n$. R is *π-algebraic of bounded degree* if there exists an integer n such that R is π-algebraic of bounded degree $\leq n$.

It follows from the proof of Proposition 2.3.7 that a K-algebra is π-algebraic of bounded degree $\leq n$ if and only if it is Jacobson radical and integral of bounded degree $\leq n$. The following natural question arises. Let K be a special PID. If a K-algebra R is π-algebraic of bounded degree, is it nil of bounded index? It is clear from the comment before Proposition 2.3.15 and the proof of Proposition 2.3.13 that the answer is positive for algebras over arbitrary fields and for algebras over K with no K-torsion.

Corollary 2.3.19. *If R is a π-algebraic F-algebra of bounded degree $\leq n$ or a π-algebraic K-algebra of bounded degree $\leq n$ with no K-torsion, where K is a special PID, then R is nil of bounded index $\leq n$.*

Perhaps surprisingly, the answer for general K-algebras is negative as the following example shows.

Example 2.3.20. Let K be a special PID which is not a field. Then by Proposition 2.3.9 K has infinitely many nonassociated irreducible elements. Choose a countable set of nonassociated irreducible elements $\{p_1, p_2, p_3, \ldots\}$ and let $R = \bigoplus_{i=1}^{\infty} p_i K / p_i^i K$. Clearly R is nil but not of bounded index. Let $a = (a_i)_i$ be an element of R. By the Chinese remainder theorem there is an element $k \in K$ such that $k \equiv a_i \pmod{p_i^i}$ for all i with $a_i \neq 0$. Then a is a zero of the monic polynomial $x^2 - kx$. This shows that R is integral of bounded degree ≤ 2, hence it is also π-algebraic of bounded degree ≤ 2.

Nevertheless the following holds for arbitrary algebras over special PIDs.

Proposition 2.3.21. *Let K be a field or a special PID. If R is a K-algebra which is π-algebraic of bounded degree then $Nil_*(R) = R$.*

Proof. Suppose P is a prime ideal of R. If K is a field then R/P is again a π-algebraic K-algebra of bounded degree. If K is a special PID then as in the proof of Theorem 2.3.16 R/P is either an algebra over some field or a K-algebra with no K-torsion and it is again π-algebraic of bounded degree. In any case R/P is nil of bounded index by Corollary 2.3.19. Thus by Theorem 2.1.20 we have $Nil_*(R/P) = R/P$, but on the other hand $Nil_*(R/P) = 0$, since P is a prime ideal. So $P = R$, which shows that $Nil_*(R) = R$. $\square$

Now we address question Q2.

Corollary 2.3.22. *Let R be a K-algebra, where K is a field or a special PID. Then the following hold:*

(i) *$Nil^*(R)$ is the largest π-algebraic ideal of R,*

(ii) *$Nil^*(R)$ is the largest integral quasi-regular ideal of R,*

(iii) *$Nil^*(R)$ is the largest quasi-regular ideal of R such that the quasi-inverse of each element is a polynomial in this element.*

Proof. If I is an ideal of R satisfying any of the above conditions then I is π-algebraic by Proposition 2.3.7 and thus nil by Proposition 2.3.15 or Theorem 2.3.16. Hence $Nil^*(R)$ is the largest such ideal. $\square$

Corollary 2.3.23. *If R is an integral ring then $J(R) = Nil^*(R)$.*

Corollary 2.3.24. *The Köthe conjecture is equivalent to the statement 'if R is a nil ring then $M_2(R)$ is an integral ring'.*

Proof. By Theorem 2.3.17 the given statement is clearly equivalent to Conjecture 2.2.6, since $M_2(R)$ is Jacobson radical if R is nil. $\square$

Corollary 2.3.25. *Every integral ring satisfies the Köthe conjecture.*

2.3.3 The structure of $\pi(R)$

In this subsection we investigate the structure of the set of all π-algebraic elements of an algebra. We restrict ourselves to algebras over fields and to rings. By Lemma 2.3.2 we know that $N(R) \subseteq \pi(R) \subseteq Q(R)$ and $(Q(R), \circ)$ is a group. It is thus natural to ask under what conditions $N(R)$ and $\pi(R)$ are (normal) subgroups of $Q(R)$ and more generally, what can be said about the structure of $\pi(R)$. In general $\pi(R)$ will not be closed under $\circ$. We will give a concrete example later on (see Example 2.3.28), but the reason for this is that the algebraic elements of R do not have any structure in general (they do not form a ring). However if R is commutative $\pi(R)$ will be closed under $\circ$. From here on $Q(R)$ will always be considered a group with operation $\circ$.

Lemma 2.3.26. *If r is a quasi-regular element of a ring or an F-algebra R then the map $x \mapsto r \circ x \circ r^{(-1)}$ is an automorphism of R.*

The proof of this lemma is an easy calculation. In fact if R is unital then $r \circ x \circ r^{(-1)} = (1-r)x(1-r^{(-1)}) = (1-r)x(1-r)^{-1}$, so the map is just the usual conjugation by $1-r$.

Proposition 2.3.27.

(i) *If R is a ring or an F-algebra then $N(R)$ is closed under conjugation and inversion. If R is commutative then $N(R)$ is a subgroup of $Q(R)$.*

(ii) *If R is an F-algebra, then $\pi_F(R)$ is closed under conjugation and inversion. If R is commutative then $\pi_F(R)$ is a subgroup of $Q(R)$.*

(iii) *If R is a ring, then $\pi(R)$ is closed under conjugation. If R is commutative then $\pi(R)$ is a submonoid of $Q(R)$.*

Proof. Let $a \in R$, $r \in Q(R)$, and let p be a polynomial. Then by Lemma 2.3.26 $r \circ p(a) \circ r^{(-1)} = p(r \circ a \circ r^{(-1)})$, so $r \circ a \circ r^{(-1)}$ is annihilated by the same polynomials as a. This shows that $N(R)$ and $\pi(R)$ (resp. $\pi_F(R)$) are closed under conjugation. The inverse of a nilpotent element is a polynomial in this element, so it is again nilpotent. Thus $N(R)$ is closed under inversion. If R is commutative then $N(R)$ is a subring of R, so it is closed under $\circ$ as well.

Let R be an F-algebra. If R is commutative then $A_F(R)$ is a subalgebra of R. Thus $A_F(R)$ is closed under $\circ$ and by Lemma 2.3.2 so is $\pi_F(R)$. If $a \in \pi_F(R)$ then the quasi-inverse of a is a polynomial in a, so it is algebraic and hence contained in $A_F(R) \cap Q(R) = \pi_F(R)$.

If R is a ring then by Proposition 2.3.7 $\pi(R)^{(-1)} = I(R) \cap Q(R)$. If R is commutative then $I(R)$ is a subring of R by Proposition 2.1.21 and hence closed under $\circ$. So $\pi(R)^{(-1)}$ is closed under $\circ$ and thus so is $\pi(R)$. $\square$

Unfortunately for a ring R the set $\pi(R)$ need not be closed under inversion. For example the quasi-inverse of $1 + \frac{1}{2} \in \pi(\mathbb{Q})$ is $1 + 2$ and it is not contained in $\pi(\mathbb{Q})$. In fact we know that $\pi(R)^{(-1)} = I(R) \cap Q(R)$.

Example 2.3.28. Let F be an algebraically closed field and $E = F(x)$ the field of rational functions over F. By Example 2.3.6 $\pi_F(M_2(E))$ consists of matrices with eigenvalues in $F \backslash \{1\}$. Take matrices

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which both lie in $\pi_F(M_2(E))$, since they are nilpotent. Then

$$A \circ B = \begin{bmatrix} -x & x \\ 1 & 0 \end{bmatrix}$$

does not have eigenvalues in F, since its trace is $-x \notin F$. So $\pi_F(M_2(E))$ is not closed under $\circ$.

Let R be either a ring or an F-algebra. For a subset S of $Q(R)$ let $\langle S \rangle$ denote the normal subgroup of $Q(R)$ generated by S. By Proposition 2.3.27 we have:

$$\begin{aligned} \langle N(R) \rangle &= \text{finite quasi-products of elements of } N(R), \\ \langle \pi_F(R) \rangle &= \text{finite quasi-products of elements of } \pi_F(R), \\ \langle \pi(R) \rangle &= \text{finite quasi-products of elements of } \pi(R) \cup \pi(R)^{(-1)}, \\ \langle \pi(R) \cap \pi(R)^{(-1)} \rangle &= \text{finite quasi-products of elements of } \pi(R) \cap I(R). \end{aligned}$$

Example 2.3.29. From Example 2.3.5 it is easy to calculate that we have $\langle \pi(\mathbb{Q}) \rangle = Q(\mathbb{Q}) = \mathbb{Q} \backslash \{1\}$ and $\langle \pi(\mathbb{Q}) \cap \pi(\mathbb{Q})^{(-1)} \rangle = \{0, 2\}$.

Example 2.3.30. Recall that a complex matrix A is called *unipotent* if $I - A$ is nilpotent, where I denotes the identity matrix. In [39] it was shown that a complex matrix is a finite product of unipotent matrices iff it has determinant 1. This shows that $\langle N(M_n(\mathbb{C})) \rangle = \{A \in M_n(\mathbb{C}) \; ; \; \det(I - A) = 1\}$.

Note that any quasi-regular ideal of R and in particular any nil ideal of R is a normal subgroup of $Q(R)$.

Proposition 2.3.31. *If R is a ring and I a nil ideal of R then:*

(i) $Q(R/I) \cong Q(R)/I$,

(ii) $\langle N(R/I)\rangle \cong \langle N(R)\rangle/I$,

(iii) $\langle \pi(R/I)\rangle \cong \langle \pi(R)\rangle/I$,

(iv) $\langle \pi(R/I)\cap\pi(R/I)^{(-1)}\rangle \cong \langle \pi(R)\cap\pi(R)^{(-1)}\rangle/I$.

Proof. The canonical map $f : R \to R/I$ preserves the operation $\circ$, since it is a ring homomorphism. It obviously maps $Q(R)$ to $Q(R/I)$, $N(R)$ to $N(R/I)$, $\pi(R)$ to $\pi(R/I)$, and $\pi(R)\cap\pi(R)^{(-1)}$ to $\pi(R/I)\cap\pi(R/I)^{(-1)}$. If $r + I \in Q(R/I)$ then $r \circ r'$ and $r' \circ r$ are elements of I for some $r' \in R$, so they are quasi-regular. Hence r is quasi-regular in R. If $r + I \in N(R/I)$ then $r^n \in I$ for some n, so r is nilpotent. If $r+I$ is π-algebraic with polynomial p then $p(r) \in I$, so $p(r)^n = 0$ for some n. Hence r is π-algebraic with polynomial $p(x)^n$. Similarly if $r + I \in \pi(R/I)\cap\pi(R/I)^{(-1)}$ then $r \in \pi(R)\cap\pi(R)^{(-1)}$. Obviously $I \subseteq Q(R), N(R), \pi(R), \pi(R)\cap\pi(R)^{(-1)}$ so f induces all the isomorphisms in the proposition. □

Of course the same holds for $\pi_F(R)$ if R is an F-algebra. In particular it follows from the proof that $\pi(R)$ is a normal subgroup of $Q(R)$ iff $\pi(R/I)$ is a normal subgroup of $Q(R/I)$.

