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# Classification of knots in lens spaces

**Doctoral thesis**

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# Klasifikacija vozlov v lečastih prostorih

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## Abstract

So far knots have been classified up to a certain number of crossings only for a handful of spaces: the 3-dimensional Euclidean space, the projective space, and the solid torus, the latter being classified only up to a so-called flip. In this thesis we append the infinite family of lens spaces to this modest list. As a side product, we refine the case of the solid torus by providing a complete classification of knots in it. In both cases we classify knots up to four crossings and up to five crossings with a few exceptions. We also establish which of the knots in the solid torus are amphichiral. We will see that for each lens space, a subset of prime knots in the solid torus gives the classification in the lens space. Since there are very few applicable invariants of links in $L(p,q)$, a necessary condition for making a classification in these spaces is to develop invariants of links in $L(p,q)$.

The first invariant we introduce is the HOMFLYPT skein module. The HOMFLYPT skein module has so far only been calculated only for $S^3$ and the solid torus. We show that the HOMFLYPT skein module of $L(p,1)$ is a free $R$-module and we present a basis of this module for each $p > 1$.

The second invariant is the Khovanov homology of the Kauffman bracket skein module of $\mathbb{R}\mathrm{P}^3$. Khovanov homology, an invariant of links in $\mathbb{R}^3$, is a graded homology theory that categorifies the Jones polynomial in the sense that the graded Euler characteristic of the homology is the Jones polynomial. Asaeda, Przytycki, and Sikora generalized this construction by defining a double graded homology theory that categorifies the Kauffman bracket skein module of links in $I$-bundles over surfaces, except for the surface $\mathbb{R}\mathrm{P}^2$, where the construction fails due to the strange behavior of links when projected to the non-orientable $\mathbb{R}\mathrm{P}^2$. We categorify the missing case of the twisted $I$-bundle over $\mathbb{R}\mathrm{P}^2$, $\mathbb{R}\mathrm{P}^2\widetilde{\times}I \approx \mathbb{R}\mathrm{P}^3 \setminus \{*\}$, by redefining the differential in the Khovanov chain complex in a suitable manner.

The classification, the calculations of the HOMFLYPT skein modules of the knots, and the calculations of the Kauffman bracket skein modules of the knots are done by a computer program that is available online at [10].



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Keywords: knot, classification, lens space, solid torus, skein module, Kauffman bracket, HOMFLYPT, Khovanov homology, categorification.

# Povzetek

Vozli so bili do sedaj klasificirani le za peščico prostorov: 3-dimenzionalni evklidski prostor, projektivni prostor in poln torus, kjer so bili vozli v slednjem klasificirani le do tako imenovanega obrata torusa. S to disertacijo smo na ta skromni seznam dodali tudi lečasti prostor $L(p,q)$. Kot stranski produkt te klasifikacije bomo vozle v polnem torusu popolnoma klasificirali, poleg tega pa bomo ugotovili tudi, kateri od teh vozlov so akiralni. V obeh primerih klasificiramo vozle do štirih križišč in do petih križišč z nekaj izjemami. Videli bomo, da za vsak lečasti prostor obstaja podmnožica pravozlov v polnem torusu, ki bo predstavljala klasifikacijo vozlov v tem lečastem prostoru. Da lahko uspešno klasificiramo vozle v lečastih prostorih, moramo pred tem izdelati dovolj močne invariante vozlov.

Kot prvo invarianto vpeljemo HOMFLYPT premenjalni modul. Do sedaj je bil HOMFLYPT premenjalni modul izračunan samo za prostora $S^3$ in polni torus $S^1 \times D$. Dokazali bomo, da je HOMFLYPT premenjalni modul prostora $L(p,1)$ prost, pri tem bomo predstavili bazo za vsak $p > 1$.