Next we investigate what can be said about addition. We will need the following proposition which may be of independent interest.

Proposition 2.3.32. *Let R be a unital ring and K a commutative subring of R with $1 \in K$ such that $R\backslash K \subseteq R^{-1}$. If K is a finite ring or a ring with factorization then one of the following holds:*

(i) *$R = K$,*

(ii) *R is a local ring with maximal ideal $m \subseteq K$ and K is a local ring with maximal ideal m,*

(iii) *R is a division ring.*

Proof. Suppose that $R \neq K$ and R is not a division ring. Then there exist $r \in R\backslash K \subseteq R^{-1}$ and $0 \neq a \in K\backslash R^{-1}$. We will prove that $K^{-1} = K \cap R^{-1}$. If K is a finite ring then for every $x \in K$ there exist positive integers k and n such that $x^n = x^{n+k}$. So if x is invertible in R then $x^k = 1$, hence it is also invertible in K. Now suppose K has factorization. Then we may assume that the element a is irreducible. Let x be arbitrary element of K that is invertible in R and set $y = x^{-1}a$. Then y is not invertible in R, since a is not. But $R\backslash K \subseteq R^{-1}$, so $y \in K$. Thus $a = xy$ is a factorization of a in K. Since a was irreducible and y is not invertible, x must be invertible in K. Now let m be the set of all elements of K that are not invertible in K. Since $R\backslash K \subseteq R^{-1}$, m is also the set of all non-invertible elements of R. If $x \in m$ and $k \in K$ then xk is not invertible in K, otherwise x would be invertible due to the commutativity of K. So $mK \subseteq m$. If $x, y \in m$ then by the above x and y are not invertible in R. By the choice of r this implies that xr and yr are not invertible in R, so $xr, yr \in K$. Thus $(x - y)r \in K$. But $x - y \in K$ and $r \notin K$, hence $x - y$ cannot be invertible in K, so $x - y \in m$. This proves that m

in an ideal in K, so K is local with maximal ideal m. Now let $x \in m$ and $s \in R$, so by the above x is not invertible in R. If $s \in K$ then $sx, xs \in m$ by what we have just proved. If $s \notin K$ then s is invertible in R. So sx and xs are not invertible in R, hence $sx, xs \in m$. This shows that m is also an ideal of R and R is local with maximal ideal m. $\square$

Remark 2.3.33. There exist examples where case (ii) of Proposition 2.3.32 occurs in a nontrivial way. Take for example $R = E[[x]]$ and $K = F + E[[x]]x \subseteq R$ where $F \subsetneq E$ are fields. Every nonunit in K is contained in $E[[x]]x$ and factors as $x^n g(x)$ for some nonnegative integer n and some $g(x)$ of the form $\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \dots$ with $\alpha_1 \neq 0$.

Theorem 2.3.34. *Let R be a ring. For any subgroup S of $Q(R)$ the following are equivalent:*

(i) *S is closed under addition,*

(ii) *S is closed under multiplication,*

(iii) *S is a subring of R.*

Proof. Assume that R is unital. Let $x, y \in Q(R)$ be arbitrary. Then $x \circ x^{(-1)} = 0$ implies $x = -x^{(-1)} + xx^{(-1)} = -(1-x)x^{(-1)}$ and similarly $y = -y^{(-1)}(1-y)$. Hence

$$\begin{aligned}
xy &= (1-x)x^{(-1)}y^{(-1)}(1-y) = (1-x)\Big(x^{(-1)} + y^{(-1)} - x^{(-1)} \circ y^{(-1)}\Big)(1-y) = \\
&= (1-x)\Big(\big(1 - x^{(-1)} \circ y^{(-1)}\big) - \big(1 - (x^{(-1)} + y^{(-1)})\big)\Big)(1-y) = \\
&= (1-x)\Big((1 - x^{(-1)})(1 - y^{(-1)}) - \big(1 - (x^{(-1)} + y^{(-1)})\big)\Big)(1-y) = \\
&= (1-x)(1-x^{(-1)})(1-y^{(-1)})(1-y) - (1-x)\big(1 - (x^{(-1)} + y^{(-1)})\big)(1-y) = \\
&= 1 - (1-x)\big(1 - (x^{(-1)} + y^{(-1)})\big)(1-y) = \\
&= x \circ (x^{(-1)} + y^{(-1)}) \circ y.
\end{aligned}$$

A direct calculation shows that $xy = x \circ (x^{(-1)} + y^{(-1)}) \circ y$ holds even if R is not unital. Replacing x with $x^{(-1)}$ and y with $y^{(-1)}$ in this equation we get $x^{(-1)}y^{(-1)} = x^{(-1)} \circ (x+y) \circ y^{(-1)}$. If we quasi-multiply by x from the left and by y from the right we get $x + y = x \circ (x^{(-1)}y^{(-1)}) \circ y$. The two equations we have derived show that (i) and (ii) are equivalent.

Assume that R is unital and 2 is invertible. Then $1 - 2 = -1$ is quasi-regular and $(-1)^{(-1)} = 1 - (1-(-1))^{-1} = \frac{1}{2}$. By one of the above equations we have

$$\begin{aligned}
-x = x(-1) &= x \circ (x^{(-1)} + (-1)^{(-1)}) \circ (-1) = x \circ \left(x^{(-1)} + \frac{1}{2}\right) \circ (-1) = \\
&= x \circ \left(x^{(-1)} + \frac{1}{2} - 1 + x^{(-1)} + \frac{1}{2}\right) = x \circ (2x^{(-1)}).
\end{aligned}$$

A direct calculation again shows that $-x = x \circ (2x^{(-1)})$ holds even if R is not unital. Hence (i) implies that S is closed under negation as well, which implies (iii). Clearly (iii) implies (i). $\square$

Corollary 2.3.35. *Let F be a field of characteristic 0 and R a commutative F-algebra. If $\pi_F(R)$ is closed under addition then $\pi_F(R) = N(R)$.*

Proof. Since R is commutative, $\pi_F(R)$ is a subgroup of $Q(R)$ by Proposition 2.3.27. If $\pi_F(R)$ is closed under addition then it is a subring of R by Theorem 2.3.34. Let $a \in R$ be π-algebraic with polynomial p and let λ be a nonzero scalar. Since F is of characteristic 0 there exists a positive integer n such that $n\lambda^{-1}$ is not a zero of p. Hence $n^{-1}\lambda a$ is π-algebraic with polynomial $p(n\lambda^{-1})^{-1}p(n\lambda^{-1}x)$. Since $\pi_F(R)$ is closed under addition and λa is a multiple of $n^{-1}\lambda a$, λa is π-algebraic as well. So $\pi_F(R)$ is in fact a subalgebra of R. Thus $\pi_F(R)$ is nil by Proposition 2.3.15 and $\pi_F(R) = N(R)$ follows. $\square$

Proposition 2.3.36. *Let R be a commutative ring. If $\pi(R)$ is closed under addition then $\pi(R) = N(R)$.*

Proof. Suppose $\pi(R)$ is closed under addition. First we show that $\pi(R)$ is closed also under negation. If a is π-algebraic then $\mathbb{N}a \subseteq \pi(R)$ since $\pi(R)$ is closed under addition. Observe that in the proof of Proposition 2.3.12 that $\mathbb{Z}$ is a special PID, we can conclude that the polynomial p is constant even if $p(n)$ is invertible only for positive integers n coprime to $p(0)$. This means that the conclusion of Proposition 2.3.13 is true even if we only assume $\mathbb{N}a \subseteq \pi(R)$. Hence there exists a positive integer n such that na is nilpotent. Consequently $-na$ is nilpotent and hence π-algebraic. Since $\pi(R)$ is closed under addition, $-a = -na + (n-1)a$ is π-algebraic as well. By Proposition 2.3.27 the commutativity of R implies that $\pi(R)$ is closed under $\circ$. Since $xy = x + y - x \circ y$, $\pi(R)$ is closed under multiplication as well. So $\pi(R)$ is a π-algebraic subring of R, hence it is nil by Theorem 2.3.16. $\square$

Corollary 2.3.37. *Let p be a prime number, F an algebraic field extension of the prime field $\mathbb{Z}/p\mathbb{Z}$, and R a commutative F-algebra. If $\pi_F(R)$ is closed under addition then $\pi_F(R) = N(R)$.*

Proof. Since F is algebraic over $\mathbb{Z}/p\mathbb{Z}$, we have $A_F(R) = A_{\mathbb{Z}/p\mathbb{Z}}(R)$, so $\pi_F(R) = \pi_{\mathbb{Z}/p\mathbb{Z}}(R)$ by Lemma 2.3.2. Now let $a \in R$ be π-algebraic over $\mathbb{Z}/p\mathbb{Z}$ with polynomial $\widehat{f}$ and let f be a polynomial with integer coefficients that represents $\widehat{f}$ such that $f(0) = 0$. Since $\widehat{f}(1) = 1$, there exists an integer k such that $f(1) = kp + 1$. If we set $F(x) = f(x) - kpx$ then $F(0) = 0$, $F(1) = 1$ and $F(a) = 0$, since $pa = 0$. So a is π algebraic over $\mathbb{Z}$. Hence $\pi_{\mathbb{Z}/p\mathbb{Z}}(R) \subseteq \pi(R)$ and clearly $\pi(R) \subseteq \pi_{\mathbb{Z}/p\mathbb{Z}}(R)$. This implies $\pi_F(R) = \pi(R)$ and so $\pi_F(R) = N(R)$ by Proposition 2.3.36. $\square$

It remains an open question, whether Corollary 2.3.37 holds over arbitrary fields of prime characteristic.

This was one extremal situation; when the set of all π-algebraic elements forms a nil subring or subalgebra. The other extremal situation is when there are no nilpotent elements. As we have mentioned before in an algebraic division algebra there are no nonzero nilpotent elements although all elements except the unit are π-algebraic. Next we investigate when something similar happens in general algebras. The question is whether $\pi_F(R) \cup \{1\}$ will form a division ring for a unital F-algebra R. We can ask a similar question for rings, however it seems more natural to consider the set $\langle\pi(R)\rangle \cup (\mathbb{Z} \cdot 1)$ instead of $\langle\pi(R)\rangle \cup \{1\}$, since the elements in $(\mathbb{Z} \cdot 1)\backslash\{1\}$ need not be automatically π-algebraic. In certain situations, though, these two sets are in fact the same.

Theorem 2.3.38. *Let R be a unital ring of characteristic 0. For any subgroup S of $Q(R)$ with $\{0, 2\} \subsetneq S$ the following are equivalent:*

(i) *$S \cup \mathbb{Z}$ is closed under addition,*

(ii) *$S \cup \mathbb{Z}$ is a division subring of R,*

(iii) *$S \cup \{1\}$ is a division subring of R.*

Proof. Obviously (ii) implies (i). Also (iii) implies (ii), since in this case $S \cup \{1\} = S \cup \mathbb{Z}$.

(i) $\Rightarrow$ (ii): If $x \in S \cup \mathbb{Z}$ then $2 \circ x = 2 - x \in S \cup \mathbb{Z}$, since $2 \in S \cap \mathbb{Z}$. So if $x \in S \cup \mathbb{Z}$ then $-x = 2 - (2 + x) \in S \cup \mathbb{Z}$ by (i). Thus $S \cup \mathbb{Z}$ is closed under negation. S and $\mathbb{Z}$ are closed under $\circ$. If $x \in S$ and $n \in \mathbb{Z}$ then $x \circ n = n \circ x = n + x - nx \in S \cup \mathbb{Z}$, since nx is a multiple of x or $-x$ and $S \cup \mathbb{Z}$ is closed under addition. So $S \cup \mathbb{Z}$ is closed under $\circ$ and hence also under multiplication, since $xy = x + y - x \circ y$. This shows that $S \cup \mathbb{Z}$ is a subring of R. Now every element in S is quasi-regular with quasi-inverse in S, thus every element in $1 - S$ is invertible in $S \cup \mathbb{Z}$. Since $S \cup \mathbb{Z}$ is a subring, we have $1 - S\backslash\mathbb{Z} = S\backslash\mathbb{Z}$. So every element in $S\backslash\mathbb{Z}$ is invertible in $S \cup \mathbb{Z}$. By Proposition 2.3.32 either $S \subseteq \mathbb{Z}$ or $S \cup \mathbb{Z}$ is a division ring. Suppose $S \subseteq \mathbb{Z}$. Then the quasi-inverse of every element in $S \subseteq \mathbb{Z}$ lies again in $S \subseteq \mathbb{Z}$, so $S \subseteq Q(\mathbb{Z}) = \{0, 2\}$, which contradicts our assumption.

(ii) $\Rightarrow$ (iii): It is enough to show that $\mathbb{Z}\backslash\{1\} \subseteq S$. Let $n \in \mathbb{Z}\backslash\{1\}$. If $n = 0$ or $n = 2$ then $n \in S$ by assumption. So suppose $n \neq 0, 2$. Since $S \cup \mathbb{Z}$ is a division ring, $1 - n$ is invertible and since $n \neq 0, 2$ and the characteristic of R is 0, $(1 - n)^{-1} \notin \mathbb{Z}$. Thus $1 - (1 - n)^{-1} \in S$ and so $(1 - (1 - n)^{-1})^{(-1)} = 1 - (1 - n) = n \in S$, since S is a subgroup of $Q(R)$. □

Theorem 2.3.39. *Let R be a unital ring of prime characteristic p. For any subgroup S of $Q(R)$ the following are equivalent:*

(i) *$S \cup \mathbb{Z}/p\mathbb{Z}$ is closed under addition,*

(ii) *$S \cup \mathbb{Z}/p\mathbb{Z}$ is a division subring of R.*

Proof. In this case $S \cup \mathbb{Z}/p\mathbb{Z}$ is automatically closed under negation, since $-x = (p - 1)x$ is a multiple of x. The proof is now the same as that of Theorem 2.3.38

except for the case $S \subseteq \mathbb{Z}/p\mathbb{Z}$, but in this case $S \cup \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}$ is automatically a division ring. $\square$

Corollary 2.3.40. *Let F be a field and R a unital commutative F-algebra. If $\pi_F(R) \cup \{1\}$ is closed under addition then it is a subfield of R.*

Proof. This follows directly from Proposition 2.3.27 and Theorems 2.3.38 and 2.3.39, since $(\mathbb{Z} \cdot 1)\backslash\{1\} \subseteq \pi_F(R)$ by Lemma 2.3.3. $\square$

Corollary 2.3.41. *Let R be a unital commutative ring of prime or 0 characteristic with $\pi(R) \neq \{0, 2\}$. If $\pi(R) \circ \pi(R)^{(-1)} \cup (\mathbb{Z} \cdot 1)$ is closed under addition then it is a subfield of R.*

Proof. The commutativity of R implies $\langle \pi(R) \rangle = \pi(R) \circ \pi(R)^{(-1)}$. Since $2 \in \pi(R)$ by Lemma 2.3.3, the result follows from Theorems 2.3.38 and 2.3.39. $\square$

Bibliography

[1] S.A. Amitsur: Algebras over infinite fields, *Proc. Am. Math. Soc.* 7 (1956), 35-48

[2] S.A. Amitsur: Radicals of polynomial rings, *Can. J. Math.* 8 (1956), 355-361

[3] J. Araujo, K. Jarosz: Biseparating maps between operator algebras, *J. Math. Anal. Appl.* 282 (2003), 48-55

[4] K.I. Beidar, W.S. Martindale III, A.V. Mikhalev: *Rings with Generalized Identities*, Marcel Dekker, Inc., 1996

[5] K.I. Beidar, Y. Fong: On Radicals of Monomial Algebras, *Commun. Algebra* 26 (1998), 3913-3919

[6] M. Brešar: Characterizing homomorphisms, derivations and multipliers in rings with idempotents, *Proc. R. Soc. Edinb., Sect. A, Math.* 137 (2007), 1-7

[7] M.A. Chebotar, Y. Fong, P.-H. Lee: On maps preserving zeros of the polynomial $xy - yx^*$, *Linear Algebra Appl.* 408 (2005), 230-243

[8] M.A. Chebotar, W.-F. Ke, P.-H. Lee: Maps characterized by action on zero products, *Pac. J. Math.* 216 (2004), 217-228

[9] M.A. Chebotar, W.-F. Ke, P.-H. Lee, N.-C. Wong: Mappings preserving zero products, *Stud. Math.* 155 (2003), 77-94

[10] J. Cui, J. Hou: Linear maps on von Neumann algebras preserving zero products or tr-rank, *Bull. Aust. Math. Soc.* 65 (2002), 79-91

[11] M. Ferrero, E.R. Puczyłowski: On rings which are sums of two subrings, *Arch. Math. (Basel)* 53 (1989), 4-10

[12] J.W. Fischer, J. Krempa: 'R^G is nil implies R is nil' is equivalent to the 'Koethe conjecture', *Houston J. Math.* 9 (1983), 177-180

[13] B.J. Gardner, R. Wiegandt: *Radical Theory of Rings*, Marcel Dekker, New York, 2004

[14] E.S. Golod: On nil algebras and finitely approximable p-groups, *Izv. Akad. Nauk SSSR, Ser. Mat.* 28 (1964), 273-276

[15] E.S. Golod, I.R. Shafarevich: On the class field tower, *Izv. Akad. Nauk SSSR, Ser. Mat.* 28 (1964), 261-272

[16] G. Köthe: Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist, *Math. Z.* 32 (1930), 161-186

[17] J. Krempa: Logical connections between some open problems concerning nil rings, *Fundam. Math.* 76 (1972), 121-130

[18] T.Y. Lam: *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999

[19] T.Y. Lam: *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 2001

[20] J. Levitzki: A theorem on polynomial identities, *Proc. Am. Math. Soc.* 1 (1950), 334-341

[21] J. Levitzki: Prime ideals and the lower radical, *Am. J. Math.* 73 (1951), 25-29

[22] M. Nagata: On the nilpotency of nil-algebras, *J. Math. Soc. Japan* 4 (1952), 296–301

[23] C. Procesi: *Rings with polynomial identities*, Marcel Dekker, New York, 1973

[24] E.R. Puczyłowski, A. Smoktunowicz: A polynomial ring that is Jacobson radical and not nil, *Isr. J. Math.* 124 (2001), 317-325

[25] A.D. Sands: Radicals and Morita contexts, *J. Algebra* 24 (1973), 335-345

[26] A. Smoktunowicz: A simple nil ring exists, *Commun. Algebra* 30 (2002), 27-59

[27] A. Smoktunowicz: On primitive ideals in polynomial rings over nil rings, *Algebr. Represent. Theory* 8 (2005), 69-73

[28] A. Smoktunowicz: Polynomial rings over nil rings need not be nil, *J. Algebra* 233 (2000), 427-436

[29] A. Smoktunowicz: Some open results related to Köthe's conjecture, *Serdica Math. J.* 27 (2001), 159-170

[30] A. Smoktunowicz: Some results in noncommutative ring theory, *Proceedings of the International Congress of Mathematicians*, Madrid, Spain, 2006

[31] B. Stenström: *Rings of quotients*, Springer-Verlag, 1975

[32] N. Stopar: Nilpotent, algebraic and quasi-regular elements in rings and algebras, preprint

[33] N. Stopar: Preserving zeros of xy and xy^*, *Commun. Algebra* 40 (2012), 2053-2065

[34] G.A. Swain: Maps preserving zeros of xy^*, *Commun. Algebra* 38 (2010), 1613-1620

[35] F.A. Szász: *Radicals of rings*, John Wiley & Sons, Chichester, 1981

[36] Y. Utumi: On quotient rings, *Osaka J. Math.* 8 (1956), 1–18.

[37] Y. Wang: A Note on Maps Characterized by Action on Zero Products, *Algebra Colloq.* 13 (2006), 685-687

[38] J.F. Watters: On the adjoint group of a radical ring, *J. Lond. Math. Soc.* 43 (1968), 725-729

[39] J.H. Weng, P.Y. Wu: Products of unipotent matrices of index 2, *Linear Algebra Appl.* 149 (1991), 111-123

[40] N.-C. Wong: Triple homomorphisms of C^*-algebras, *Southeast Asian Bull. Math.* 29 (2005), 401-407

[41] W.J. Wong: Maps on simple algebras preserving zero products. I: The associative case, *Pac. J. Math.* 89 (1980), 229-247

[42] X. Yonghua: On the Koethe problem and the nilpotent problem, *Sci. Sin., Ser. A* 26 (1983), 901-908

Daljši povzetek

1 Prakolobarji

Prakolobarji sodijo med najpomembnejše tipe kolobarjev v moderni matematiki, saj so najprimernejša posplošitev komutativnih celih kolobarjev v kontekst nekomutativnih kolobarjev. Pomembni so v mnogih različnih področjih matematike, kot so na primer algebraična geometrija, teorija radikalov, teorija polinomskih in funkcijskih identitet in mnoga druga.

Definicija 1.1. Kolobar R je *prakolobar*, če je $R \neq 0$ in za vsaka ideala $I, J \triangleleft R$, za katera je $IJ = 0$, velja $I = 0$ ali $J = 0$. Ideal P kolobarja R je *praideal*, če je R/P prakolobar.

Ni težko dokazati, da je $P \triangleleft R$ praideal natanko tedaj, ko za vsaka elementa $a, b \in R$, za katera je $aRb = 0$, velja $a = 0$ ali $b = 0$.