Druga invarianta je homologija Hovanova Kauffmanovega oklepajskega premenjalnega modula prostora $\mathbb{R}\mathrm{P}^3$. Homologija Hovanova je stopničasta homološka teorija, ki kategorificira Jonesov polinom v smislu, da je Eulerjeva karakteristika homologije enaka Jonesovimu polinomu. Asaeda, Przytycki in Sikora so to homološko teorijo posplošili tako, da so uvedli dvojnostopničasto homološko teorijo, ki kategorificira Kauffmanov oklepajski premenjalni modul $I$-svežnjev nad ploskvami. Zaradi nenavadnega obnašanja spletov, ki jih projiciramo na neorientabilno ploskev $\mathbb{R}\mathrm{P}^2$, teorija odpove pri zvitem $I$-svežnju $\mathbb{R}\mathrm{P}^2 \widetilde{\times} I \approx \mathbb{R}\mathrm{P}^3 \setminus \{*\}$. Pokažemo, da je diferencial v Hovanovem verižnem kompleksu mogoče ustrezno popraviti, da teorija deluje tudi za projektivni prostor.

Za klasifikacijo, za izračun HOMFLYPT premenjalnih modulov vozlov in za izračun Kauffmanovih oklepajskih premenjalnih modulov vozlov, smo napisali računalniški program, ki je dostopen preko spleta [10].

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Kjučne besede: vozel, klasifikacija, lečast prostor, poln torus, premenjalni modul, Kauffmanov oklepaj, homologija Hovanova, kategorifikacija.

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# Preface

Knot theory plays an important role in the theory of 3-dimensional manifolds. Although it is considered as a part of geometric topology, it not only appears in many fields of mathematics, but also in fields such as physics, chemistry and biology.

While knots have been studied in various aspects throughout the whole human history, the theory's most intriguing results have been obtained over the last three decades. The importance of the theory can perhaps be demonstrated by the fact that four mathematicians have already received Fields medals for their results in this theory: V. Jones, E. Witten, V. G. Drinfeld, and M. L. Kontsevich.

In this thesis, we study the theory of knots from the perspective of geometric topology with the chapters grouped into three parts:

In Chapter 2 and Chapter 3 we overview the existing theory of knots and links in 3-manifolds.

In Chapter 4 and Chapter 5 we develop new invariants of lens spaces, namely, the HOMFLYPT skein module of lens spaces $L(p,1)$ (and conjecture about $L(p,q)$, $q > 1$) [14] and Khovanov homology of the Kauffman bracket skein module of $\mathbb{R}\mathrm{P}^3 \approx L(2,1)$ [11].

Chapter 6 is devoted to the classification of knots in the solid torus and in $L(p,q)$. As a product we produce knot tables that are presented in the appendices.

Section 3.6 and Chapter 4 is the result of the joint work with the author's coadviser dr. Maciej Mroczkowski [13, 14]. Chapter 6 is the result of the joint work with the author's coadviser (classification of knots in the solid torus up to a flip) [12] and was generalized by the author for the purpose of this paper (classification of knots in $L(p,q)$). The computer algorithm presented in Chapter 6 was primarily written by the coadviser and the author, for the purpose of calculating the Kauffman bracket skein module of knots and the classification of knots in the solid torus (up to flips) [12], and was rewritten by the author for the calculations of the HOMFLYPT skein modules of links and the classification of links in the solid torus and $L(p,q)$.

# 1

# Introduction

From the mathematical perspective, knots were first mentioned in a 1771 paper by A. T. Vandermonde [56], where braids and knots are specifically placed as a subject of the geometry of position.

The first notable knot invariant, the Gauss linking integral, was introduced by C. F. Gauss in 1833. Although the construction was inspired by astronomy, the result is nowadays known as the linking number. It was not until 1885 when P. G. Tait developed the first knot table [52]. The motivation behind such a classification was Lord Kelvin's (apparently mistaken) theory that atoms are knots in the aether.

Following the advancement of topology, knot theory became a widespread field of study at the beginning of the 20th century with early pioneers of modern knot theory being K. Reidemeister, J. W. Alexander, M. Dehn, and R. Fox, to name a few. In 1984 V. Jones furthermore popularized the subject with the discovery of the Jones polynomial [28], which led to the discovery of other polynomials, such as the HOMFLYPT polynomial and various Kauffman polynomials.