Praideali in prakolobarji imajo veliko ugodnih lastnosti. Omenimo jih nekaj, ki jih bomo potrebovali. Prva taka lastnost je, da sta levi in desni anihilator prakolobarja vedno enaka nič. Druga lastnost se nanaša na zvezo med ideali v kolobarjih in ideali v algebrah. Vsaka algebra R nad komutativnim enotskim kolobarjem je v posebnem kolobar. Vsak praideal kolobarja R je avtomatično zaprt za množenje s skalarjem in je zato tudi ideal algebre R. Za splošne ideale to ne velja. V poljubnem kolobarju, ali splošneje v poljubni algebri nad glavnim kolobarjem, je množica vseh torzijskih elementov lahko zelo zapletena. To je mnogokrat ovira pri posploševanju rezultatov iz algeber nad komutativnimi obsegi na algebre nad glavnimi kolobarji. V prakolobarjih in praalgebrah so torzijski elementi veliko bolj obvladljivi. Na primer, če je v prakolobarju $na = 0$ za nek element $a \neq 0$ in neko naravno število n, potem bo $nb = 0$ za vsak element b. Nazadnje omenimo še, da je lastnost biti prakolobar mnogokrat lahko nadomestek za obstoj enote v kolobarju. Za dodatne lastnosti prakolobarjev naslavljamo bralca na [19].

Prakolobarji so med drugim primerni tudi za konstrukcije raznoraznih kolobarjev kvocientov. Glavno vlogo v teh konstrukcijah igrajo tako imenovani *gosti* ideali.

Definicija 1.2. Desni ideal J kolobarja R je *gost*, če za vsaka $a, b \in R$, kjer je $b \neq 0$, obstaja $r \in R$, da velja $br \neq 0$ in $ar \in J$.

S pomočjo gostih desnih idealov konstruiramo prvi kolobar kvocientov, ki je v nekem smislu največji. Naj bo R prakolobar in označimo z S množico vseh parov

$(f; J)$, kjer je J gost desni ideal kolobarja R in $f : J \to R$ homomorfizem desnih R-modulov. Definirajmo relacijo $\sim$ na S, kjer je $(f; J) \sim (g; K)$ natanko tedaj, ko se f in g ujemata na $J \cap K$. Ni težko preveriti, da je $\sim$ ekvivalenčna relacija na S. Ekvivalenčni razred elementa $(f; J) \in S$ označimo z $[f; J]$, množico vseh ekvivalenčnih razredov v S pa s $Q_{mr}(R)$. Na $Q_{mr}(R)$ definiramo seštevanje in množenje s predpisoma

$$[f; J] + [g; K] = [f + g; J \cap K],$$
$$[f; J] \cdot [g; K] = [f \circ g; g^{-1}(J)],$$

kjer $f \circ g$ označuje kompozitum preslikav f and g. Ti dve operaciji zadoščata vsem aksiomom kolobarja.

Definicija 1.3. Kolobar $Q_{mr}(R)$ imenujemo *maksimalen desni kolobar kvocientov* kolobarja R.

Kolobar $Q_{mr}(R)$ je prvi konstruiral Utumi [36] leta 1956 v zgoraj opisani obliki. Za drugačen, bolj homološki pristop h konstrukciji, naslavljamo bralca na [18, §13], kjer je tudi razloženo, zakaj ta kolobar imenujemo *maksimalen* desni kolobar kvocientov. Izkaže se, da je maksimalen desni kolobar kvocientov prakolobarja spet prakolobar, operator Q_{mr} pa je idempotenten, kar pomeni, da je $Q_{mr}(Q_{mr}(R)) = Q_{mr}(R)$ (glej [4, Proposition 2.1.10]).

V nadaljevanju bomo definirali *Martindalove kolobarje kvocientov* poimenovane po W.S. Martindalu, ki jih je vpeljal leta 1969. V nasprotju z maksimalnimi kolobarji kvocientov, so ti definirani s pomočjo dvostranskih idealov. Če je R prakolobar, potem je dvostranski ideal $I \lhd R$ gost kot desni ideal natanko tedaj, ko je neničeln. Za prakolobar R definiramo

$$Q^r(R) = \{q \in Q_{mr}(R) \ ; \ qI \subseteq R \text{ za nek } 0 \neq I \lhd R\}$$

in

$$Q^s(R) = \{q \in Q_{mr}(R) \ ; \ qI \cup Iq \subseteq R \text{ za nek } 0 \neq I \lhd R\}.$$

Ni težko preveriti, da sta $Q^r(R)$ in $Q^s(R)$ podkolobarja v $Q_{mr}(R)$.

Definicija 1.4. Kolobar $Q^r(R)$ imenujemo *Martindalov desni kolobar kvocientov* kolobarja R, kolobar $Q^s(R)$ pa imenujemo *Martindalov simetrični kolobar kvocientov* kolobarja R.

Na analogen način definiramo *maksimalen levi kolobar kvocientov* $Q_{ml}(R)$ in *Martindalov levi kolobar kvocientov* $Q^l(R)$ kot podkolobar v $Q_{ml}(R)$. Izkaže se, da je analog kolobarja $Q^s(R)$ znotraj $Q_{ml}(R)$ izomorfen kolobarju $Q^s(R)$, torej lahko na $Q^s(R)$ gledamo tudi kot na podkolobar v $Q_{ml}(R)$. Kolobarje Q_{mr}, $Q^r(R)$ in $Q^s(R)$ lahko definiramo tudi aksiomatično (glej [4, Proposition 2.1.7, 2.2.1 in 2.2.3]).

Zaradi simetrije kolobar $Q^s(R)$ od kolobarja R podeduje veliko več strukture kot kolobarja $Q^r(R)$ in $Q_{mr}(R)$. Na primer, če je R prakolobar z involucijo, potem lahko involucijo na enoličen način razširimo do involucije na $Q^s(R)$ (glej [4, Proposition 2.5.4]).

Definicija 1.5. Center kolobarja $Q^s(R)$ označimo s $C(R)$ in imenujemo *razširjeni centroid* kolobarja R.

Izkaže se, da se $C(R)$ ujema s centri ostalih zgoraj definiranih kolobarjev kvocientov (glej [4, Remark 2.3.1]). Za vsak prakolobar R je $C(R)$ v resnici komutativen obseg. Nekaj konkretnih primerov različnih kolobarjev kvocientov lahko bralec najde v [18].

V nadaljevanju bomo obravnavali konkretna primera ohranjevalcev na prakolobarjih. Teorija ohranjevalcev je že desetletja aktivno raziskovalno področje z aplikacijami v mnogih vejah matematike in matematične fizike. Ohlapno rečeno je *ohranjevalec* preslikava med dvema kolobarjema, ki ohranja določeno lastnost, relacijo ali podmnožico. Cilj teorije je opisati, kako preslikave, ki ohranjajo neko lastnost, izgledajo. Najtemeljiteje raziskani ohranjevalci so ohranjevalci na matričnih algebrah. Klasični primeri so ohranjevalci komutativnosti, ohranjevalci ranga, ohranjevalci sosednosti, ohranjevalci determinante, ohranjevalci spektra, ohranjevalci obrnljivih elementov in drugi.

Najprej se bomo posvetili ohranjevalcem ničelnega produkta.

Definicija 1.6. Naj bosta A in B kolobarja. Preslikava $\theta : A \to B$ *ohranja ničelni produkt*, če je $\theta(x)\theta(y) = 0$ za vse $x, y \in A$, za katere je $xy = 0$. Pravimo tudi, da je θ *ohranjevalec ničelnega produkta.*

Ohranjevalce ničelnega produkta so obravnavali mnogi avtorji v mnogih različnih kontekstih. Omenimo najprej rezultat za matrične algebre, iz katerega je razvidno, kakšna je pričakovana oblika ohranjevalca ničelnega produkta. Rezultat so dokazali Chebotar idr. [9, Corollary 2.4].

Izrek 1.7. *Naj bo F algebraično zaprt komutativen obseg karakteristike nič in $\theta : M_n(F) \to M_r(F)$ linearna preslikava, ki ohranja ničelni produkt, kjer sta n in r naravni števili, za kateri velja $n \geq 2$ ter $n \geq r$. Potem je množenje na $\operatorname{Im}\theta$ trivialno ali pa je $n = r$ in obstaja obrnljiva matrika $A \in M_n(F)$ ter skalar $\lambda \in F$, da velja $\theta(X) = \lambda AXA^{-1}$ za vse $X \in M_n(F)$.*

Torej, če množenje na $\operatorname{Im}\theta$ ni trivialno, potem je preslikava θ skalarni večkratnik homomorfizma algeber. Tudi v splošnem je pričakovana oblika ohranjevalca ničelnega produkta podobna, se pravi, homomorfizem pomnožen s centralnim elementom. Ker pogosto zahtevamo, da je preslikava surjektivna, je pripadajoči centralni element ponavadi obrnljiv. Vsaka taka preslikava očitno ohranja ničelni produkt.

Ohranjevalci ničelnega produkta so bili obravnavani tudi v drugih kontekstih. Wong [41] je karakteriziral bijektivne semilinearne ohranjevalce ničelnega produkta na enostavnih končno razsežnih algebrah in tudi na določenem razredu primitivnih algeber. Ajauro in Jarozs [3] sta obravnavala ohranjevalce ničelnega produkta na podalgebrah Banachove algebre omejenih linearnih operatorjev in na prostorih zveznih funkcij z vrednostmi v operatorskih algebrah. Cui in Hou [10] sta karakterizirala surjektivne omejene linearne ohranjevalce ničelnega produkta na von Neumannovih algebrah. Chebotar idr. [9] so karakterizirali surjektivne omejene linearne ohranjevalce ničelnega produkta na enotskih C^*-algebrah in na nekaterih standardnih operatorskih algebrah. Leta 2004 so Chebotar idr. [8] posplošili nekatere od zgornjih rezultatov z obravnavo bijektivnih aditivnih ohranjevalcev ničelnega produkta na prakolobarjih z netrivialnimi idempotenti. Med drugim so dokazali naslednji izrek ([8, Theorem 1]).

Izrek 1.8. *Naj bosta A in B prakolobarja in $\theta : A \to B$ bijektivna aditivna preslikava, ki ohranja ničelni produkt. Denimo, da $Q_{mr}(A)$ vsebuje netrivialen idempotent e, za katerega velja $eA \cup Ae \subseteq A$.*

(i) *Če je $1 \in A$, potem velja $\theta(xy) = \lambda\theta(x)\theta(y)$ za vse $x, y \in A$, kjer je $\theta(1) \in Z(B)$ in $\lambda = 1/\theta(1) \in C(B)$. V posebnem, če je $\theta(1) = 1$, potem je θ izomorfizem kolobarjev.*

(ii) *Če je $\deg B \geq 3$, potem obstaja tak $\lambda \in C(B)$, da velja $\theta(xy) = \lambda\theta(x)\theta(y)$ za vse $x, y \in A$.*

Ker izrek opisuje ohranjevalce ničelnega produkta na razredu splošnih kolobarjev, je potreben dodaten pogoj obstoja netrivialnega idempotenta, ki zagotovi, da v kolobarju A obstaja dovolj ničelnih produktov. V nasprotnem bi lahko bil A cel kolobar in v tem primeru bi vsaka aditivna preslikava na prazno ohranjala ničelni produkt.

Wang [37] je pokazal, da tehnična predpostavka $\deg B \geq 3$ v Izreku 1.8 ni potrebna. Poleg tega je Brešar [6] predpostavko, da je A prakolobar, zamenjal s šibkejšo predpostavko, ki implicira, da A vsebuje necentralen idempotent. V naslednjem izreku še dodatno posplošimo te rezultate, tako da obravnavamo surjektivne (ne nujno injektivne) aditivne ohranjevalce ničelnega produkta. Ta rezultat je vsebovan v [33].

Izrek 1.9. *Naj bo A kolobar, B pa prakolobar. Naj bo $\theta : A \to B$ surjektivna aditivna preslikava, ki ohranja ničelni produkt. Nadalje, naj bo R enotski kolobar, ki vsebuje A kot podkolobar, in e idempotent v R, za katerega velja $eA \cup Ae \subseteq A$. Označimo $f = 1 - e$. Če je $e \in A$, $f \in A$ ali $A = \sum A^2$, potem velja ena od naslednjih možnosti:*

(i) $\theta(eA + Ae + AeA) = 0$,

(ii) $\theta(fA + Af + AfA) = 0$,

(iii) *obstaja $0 \neq \lambda \in C(B)$, da velja $\theta(xy) = \lambda\theta(x)\theta(y)$ za vse $x, y \in A$.*

Potrebno je omeniti, da Izrek 1.9 nekaj pove le v primeru, ko idempotent e deluje netrivialno na A, tj. ne kot 0 ali 1. To na nek način pomeni, da mora biti idempotent e smiselno povezan s kolobarjem A. V Izreku 1.8 je bil na primer e element maksimalnega desnega kolobarja kvocientov kolobarja A.

V določenih situacijah lahko zagotovimo, da bo izpolnjen pogoj (iii) iz Izreka 1.9. Denimo, da e deluje netrivialno na A. Potem sta $eA + Ae + \sum AeA$ in $fA + Af + \sum AfA$ neničelna ideala v A. Torej, če je A enostaven kolobar ali pa če je

θ injektivna preslikava, potem bo izpolnjen pogoj (iii). To v posebnem pokaže, da v [6, Corollary 4.3] ni potrebna nobena dodatna predpostavka na kolobar A razen obstoja netrivialnega idempotenta.

Vsaka preslikava θ, ki ustreza pogoju (iii) iz Izreka 1.9, res ohranja ničelni produkt. Po drugi strani pa pogoja (i) oziroma (ii) nista zadostna, da bi preslikava θ ohranjala ničelni produkt. Kljub temu, enostaven primer pokaže, da sta zaključka v (i) oziroma (ii) optimalna.

Zgled 1.10. Naj bo T kolobar, S cel kolobar, B prakolobar in $A = S \oplus T$. Vložimo kolobarja S in T v enotska kolobarja S^1 in T^1. Potem je kolobar A vložen v $R = S^1 \oplus T^1$ in $e = (1, 0) \in R$ je idempotent, za katerega velja $eA + Ae + AeA = S$ in $fA + Af + AfA = T$. Za poljubno (aditivno surjektivno) preslikavo $\phi : S \to B$ s $\phi(0) = 0$ preslikava $\theta : A \to B$, definirana s $\theta(s, t) = \phi(s)$ za vse $s \in S$, $t \in T$, ohranja ničelni produkt in izpolnjuje pogoj (ii) iz Izreka 1.9. To pokaže, da sta pogoja (i) oziroma (ii) iz Izreka 1.9 največ kar lahko izluščimo iz obstoja enega samega idempotenta.

Pogoj (ii) iz Izreka 1.9 med drugim implicira, da velja $\theta(x) = \theta(exe)$ za vse $x \in A$. Seveda tudi zožitev preslikave θ na podkolobar $eAe \subseteq A$ ohranja ničelni produkt. Toda to ne zreducira problema na podkolobar eAe, saj preslikava $\theta(x) = \psi(exe)$ ne ohranja nujno ničelnega produkta, tudi če ga preslikava $\psi : eAe \to B$ ohranja.

V nadaljevanju bomo obravnavali varianto ohranjevalcev ničelnega produkta za kolobarje z involucijo. *Involucija* je antiavtomorfizem reda ≤ 2.

Definicija 1.11. Naj bosta A in B kolobarja z involucijo. Preslikava $\theta : A \to B$ *ohranja ničle xy^**, če je $\theta(x)\theta(y)^* = 0$ za vse $x, y \in A$, za katere je $xy^* = 0$. Pravimo tudi, da je θ ohranjevalec ničel xy^*.

Ohranjevalci ničel xy^* niso bili tako temeljito raziskani kot ohranjevalci ničelnega produkta, saj so se v literaturi pojavili šele pred kratkim. Kljub temu je nekaj znanih rezultatov s tega področja. Spet najprej omenimo osnoven rezultat za matrike, ki ga je dokazal Swain [34, Corollary 5].

Izrek 1.12. *Naj bo F komutativen obseg, $*$ involucija na $M_n(F)$, kjer je $n \geq 2$ naravno število, in $\theta : M_n(F) \to M_n(F)$ bijektivna linearna preslikava, ki ohranja ničle xy^*. Potem obstajata obrnljivi matriki $B, U \in M_n(F)$, pri čemer je $U^* = U^{-1}$, da velja $\theta(X) = BXU$ za vse $X \in M_n(F)$.*

Preslikavo θ lahko zapišemo tudi v obliki $\theta(X) = CU^{-1}XU$ za vse $X \in M_n(F)$, kjer je $C = BU \in M_n(F)$. Taka preslikava θ je torej kompozitum $*$-homomorfizma algebre $M_n(F)$ in levega množenja z nekim elementom iz $M_n(F)$. V splošni situaciji je pričakovana oblika ohranjevalca ničel xy^* enaka, se pravi, $*$-homomorfizem pomnožen z leve z nekim elementom kolobarja. Ker pogosto zahtevamo, da je preslikava surjektivna (ali celo bijektivna), je pripadajoči element ponavadi obrnljiv.

Swain [34] je obravnaval ohranjevalce ničel xy^* tudi na prakolobarjih z involucijo. Karakteriziral je bijektivne aditivne preslikave $\theta : A \to A$, ki ohranjajo ničle xy^*, v primeru, ko je A enotski prakolobar z involucijo, ki je generiran z idempotenti.

Primeri kolobarjev, ki so generirani z idempotenti, so enostavni kolobarji z netrivialnimi idempotenti in kolobarji $n \times n$ matrik nad enotskimi kolobarji, kjer je $n \geq 2$ (glej [6] za podrobnosti). Predpostavka, da je kolobar generiran z idempotenti, je dokaj močna, toda, kot je že Swain omenil, bo morda težko priti do podobne karakterizacije v poljubnih prakolobarjih z involucijo, ki vsebujejo netrivialen idempotent.

V zadnjem desetletju se je v literaturi pojavilo tudi nekaj sorodnih problemov. Wong [40] je na primer obravnaval linearne preslikave θ na C^*-algebrah, za katere velja $\theta(x)\theta(y)^* = \theta(x)^*\theta(y) = 0$ za vse x, y, za katere je $xy^* = x^*y = 0$. Takim preslikavam je rekel *ohranjevalci disjunktnosti*. Obravnavani so bili tudi ohranjevalci ničel nekaterih drugih $*$-polinomov, kot je na primer $xy - yx^*$ (glej [7]).

Med omenjenima problemoma ohranjevalcev na kolobarjih z in brez involucije je ena bistvena razlika. Pogoj za ohranjevalce ničelnega produkta je popolnoma simetričen, medtem ko pogoj za ohranjevalce ničel xy^* ni simetričen, saj $*$ nastopa le na desni strani. Ta izguba simetrije v kontekstu kolobarjev z involucijo ima določene posledice. Kot prvo, razred pričakovanih rešitev je tukaj nekoliko večji. V obeh situacijah so pričakovane rešitve morfizmi pomnoženi z nekim elementom kolobarja, toda v kontekstu brez involucije je ta element centralen, medtem ko je v kontekstu z involucijo lahko poljuben. Poleg tega so rezultati v kontekstu z involucijo ponavadi manj splošni in pogosto so potrebne dodatne predpostavke, da lahko karakteriziramo ohranjevalce ničel xy^*. Na primer, v prej omenjenem Swainovem rezultatu o prakolobarjih z involucijo (glej [34]), je bila dodatna predpostavka to, da je kolobar generiran z idempotenti. Naš cilj v preostanku tega poglavja bo predstaviti nekaj rezultatov, v katerih se izognemo tej močni dodatni predpostavki in raje dodamo predpostavke na preslikavo samo. Večina teh rezultatov je zajetih v [33].

Najprej pokažemo, kako lahko problem ohranjevalcev ničel xy^* v resnici zagledamo kot posplošitev problema ohranjevalcev ničelnega produkta. Naj bosta A in B kolobarja z involucijo ter $\theta : A \to B$ poljubna preslikava. Definirajmo preslikavo $\phi : A \to B$ s predpisom $\phi(x) = \theta(x^*)^*$ za vse $x \in A$. Potem preslikava θ ohranja ničle xy^* natanko tedaj, ko velja $\theta(x)\phi(y) = 0$ za vse $x, y \in A$, za katere je $xy = 0$. Ta pogoj je posplošitev pogoja za ohranjevalce ničelnega produkta, saj vsebuje dve preslikavi namesto le ene. Dejstvo, da je preslikava ϕ tesno povezana s preslikavo θ v naših dokazih ne igra pomembnejše vloge, zato lahko večino rezultatov formuliramo za poljubne pare preslikav θ in ϕ. To bomo storili izrecno le v najzanimivejšem primeru.

Naš prvi rezultat pokaže, da lahko podobno kot v primeru brez involucije, v [34, Theorem 4] izpustimo predpostavko o injektivnosti preslikave.

Trditev 1.13. *Naj bo A enotski kolobar z involucijo, ki je generiran z idempotenti, in B prakolobar z involucijo. Naj bo $\theta : A \to B$ surjektivna aditivna preslikava, ki ohranja ničle xy^*. Potem obstaja $*$-homomorfizem $h : A \to Q^s(B)$, da velja $\theta(x) = \theta(1)h(x)$ za vse $x \in A$.*

Naš glavni rezultat opiše pare preslikav θ in ϕ, ki zadoščajo pogoju

$$\theta(x)\phi(y) = 0 \quad \text{natanko tedaj, ko} \quad xy = 0. \tag{1.2}$$

Izrek 1.14. *Naj bo A enotski prakolobar z netrivialnim idempotentom in B prakolobar. Naj bosta $\theta, \phi : A \to B$ taki surjektivni aditivni preslikavi, da za vsaka $x, y \in A$ velja $\theta(x)\phi(y) = 0$ natanko tedaj, ko velja $xy = 0$. Potem je $\theta(1)$ obrnljiv v $Q^r(B)$, $\phi(1)$ obrnljiv v $Q^l(B)$ in obstaja injektiven homomorfizem $h : A \to Q^s(B)$, da velja $\theta(x) = \theta(1)h(x)$ in $\phi(x) = h(x)\phi(1)$ za vse $x \in A$.*

Zaključki Izreka 1.14 so hkrati tudi zadostni, da preslikavi θ in ϕ zadoščata pogoju (1.2). Kot posledico Izreka 1.14, dobimo naslednji izrek.