Another major breakthrough in the study of knots appeared in the late 1990s by the seminal work of M. Khovanov, who managed to construct a homological theory that generalizes the Jones polynomial. Similarly, in 2002 P. Ozsváth and Z. Szabó generalized the Alexander polynomial by introducing knot Floer homology [40].

A systematic study of knots in spaces other than the 3-dimensional Euclidean space started in 1987 when Przytycki and Turaev introduced the study of skein modules [44, 55] and continued in 1991 with Yu. V. Drobotukhina's classification of knots in the projective space [17]. Since then, a considerable part of contemporary knot theory has been devoted to the study of knots in various 3-dimensional spaces [46].

# 2

# An overview of knot theory in $\mathbb{R}^3$

## 2.1 Basic definitions

Standard terminology, as well as the definitions of the polynomials in the next section can be found in [1, 7, 34].

Knot theory is inspired by what we perceive as a "knot" in our daily life: an entangled piece of string; although the theory is for the most part interested in "knots" that are connected, i.e. joined together at the ends.

A (mathematical) **knot** is an embedding of the circle $S^1$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ or in the 3-sphere $S^3$. A **link** is a embedding of a disjoint union of $n$ circles into $\mathbb{R}^3$ ($S^3$).

Studying the embeddings of circles has little in common of the intuitive idea of what a knot should be. The topological object that we will be interested in, is a class of embeddings which more naturally corresponds to the ideas portrayed above. We first set up some definitions.

Two embeddings $f_0, f_1 : X \to Y$ are **isotopic** if they can be connected through embeddings, more precisely, if there exists a homotopy

$$H : X \times I \longrightarrow Y,$$

such that $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$ and for each $t \in I$, $H(-,t)$ is an embedding.

We see that isotopy still does not apply to our situation, since any two knots can be shown to be isotopic: any "knotted" areas of a knot can be continuously contracted to a point (see Figure 2.1) [7].

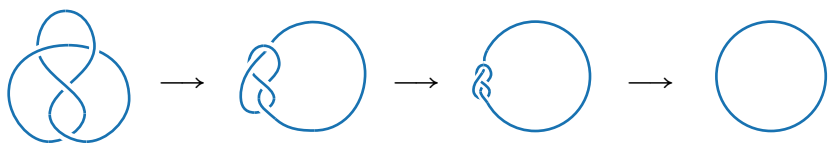


Figure 2.1: Contracting a "knotted" area to a point.

What we wish to achieve is, not only to isotope the embedding, but carry the whole ambient space along with us. This is achieved through ambient isotopy.

Two embeddings $f_0, f_1 : X \to Y$ are **ambient isotopic**, if there exists a homeomorphism (or diffeomorphism in the smooth category) $H : Y \to Y$ which is isotopic to the identity map $\mathrm{Id}_Y$, with the property that $f_1 = H \circ f_0$.

In the smooth category, ambient isotopy and isotopy coincide [22]. Rather than carrying the adjective "ambient" throughout the text, we will, from now on, work only in the category of smooth manifolds. This also eliminates any pathological behavior of knots, such as the existence of **wild knots** (Figure 2.2).

Figure 2.2: An example of a wild knot.

An isotopy forms an equivalence relation on the set of knots and we will often use the word "knot" when referring to a knot's isotopy class. A **knot invariant** is a function on the knot that is invariant under isotopy.

To have a working theory of knots, we must introduce suitable knot diagrams.

Let $p$ be a projection of the knot $K$ to a plane $\Sigma \subset \mathbb{R}^3$. We require $p$ to satisfy the following properties (see Figure 2.3):

- no three points project to the same point,
- all double points meet transversely,
- no cusps occur.

(a) triple point (b) self tangency (c) a cusp

Figure 2.3: Forbidden projections in a regular diagram.

A projection satisfying above conditions is called a **regular projection**. Throughout this thesis we will consider every projection to be regular.

Double points in a projection are called **crossings**. The edges of a knot's projection are called **arcs** and faces are called **regions**. The **shadow of a knot** is the image of the (regular) projection of a knot (Figure 2.4(a)), a **knot diagram** is the image containing the additional information whether the crossing is an over- or undercrossing (Figure 2.4(b)). A knot can be reconstructed from its diagram up to isotopy.

A **tangle** is the embedding of a disjoint union of arcs and circles into the 3-ball $I^3$, where the embedding sends the endpoints of the arcs to the boundary of $I^3$. A $n$-**tangle** is a tangle consisting of $n$ arcs (and possibly some circles) where $n$ endpoints lie in $\{0\} \times I^2$ and $n$ endpoints lie in $\{1\} \times I^2$ (Figure 2.4(c)) [8].

(a) a shadow (b) a knot diagram (c) a 2-tangle

Figure 2.4: A shadow, a knot diagram and a tangle.

Knot diagrams can be transformed to other diagrams using Reidemeister moves R-I, R-II, and R-III presented in Figure 2.5 [49]. Each move is a local transformation on a small region of the diagram. For these moves the following Reidemeister theorem holds:

(a) R-I (b) R-II (c) R-III

Figure 2.5: Reidemeister moves.

**Theorem 2.1.1** (Reidemeister [49])**.** *Two diagrams represent equivalent knots if and only if there exists a finite sequence of Reidemeister moves* R-I, R-II, *and* R-III *that transform one diagram to the other.*

The proof is quite elementary in nature: we have to study what happens with the knot's projection when we isotope the knot in a way that in one step of the isotopy a forbidden singularity from Figure 2.3 appears in the projection.

A **flype**[1] is a local transformation of a knot diagram presented in Figure 2.6. A flype is geometrically interpreted by flipping a 2-tangle.

Figure 2.6: A flype.

The knot that allows a diagram with no crossings is called the **unknot** $O$ (Figure 2.7(a)), although sometimes we will refer to it as the **trivial knot**. A **kink** is the loop created by R-I (see the right-hand side of Figure 2.5(a)).

If a knot has been given an orientation, we call such a knot an **oriented knot**. In a diagram, the orientation is indicated by one or more arrows (Figure 2.7(b)). When talking about isotopy between oriented knots, we require the isotopy to preserve the orientation of the knot.

[1] An old Scottish word meaning *to turn* or *to fold back* [18].

(a) the unknot (b) an oriented knot

Figure 2.7: The unknot and an oriented knot.

If we reverse the orientation of an oriented knot $K$, the resulting knot is called the **reverse** of $K$ and is denoted by $-K$ (Figure 2.8).

(a) $K$ (b) $-K$

Figure 2.8: The reverse of a knot.

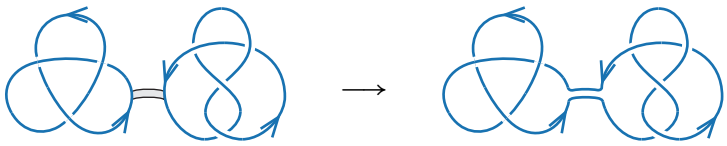
In a diagram of an oriented knot, each crossing can be assigned an integer +1 or −1, according to the right-hand rule pictured in Figure 2.9. The sum of all the crossing signs of a diagram $D$ is called the **writhe** $wr(D)$ of $D$.

(a) positive crossing (b) negative crossing

Figure 2.9: The sign of a crossing.

The minimal number of crossings a knot $K$ can have in any of its diagrams is called the **crossing number** $cr(K)$ of $K$.

The **connected sum** of two oriented knots $K_1$ and $K_2$ is formed by removing a small arc from each knot and connecting the four endpoints by two new arcs bounding a band in such a way that the orientation stays consistent with the original knots, the result being a single (oriented) knot $K_1 \# K_2$ (Figure 2.10).