Izrek 1.15. *Naj bo A enotski prakolobar z involucijo, ki vsebuje netrivialen idempotent, in B prakolobar z involucijo. Naj bo $\theta : A \to B$ taka surjektivna aditivna preslikava, za za vsaka $x, y \in A$ velja $\theta(x)\theta(y)^* = 0$ natanko tedaj, ko velja $xy^* = 0$. Potem je $\theta(1)$ obrnljiv v $Q^r(B)$ in obstaja injektiven $*$-homomorfizem $h : A \to Q^s(B)$, da velja $\theta(x) = \theta(1)h(x)$ za vse $x \in A$.*

Dodatna predpostavka, da θ ohranja ničle xy^* tudi v obratno smer, nam je omogočila, da smo se znebili predpostavke, da je kolobar A generiran z idempotenti. Ni težko preveriti, preslikava θ iz Izreka 1.15 zadošča identiteti

$$\theta(xy^*) = \theta(x)\theta(y)^* r \qquad \text{za vse } x, y \in A,$$

kjer je r inverz elementa $\theta(1)^*$ v $Q^l(B)$. To je še ena oblika, iz katere je razvidno, da taka preslikava tudi res ohranja ničle xy^* v obe smeri.

Izrek 1.15 torej karakterizira preslikave, ki ohranjajo ničle xy^* v obe smeri. Ostaja vprašanje, kaj lahko povemo o preslikavah na kolobarjih z netrivialnimi idempotenti, ki ohranjajo ničle xy^*. Naslednji rezultat opiše take preslikave v posebnem primeru, ko je $\theta(1)$ centralen element v B. V posebnem je to res, če privzamemo, da je tudi B enotski kolobar in velja $\theta(1) = 1$. Označimo z I_e ideal generiran z elementom e.

Trditev 1.16. *Naj bo A enotski kolobar z involucijo, ki vsebuje netrivialen idempotent e, in B prakolobar z involucijo. Naj bo $\theta : A \to B$ surjektivna aditivna preslikava, ki ohranja ničle xy^*, in denimo, da je $\theta(1) \in Z(B)$. Potem velja ena od naslednjih možnosti:*

(i) $\theta(I_e \cap I_{1-e} + I_{e^*} \cap I_{1-e^*}) = 0$,

(ii) $\theta(xy^*) = \lambda\theta(x)\theta(y)^*$ *za vse $x, y \in A$, kjer je $\lambda = 1/\theta(1)^* \in C(B)$.*

Če v Izreku 1.16 dodatno predpostavimo še, da je θ injektivna preslikava, potem pogoj (i) implicira $I_e \cap I_{1-e} = 0$. Ker je $I_e + I_{1-e} = A$, to pomeni, da je $A = I_e \oplus I_{1-e}$. Če je A prakolobar, se to ne more zgoditi in zato taka preslikava avtomatično zadošča pogoju (ii).

2 Nilkolobarji

Element a kolobarja R je *nilpotent*, če je $a^n = 0$ za neko naravno število n. Najmanjše naravno število n, za katerega je $a^n = 0$, imenujemo *indeks nilpotentnosti* elementa a ali kar *indeks* elementa a. Množico vseh nilpotentnih elementov kolobarja R označimo z $N(R)$.

Definicija 2.1. Kolobar R je *nilkolobar*, če je vsak njegov element nilpotent. V tem primeru pravimo tudi, da je kolobar R *nil*.

Podobno je ideal $I \triangleleft R$ *nilideal*, če je vsak njegov element nilpotent. V vsakem kolobarju R je vsota vseh nilidealov spet nilideal in je zato to največji nilideal kolobarja R.

Definicija 2.2. *Zgornji nilradikal* kolobarja R je največji nilideal kolobarja R in ga označimo z $Nil^*(R)$.

Kolobar R je *lokalno nilpotenten*, če je vsak končno generiran podkolobar $S \subseteq R$ nilpotenten, tj. $S^n = 0$ za neko naravno število n. Ideal $I \triangleleft R$ je *lokalno nilpotenten*, če je lokalno nilpotenten kot kolobar. Vsota vseh lokalno nilpotentnih idealov kolobarja R je spet lokalno nilpotenten ideal.

Definicija 2.3. *Levitzkijev radikal* kolobarja R (imenovan tudi *lokalno nilpotenten radikal*) je največji lokalno nilpotenten ideal kolobarja R in ga označimo z $L(R)$.

Definicija 2.4. *Spodnji nilradikal* kolobarja R (imenovan tudi *praradikal*) je presek vseh praidealov kolobarja R in ga označimo $Nil_*(R)$. V posebnem je $Nil_*(R) = R$, če kolobar R nima praidealov.

Za poljuben kolobar $(R, +, \cdot)$ definiramo operacijo $\circ$ on R s predpisom

$$a \circ b = a + b - ab.$$

Operaciji $\circ$ pravimo *kvazimnoženje*. Ni težko preveriti, da je $(R, \circ)$ monoid z enoto 0. Element $a \in R$ je *kvaziregularen*, če je obrnljiv v $(R, \circ)$, tj. če obstaja $a' \in R$, da velja $a' \circ a = a \circ a' = 0$. V tem primeru elementu a' pravimo *kvaziinverz* elementa a. Če ima kolobar R enoto, je slednje ekvivalentno temu, da je $1 - a$ obrnljiv v $(R, \cdot)$ z inverzom $1 - a'$. Množico vseh kvaziregularnih elementov kolobarja R označimo s $Q(R)$. $(Q(R), \circ)$ je očitno grupa, saj je to grupa obrnljivih elementov monoida $(R, \circ)$. Za vsak $a \in Q(R)$ in vsak $n \in \mathbb{Z}$ označimo z $a^{(n)}$ n-to potenco elementa a v grupi $(Q(R), \circ)$. V posebnem je $a^{(0)} = 0$ in $a^{(-1)}$ je kvaziinverz elementa a. Ideal $I \triangleleft R$ je *kvaziregularen*, če je $I \subseteq Q(R)$.

Definicija 2.5. *Jacobsonov radikal* kolobarja R je največji kvaziregularen ideal kolobarja R in ga označimo z $J(R)$.

Vsak nilpotenten element je kvaziregularen, namreč če je $x^n = 0$, potem je $-x - x^2 - \ldots - x^{n-1}$ kvaziinverz elementa x. Torej velja $N(R) \subseteq Q(R)$ in $Nil^*(R) \subseteq J(R)$. Izkaže se, da Jacobsonov radikal kolobarja R vsebuje celo vse enostranske kvaziregularne ideale kolobarja R.

Definicija 2.6. Kolobar R je *Jacobsonovo radikalen*, če je $J(R) = R$.

Jacobsonov radikal kolobarja R lahko opišemo tudi s pomočjo primitivnih idealov. Kolobar R je *primitiven*, če obstaja enostaven zvest levi R-modul. Ideal $I \lhd R$ je *primitiven*, če je R/I primitiven kolobar. Jacobsonov radikal $J(R)$ je enak preseku vseh primitivnih idealov kolobarja R. Izkaže se, da je vsak primitiven ideal praideal, vsak komutativen primitiven kolobar pa je obseg (glej [19]).

Omenimo še, da za vsak kolobar R velja

$$Nil_*(R) \subseteq L(R) \subseteq Nil^*(R) \subseteq J(R). \tag{2.3}$$

V splošnem je vsaka od teh vsebovanosti lahko stroga. Primere kolobarjev, ki to pokažejo, lahko bralec najde v [13] in [19].

Definicija 2.7. Kolobar R je *nil omejenega indeksa* $\leq n$, če je vsak element v R nilpotent indeksa $\leq n$. Kolobar R je *nil omejenega indeksa*, če obstaja naravno število n, da je R nil omejenega indeksa $\leq n$.

Naj bo K komutativen enotski kolobar in R poljubna K-algebra. Element $a \in R$ je *algebraičen* nad K, če obstaja neničeln polinom $p \in K[x]$, za katerega velja $p(0) = 0$ in $p(a) = 0$. Če obstaja polinom p s temi lastnostmi, ki je hkrati moničen (tj. vodilni koeficient je enak 1), potem pravimo, da je element a *celosten* nad K. Množico vseh algebraičnih elementov algebre R označimo z $A_K(R)$, množico vseh celostnih elementov pa z $I_K(R)$. K-algebra R je *algebraična (celostna)* nad K, če je vsak element v R algebraičen (celosten) nad K. V primeru, ko je $K = \mathbb{Z}$, bomo pisali tudi $A(R) = A_{\mathbb{Z}}(R)$ in $I(R) = I_{\mathbb{Z}}(R)$. Seveda je vsak nilpotenten element tudi celosten, torej velja $N(R) \subseteq I_K(R) \subseteq A_K(R)$.

Naj bo a celosten element algebre R. Najmanjše naravno število n, za katerega obstaja moničen polinom p stopnje n z lastnostmi $p(0) = 0$ in $p(a) = 0$, imenujemo *stopnja celostnosti* elementa a ali kar *stopnja* elementa a. Podobno kot nilkolobar omejenega indeksa definiramo tudi *celostno algebro omejene stopnje.*

Eden najpomembnejših problemov s področja nilkolobarjev je Köthejeva domneva. Leta 1930 je Köthe [16] domneval naslednje.

Köthejeva domneva 2.8. *Kolobar, ki nima neničelnih nilidealov, nima niti neničelnih nil enostranskih idealov.*

Köthejeva domneva je pomembna, ker bi iz nje sledilo, da zgornji nilradikal $Nil^*(R)$ kolobarja R vsebuje celo vse nil enostranske ideale kolobarja R. Čeprav je vprašanje o resničnosti Köthejeve domneve še vedno odprto, je bil od leta 1930 narejen precejšnji napredek na tem področju. Znanih je veliko razredov kolobarjev, ki zadoščajo Köthejevi domnevi. Med take kolobarje spadajo komutativni kolobarji ter splošneje kolobarji s polinomsko identiteto (Levitzki [20]), levo artinski kolobarji in celo levo noetherski kolobarji (Levitzki). Poleg tega so take tudi algebraične (in v posebnem končno dimenzionalne) algebre nad komutativnimi obsegi ([35, str. 144]) ter monomske algebre (Beidar in Fong [5]). Morda najbolj pomemben primer algeber, ki zadoščajo Köthejevi domnevi, so algebre nad neštevnimi komutativnimi obsegi (Amitsur [1], [2]). Vsi ti primeri, še posebej zadnji, nakazujejo, da bi Köthejeva domneva lahko bila resnična. Toda obstaja tudi nekaj bolj nedavnih rezultatov, ki namigujejo, da bi bilo morda mogoče najti protiprimer k domnevi.

Znanih je veliko izjav, ki so ekvivalentne Köthejevi domnevi. Te izjave bomo formulirali kot domneve, torej vsaka domneva v nadaljevanju bo ekvivalentna Köthejevi domnevi. Začnimo z dvema bolj osnovnima.

Domneva 2.9. *Vsak nil enostranski ideal poljubnega kolobarja R je vsebovan v $Nil^*(R)$.*

Domneva 2.10. *Vsota dveh nil levih idealov poljubnega kolobarja je nil.*

Izkaže se, da ima Köthejeva domneva veliko opraviti s problemom opisa Jacobsonovega radikala polinomskih kolobarjev. Najpomembnejši rezultat, ki govori o tem problemu, je naslednji Amitsurjev izrek [2, Theorem 1].