$K_1$ $K_2$ $K_1 \# K_2$

Figure 2.10: The construction of a connected sum of two knots.

We call a knot **prime** if it cannot be written as the connected sum of two non-trivial knots and **composite** otherwise. In the case of unoriented knots, the connected sum in not well-defined: in general there are two distinct equivalence classes of the resulting knot. The equivalence class depends on the relative orientations chosen in order to perform the connected sum operation.

The following theorem regarding connected sums was shown by Schubert [51]:

**Lemma 2.1.1** (Unique decomposition of knots)**.** *Every knot can be uniquely decomposed, up to the order in which the decomposition is performed, as the connected sum of prime knots.*

If we embed a ribbon $S^1 \times I$ in $\mathbb{R}^3$ instead of a circle, such an embedding is called a **framed knot** (Figure 2.11). One often represents diagrams of framed knots with unframed diagrams, in this case, **blackboard framing** is assumed. Blackboard framing corresponds to a ribbon lying flat on the projection plane. In case the knot has a different framing than the blackboard one, we specify the (relative) framing number, i.e. additional left or right twists. If we assume the ribbon $S^1 \times I$ is oriented, we speak of oriented framed knots. We present such knots with ordinary oriented knot diagrams, where blackboard framing is again assumed and the orientation is induced by $K \times \{0\}$.

Figure 2.11: A framed knot.

**Theorem 2.1.2** (Kauffman [30])**.** *Two diagrams of framed knots represent equivalent framed knots if and only if there exits a finite sequence of Reidemeister moves* R-II *and* R-III *that transform one diagram to the other.*

Figure 2.12 demonstrates that a framed knot is not invariant under R-I.

Figure 2.12: Removal of a kink changes the framing of a knot.

The equivalence relation of knot diagrams generated by only Reidemeister moves R-II and R-III is called **regular isotopy**.

If we reflect a knot $K$ through a plane, the reflection is called the **mirror** of $K$ and is denoted by $\overline{K}$. To construct a mirror knot from a knot diagram, we exchange all overcrossings with undercrossings and vice versa (Figure 2.13). A knot is **amphichiral** if it is equivalent to its mirror image and **chiral** if it is not.

Let us for a moment assume, that a knot $K$ lies in an arbitrary 3-manifold $M$. We call a knot **affine**, if it lies inside a 3-ball $B^3 \subset M$ and **non-affine** otherwise. Note that all knots in $S^3$ are affine.

(a) $K$ (b) $\overline{K}$

Figure 2.13: The mirror of a knot.

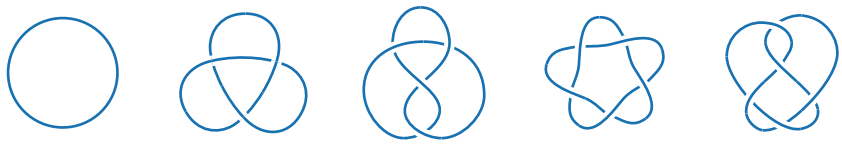
### 2.1.1 Knot tables

One of the earliest motivations in knot theory was the classification of knots. Such a classification is usually presented by providing a **knot table**. In such a table, we denote each knot by a symbol $n_k$, where $n$ presents the crossing number and $k$ represents the index of the knot in the table among knots with $n$ crossings (i.e. $n_k$ lies in the $k$-th place among $n$-crossing knots). If, for a given 3-manifold, the mirror operation is well-defined, we only put one of the possible pair in the table (and preferably provide information if the knot is amphichiral or not). If the operation of connected sums is well-defined in the manifold, we do not, due to Lemma 2.1.1, include connected sums in a knot table. Similarly, a classification of links is given by a **link table**.

There are various knot and link tables of knots and links in $S^3$, but the most widely used convention is to use the Rolfsen knot table for knots and the Thistlethwaite link table for links, both of which can be found at [5].

As an example, the first few knots in the Rolfsen knot table are presented in Table 2.1.