Izrek 2.11. *Za vsak kolobar R je $J(R[x]) = N[x]$, kjer je $N = J(R[x]) \cap R$ nilideal v R.*

Krempa [17, Theorem 1] je s pomočjo matričnih kolobarjev karakteriziral kdaj je kolobar $R[x]$ Jacobsonovo radikalen.

Izrek 2.12. *Za kolobar R je polinomski kolobar $R[x]$ Jacobsonovo radikalen natanko tedaj, ko je $M_n(R)$ nilkolobar za vsako naravno število n.*

S pomočjo Izreka 2.12 je Krempa [17] dokazal, da so naslednje izjave ekvivalentne Köthejevi domnevi. Prvi dve je neodvisno odkril tudi Sands [25].

Domneva 2.13. *Za vsak nilkolobar R je tudi $M_2(R)$ nilkolobar.*

Domneva 2.14. *Za vsak nilkolobar R je tudi $M_n(R)$ nilkolobar za vsako naravno število n.*

Domneva 2.15. *Za vsak nilkolobar R je kolobar $R[x]$ Jacobsonovo radikalen.*

Opomba 2.16. Domneva 2.15 poveže Köthejevo domnevo s polinomskimi kolobarji. Natančneje, če bi bila domneva resnična, potem bi za vsak kolobar R veljalo $J(R[x]) = Nil^*(R)[x]$ (primerjaj z Izrekom 2.11). S tem bi torej dobili popoln opis Jacobsonovega radikala kolobarja polinomov $R[x]$ s pomočjo baznega kolobarja R. Poleg tega iz [13, Proposition 4.9.1] sledi, da je zgornji nilradikal kolobarja $M_n(R)$ oblike $M_n(I)$ za nek ideal $I \triangleleft R$. Torej, če bi bila Domneva 2.14 resnična, potem bi za vsak kolobar R veljalo $Nil^*(M_n(R)) = M_n(Nil^*(R))$ za vsako naravno število n.

Leta 2005 je Smoktunowicz [27] podala naslednjo veliko šibkejšo izjavo od Domneve 2.15, ki je še vedno ekvivalentna Köthejevi domnevi.

Domneva 2.17. *Za vsak nilkolobar R kolobar $R[x]$ ni primitiven.*

Naslednja izjava ima bolj grupno teoretičen priokus. Njeno ekvivalentnost s Köthejevo domnevo sta dokazala Fisher in Krempa [12]. Za podgrupo G grupe avtomorfizmov kolobarja R označimo z R^G *podkolobar fiksnih točk* pri delovanju G na R, tj. $R^G = \{r \in R \ ; \ g(r) = r \text{ za vse } g \in G\}$. Element $r \in R$ je *aditivna* $|G|$*-torzija*, če je $r \neq 0$ in $|G|r = 0$.

Domneva 2.18. *Naj bo R kolobar in G končna podgrupa grupe avtomorfizmov kolobarja R, tako da R nima aditivne $|G|$-torzije. Če je R^G nilkolobar, potem je tudi R nilkolobar.*

Več informacij o ozadju te izjave lahko bralec najde v [12]. Naš naslednji rezultat podaja še eno izjavo ekvivalentno Köthejevi domnevi, ki je kombinacija prejšnjih.

Domneva 2.19. *Naj bo R kolobar, za katerega je $R[x]$ primitiven kolobar, in $G \neq 1$ končna podgrupa grupe avtomorfizmov kolobarja R. Če je R^G nilkolobar, potem ima R aditivno $|G|$-torzijo.*

Krempa [17] je dokazal, da je dovolj obravnavati Köthejevo domnevo v razredu algeber nad komutativnimi obsegi.

Domneva 2.20. *Vsak nil enostranski ideal poljubne algebre R nad poljubnim komutativnim obsegom F je vsebovan v $Nil^*(R)$.*

Z naslednjo izjavo dokažemo, da se lahko omejimo celo na praalgebre.

Domneva 2.21. *Vsak nil enostranski ideal poljubne praalgebre R nad poljubnim komutativnim obsegom F je vsebovan v $Nil^*(R)$.*

Amitsur [1], [2] je dokazal, da vse algebre nad neštevnimi komutativnimi obsegi zadoščajo Köthejevi domnevi, zato se lahko še dodatno omejimo na algebre nad števnimi komutativnimi obsegi.

Opomba 2.22. Domneva 2.21 je verzija Domneve 2.9 za praalgebre nad komutativnimi obsegi. Posledično lahko pokažemo, da so tudi verzije Domnev 2.10, 2.13, 2.14 in 2.15 za praalgebre nad komutativnimi obsegi ekvivalentne Köthejevi domnevi.

Več informacij o Köthejevi domnevi in sorodnih problemih lahko bralec najde v [29] in [30].

Sodeč po Opombi 2.16 je naravno poskušati poiskati alternativen opis Jacobsonovega radikala polinomskih kolobarjev. Naslednji izrek, ki je del splošnejše teorije radikalov polinomskih kolobarjev (glej [13, §4.9]), podaja en tak opis.

Izrek 2.23. *Za poljuben kolobar R velja $J(R[x]) = N[x]$, kjer je*

$$N = \bigcap \{P \triangleleft R \ ; \ P \text{ praideal in } J((R/P)[x]) = 0\}.$$

V naslednji trditvi podamo še eno zanimivo lastnost Jacobsonovega radikala polinomskih kolobarjev, ki za splošnejše kolobarje ne velja.

Trditev 2.24. *Za vsak podkolobar S kolobarja R velja $J(R[x]) \cap S[x] \subseteq J(S[x])$.*

Kot posledico dobimo naslednje.

Posledica 2.25. *Jacobsonov radikal kolobarja polinomov nad poljubnim kolobarjem je unija Jacobsonovo radikalnih kolobarjev polinomov nad končno generiranimi kolobarji.*

Pri obravnavi resničnosti Domneve 2.15 se lahko v resnici omejimo na končno generirane kolobarje oziroma algebre.

Trditev 2.26. *Köthejeva domneva je ekvivalentna izjavi 'za vsak števen komutativen obseg F in vsako končno generirano nil praalgebro R nad F je algebra $R[x]$ Jacobsonovo radikalna'.*

V nadaljevanju nas bodo zanimale povezave med pojmi nilpotentnost, algebraičnost in kvaziregularnost. Naša motivacija izhaja iz naslednjih dveh vprašanj:

Q1. Algebraične kolobarje in algebre ponavadi štejemo med kolobarje in algebre, ki imajo lepe lastnosti. Na primer, algebraična algebra nad komutativnim obsegom, ki je brez deliteljev niča, je obseg. Po drugi stani pa nilkolobarje in nilalgebre, ki so seveda tudi algebraični, štejemo med kolobarje in algebre, s katerimi je težko delati. Zato se naravno pojavi vprašanje, zakaj so nilkolobarji in nilalgebre slabi med vsemi algebraičnimi.

Odgovor za algebre nad komutativnimi obsegi je dobro znan, namreč ker so Jacobsonovo radikalne. Ta rezultat bomo posplošili na algebre nad določenimi glavnimi kolobarji in v posebnem na kolobarje.

Q2. Ali lahko nilpotentne elemente med vsemi kvaziregularnimi elementi karakteriziramo z lastnostjo 'kvaziinverz elementa a je polinom v a'?

Po elementih to dokaj očitno ne bo mogoče,lahko pa bomo na ta način karakterizirali zgornji nilradikal. Večina teh rezultatov je vsebovanih v [32].

Od tu dalje bomo s K vedno označevali enotski komutativen kolobar, z F komutativen obseg, z R pa algebro nad K ali F. Zgornji dve vprašanji motivirata naslednjo definicijo, ki go igrala glavno vlogo v naši obravnavi.

Definicija 2.27. Element a K-algebre R je *π-algebraičen* (nad K), če obstaja polinom $p \in K[x]$, za katerega velja $p(0) = 0$, $p(1) = 1$ in $p(a) = 0$. V tem primeru pravimo tudi, da je element *a π-algebraičen s polinomom p*. Podmnožica $S \subseteq R$ je *π-algebraična*, če je vsak njen element π-algebraičen. Množico vseh π-algebraičnih elementov K-algebre R označimo s $\pi_K(R)$.

V posebnem, ko je R le kolobar in $K = \mathbb{Z}$, bomo pisali tudi $\pi(R) = \pi_{\mathbb{Z}}(R)$. Naslednja lema podaja zvezo med nilpotentnimi, π-algebraičnimi in kvaziregularnimi elementi.

Lema 2.28. *Za vsako K-algebro R velja $N(R) \subseteq \pi_K(R) \subseteq A_K(R) \cap Q(R)$. Za vsako F-algebro R velja $N(R) \subseteq \pi_F(R) = A_F(R) \cap Q(R)$. Kvaziinverz π-algebraičnega elementa je polinom v tem elementu.*

Oglejmo si nekaj primerov.

Zgled 2.29. Za vsak končen kolobar R velja $\pi(R) = Q(R)$ in $J(R) = Nil^*(R)$.

Zgled 2.30. Za vsak komutativen obseg F velja $\pi_F(F) = F \backslash \{1\} = Q(F)$. V posebnem je $\pi_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q} \backslash \{1\} = Q(\mathbb{Q})$, po drugi strani pa velja $\pi(\mathbb{Q}) = \{1 + \frac{1}{n} \,;\, n \in \mathbb{Z} \backslash \{0\}\}$.

Zgled 2.31. Naj bosta $F \subseteq E$ komutativna obsega in $M_n(E)$ algebra $n \times n$ matrik nad E. Potem velja

$$\begin{aligned} N(M_n(E)) &= \text{matrike z lastnimi vrednostmi } 0, \\ \pi_F(M_n(E)) &= \text{matrike z lastnimi vrednostmi v } \overline{F} \backslash \{1\}, \\ Q(M_n(E)) &= \text{matrike z lastnimi vrednostmi v } \overline{E} \backslash \{1\}, \end{aligned}$$

kjer $\overline{F} \subseteq \overline{E}$ označujeta algebraični zaprtji obsegov F in E.

Naslednja trditev podaja natančno zvezo med π-algebraičnimi in celostnimi elementi.

Trditev 2.32. *Naj bo a element K-algebre R. Naslednje trditve so ekvivalentne:*

(i) *a je π-algebraičen,*

(ii) *a je kvaziregularen in $a^{(-1)}$ je celosten,*

(iii) *a je kvaziregularen in $a^{(-1)}$ je polinom v a.*

Trditev 2.32 med drugim pove, da velja $\pi_K(R) = (Q(R) \cap I_K(R))^{(-1)}$ (primerjaj z Lemo 2.28). Videli bomo, da obstaja tudi tesna zveza med π-algebraičnimi in nilpotentnimi elementi (glej Trditev 2.36), čim kolobar K zadošča določeni lastnosti podani z naslednjo definicijo.

Definicija 2.33. Pravimo, da je glavni kolobar K *poseben*, če ne obstaja nekonstanten polinom $p \in K[x]$ s $p(0) \neq 0$, za katerega bi bil $p(k)$ obrnljiv v K za vse $k \in K$, ki so tuji $p(0)$.

Vsak poseben glavni kolobar zadošča naslednjemu pogoju.

Trditev 2.34. *Za vsak poseben glavni kolobar K velja $J(K) = 0$. V posebnem, če je K poseben glavni kolobar, ki ni obseg, potem ima K neskončno mnogo neasociiranih nerazcepnih elementov.*

Tukaj je nekaj primerov posebnih glavnih kolobarjev.

Trditev 2.35.

(i) *Komutativen obseg je poseben glavni kolobar natanko tedaj, ko je algebraično zaprt.*

(ii) *Kolobar celih števil $\mathbb{Z}$ in kolobar Gaussovih števil $\mathbb{Z}[i]$ sta posebna glavna kolobarja.*

(iii) *Za vsak komutativen obseg F je kolobar polinomov $F[x]$ poseben glavni kolobar.*

Naslednja trditev podaja zvezo med π-algebraičnimi in nilpotentnimi elementi, ki je ključnega pomena.

Trditev 2.36. *Naj bo K poseben glavni kolobar in R poljubna K-algebra. Če za element $a \in R$ velja $Ka \subseteq \pi_K(R)$, potem obstaja $0 \neq k \in K$, da je ka nilpotent. V posebnem, če R nima K-torzije, potem je a nilpotent.*

Zaključki Trditve 2.36 ne veljajo nujno, če K ni poseben glavni kolobar. Ne veljajo na primer, če je K komutativen obseg, ki ni algebraično zaprt.

Zgled 2.37. Naj bo F komutativen obseg, ki ni algebraično zaprt in $p \in F[x]$ nekonstanten polinom, ki nima ničel v F. Naj bo a neka ničla polinoma p v algebraičnem zaprtju obsega F. Potem je za vsak $0 \neq \lambda \in F$ element λa π-algebraičen s polinomom $p(\lambda^{-1})^{-1}p(\lambda^{-1}x)x$, toda ne obstaja $0 \neq \lambda \in F$, za katerega bi bil λa nilpotent.

S pomočjo Trditve 2.36 dokažemo naslednja dva izreka, ki odgovorita na vprašanje Q1 na dva različna načina.

Izrek 2.38. *Če je K poseben glavni kolobar, potem je vsaka π-algebraična K-algebra nilalgebra.*

Morda je zanimivo, da ima ta razmeroma velika družina polinomov z vsoto koeficientov 1 enak učinek kot razmeroma majhna družina polinomov $\{x, x^2, x^3, x^4, \ldots\}$.

Izrek 2.39. *Če je K poseben glavni kolobar, potem je vsaka celostna Jacobsonovo radikalna K-algebra nilalgebra.*

Pogoj celostnosti v Izreku 2.39 je nujen, saj algebraična Jacobsonovo radikalna K-algebra ni nujno nilalgebra. Tako je na primer kolobar $R = \{\frac{2m}{2n-1}\ ;\ m, n \in \mathbb{Z}\}$ kot podkolobar racionalnih števil algebraičen nad $\mathbb{Z}$ in Jacobsonovo radikalen, toda ni nilkolobar.

Definicija 2.40. K-algebra R je *π-algebraična omejene stopnje* $\leq n$, če je vsak element v R π-algebraičen s polinomom stopnje $\leq n$. K-algebra R je *π-algebraična omejene stopnje*, če obstaja naravno število n, da je R π-algebraična omejene stopnje $\leq n$.

Postavi se naslednje naravno vprašanje. Če je K-algebra R π-algebraična omejene stopnje, ali je potem tudi nil omejenega indeksa? Delen odgovor na to vprašanje podaja naslednja posledica.

Posledica 2.41. *Naj bo R π-algebraična F-algebra omejene stopnje $\leq n$, kjer je F komutativen obseg, ali π-algebraična K-algebra omejene stopnje $\leq n$ brez K-torzije, kjer je K poseben glavni kolobar. Potem je R nilalgebra omejenega indeksa $\leq n$.*

Odgovor za splošne K-algebre je morda presenetljivo negativen, kot pokaže naslednji primer.

Zgled 2.42. Naj bo K poseben glavni kolobar, ki ni komutativen obseg. Po Trditvi 2.34 ima K neskončno mnogo neasociiranih nerazcepnih elementov. Izberimo števno množico neasociiranih nerazcepnih elementov $\{p_1, p_2, p_3, \ldots\}$ in postavimo $R = \bigoplus_{i=1}^{\infty} p_i K / p_i^i K$. K-algebra R je očitno nilalgebra, ki pa ni omejenega indeksa. Naj bo $a = (a_i)_i$ poljuben element algebre R. Po Kitajskem izreku o ostankih obstaja element $k \in K$, za katerega velja $k \equiv a_i \pmod{p_i^i}$ za vse i, za katere je $a_i \neq 0$. Torej je element a celosten, saj je ničla polinoma $x^2 - kx$. To pomeni, da je algebra R celostna omejene stopnje ≤ 2 in posledično tudi π-algebraična omejene stopnje ≤ 2.

Kljub temu za poljubno algebro nad posebnim glavnim kolobarjem velja nekoliko šibkejši zaključek.

Trditev 2.43. *Naj bo K komutativen obseg ali poseben glavni kolobar. Potem za vsako π-algebraično K-algebro R omejene stopnje velja $Nil_*(R) = R$.*

Naslednja posledica odgovori na vprašanje Q2.

Posledica 2.44. *Naj bo R poljubna K-algebra, kjer je K komutativen obseg ali poseben glavni kolobar. Potem velja naslednje:*

(i) *$Nil^*(R)$ je največji π-algebraičen ideal v R,*

(ii) *$Nil^*(R)$ je največji celosten kvaziregularen ideal v R,*

(iii) *$Nil^*(R)$ je največji kvaziregularen ideal v R, za katerega je kvaziinverz vsakega elementa polinom v tem elementu.*

Posledica 2.44 nam da še naslednje.

Posledica 2.45. *Za vsak celosten kolobar R velja $J(R) = Nil^*(R)$.*

Posledica 2.46. *Vsak celosten kolobar zadošča Köthejevi domnevi.*

Za konec bomo raziskali strukturo množice vseh π-algebraičnih elementov algebre. Omejili se bomo na algebre nad komutativnimi obsegi in na kolobarje.

Ker je $\pi_K(R)$ podmnožica v $Q(R)$, se je naravno vprašati ali je $\pi_K(R)$ podgrupa (edinka) v grupi $(Q(R), \circ)$. Izkaže se, da je $\pi_K(R)$ vedno zaprta za konjugiranje. V splošnem $\pi_K(R)$ ni zaprta za operacijo $\circ$, toda če je R komutativna, potem je $\pi_K(R)$ zaprta za $\circ$. Če je $K = F$ komutativen obseg, potem je $\pi_F(R)$ zaprta tudi za invertiranje. Skratka, če je R komutativna algebra nad komutativnim obsegom F, potem je $\pi_F(R)$ podgrupa v $Q(R)$. Za kolobar R množica $\pi(R)$ ni nujno zaprta za invertiranje, vemo namreč že, da velja $\pi(R)^{(-1)} = I(R) \cap Q(R)$ (glej Trditev 2.32).

Poglejmo si, kaj lahko povemo o seštevanju. Naslednji izrek podaja tesno povezavo med operacijami seštevanja, množenja in kvazimnoženja.

Izrek 2.47. *Naj bo R kolobar. Za vsako podgrupo S grupe $Q(R)$ so naslednje trditve ekvivalentne:*

(i) *S je zaprta za seštevanje,*

(ii) *S je zaprta za množenje,*

(iii) *S je podkolobar v R.*

Izkaže se, da lahko s pomočjo kvazimnoženja množenje dejansko izrazimo s seštevanjem in obratno. Za vsaka $x, y \in Q(R)$ namreč veljata formuli

$$xy = x \circ (x^{(-1)} + y^{(-1)}) \circ y \qquad \text{in} \qquad x + y = x \circ (x^{(-1)} y^{(-1)}) \circ y.$$

Izrek 2.47 nam pomaga dokazati naslednje posledice.

Posledica 2.48. *Naj bo F komutativen obseg karakteristike 0 in R komutativna F-algebra. Če je $\pi_F(R)$ zaprta za seštevanje, potem je $\pi_F(R) = N(R)$.*

Trditev 2.49. *Naj bo R komutativen kolobar. Če je $\pi(R)$ zaprta za seštevanje, potem je $\pi(R) = N(R)$.*

Posledica 2.50. *Naj bo p praštevilo, F algebraična razširitev obsega $\mathbb{Z}/p\mathbb{Z}$ in R komutativna F-algebra. Če je $\pi_F(R)$ zaprta za seštevanje, potem je $\pi_F(R) = N(R)$.*

Vprašanje ali Posledica 2.50 velja za poljubne komutativne obsege praštevilske karakteristike ostaja odprto. Opazimo lahko, da veljajo tudi obrati teh trditev, saj je v komutativnem množica $N(R)$ celo ideal v R.

Zgornje posledice obravnavajo ekstremen primer, ko v inkluziji $N(R) \subseteq \pi(R)$ velja enakost. Druga ekstremna situacija nastopi, ko je v algebri veliko π-algebraičnih elementov a malo nilpotentov. Na primer v obsegu, ki je algebraičen nad svojim centrom, je vsak element razen enote π-algebraičen, neničelnih nilpotentov pa ni. Zanima nas, kdaj se nekaj podobnega zgodi v splošnih algebrah. Vprašanje, ki si ga zastavimo je, kdaj bo $\pi_F(R) \cup \{1\}$ obseg, če je R enotska F-algebra. Nanj nam pomagata odgovoriti naslednja izreka.

Izrek 2.51. *Naj bo R enotski kolobar karakteristike 0. Za vsako podgrupo S grupe $Q(R)$, za katero velja $\{0, 2\} \subsetneq S$, so naslednje trditve ekvivalentne:*

(i) *$S \cup \mathbb{Z}$ zaprta za seštevanje,*

(ii) *$S \cup \mathbb{Z}$ je podobseg v R,*

(iii) *$S \cup \{1\}$ je podobseg v R.*

Izrek 2.52. *Naj bo R enotski kolobar praštevilske karakteristike p. Za vsako podgrupo S grupe $Q(R)$ so naslednje trditve ekvivalentne:*

(i) *$S \cup \mathbb{Z}/p\mathbb{Z}$ je zaprta za seštevanje,*

(ii) *$S \cup \mathbb{Z}/p\mathbb{Z}$ je podobseg v R.*

S pomočjo teh dveh izrekov dokažemo naslednji posledici.

Posledica 2.53. *Naj bo F komutativen obseg in R enotska komutativna F-algebra. Če je $\pi_F(R) \cup \{1\}$ zaprta za seštevanje, potem je podobseg v R.*

Posledica 2.54. *Naj bo R enotski komutativen kolobar ničelne ali praštevilske karakteristike, za katerega velja $\pi(R) \neq \{0, 2\}$. Če je $\pi(R) \circ \pi(R)^{(-1)} \cup (\mathbb{Z} \cdot 1)$ zaprta za seštevanje, potem je podobseg v R.*