Table 2.1: Table of knots in $S^3$ up to 5 crossings.



| $0_1$ | $3_1$ | $4_1$ | $5_1$ | $5_2$ |
|---|---|---|---|---|

For the case of classifying knots in 3-manifolds that are not $S^3$, a knot table for knots in $\mathbb{R}P^3$ can be found in [17] and a partial classification of knots in the solid torus can be found in [12].

## 2.2 Knot polynomials

A **knot polynomial** is a polynomial that is a knot invariant. The first knot polynomial, the Alexander polynomial, was discovered by J. W. Alexander II in 1923 [2] and the second one was the Jones polynomial discovered in the early 1980s by V. Jones [28]. Jones's discovery quickly led to an outburst of various other polynomials, most notably the Conway's bracket polynomial, the Kauffman polynomial, and the HOMFLYPT polynomial [1, 34].

We outline the definitions of some of the most commonly used polynomials that we will refer to throughout the thesis.

Let $L_+$, $L_-$, and $L_0$ be oriented links that are identical except inside a small 3-ball, where their projections look like those presented in Figure 2.14. Such a triple of links is referred to as the **skein triple** and a relation involving such a triple is called a **skein relation**.

(a) $L_+$ (b) $L_-$ (c) $L_0$

Figure 2.14: The skein triple.

The **normalized Alexander polynomial**[2] $\Delta(L)$ of an oriented link $L$ is a Laurent polynomial in $t^{1/2}$ and is characterized by the following conditions:

$$\Delta(O) = 1, \qquad \text{(normalization)}$$
$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0), \qquad \text{(skein relation)}$$

with $O$ being the unknot.

The **Alexander-Conway polynomial** $\nabla(L)$ is a polynomial in $z$ and is characterized by:

$$\nabla(O) = 1, \qquad \text{(normalization)}$$
$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0). \qquad \text{(skein relation)}$$

The **Jones polynomial** $J(L)$ of an oriented link $L$ is a Laurent polynomial in $t^{1/2}$ and is defined by the following relations:

$$J(O) = 1, \qquad \text{(normalization)}$$
$$t^{-1}J(L_+) - tJ(L_-) = (t^{1/2} - t^{-1/2})J(L_0). \qquad \text{(skein relation)}$$

The **Kauffman bracket**[3] $\langle \cdot \rangle$ is an unoriented framed link polynomial invariant in variable $A$ [31]. The bracket is defined by the state-sum formula, for which we first need to set up a few definitions.

Let $D$ be an oriented link diagram with $n$ crossings labeled arbitrarily from 1 to $n$ and denote this set of crossings by $\mathcal{X}$. The number of positive crossings in $D$ is marked by $n_+$ and the number of negative crossings is marked by $n_-$.

Each crossing can be smoothened by a smoothening of type 0 or 1 according to Figure 2.15. We call $\{0,1\}^{\mathcal{X}}$ the **discrete cube** of $D$ and a vertex $s \in \{0,1\}^{\mathcal{X}}$ a **(Kauffman) state** of $D$. Each state corresponds to a diagram with each crossing smoothened either by a type 0 or a type 1 smoothening. For convenience, this complete smoothening is also called a **state** of $D$. Each state is just a collection of disjoint closed loops which are called **circles**, the number of circles in a state $s$ is denoted by $|s|$.

[2] Alexander's original polynomial is unique only up to multiplication by the Laurent monomial $\pm t^{n/2}$, so usually one fixes a certain normalization, which may not necessarily be the one used in our characterization.

[3] Also known as the Bracket polynomial.

(a) type 0 smoothening (b) type 1 smoothening

Figure 2.15: Two types of smoothenings.

By $\#0(s)$ we denote the number of 0 factors in $s$ and by $\#1(s)$ the number of 1 factors in $s$.

For a given diagram $D$ of the link $L$, the Kauffman bracket $\langle D\rangle$ is defined by the state-sum formula:

$$\langle D\rangle = \sum_{s\in\{0,1\}^{\mathcal{X}}} A^{\#0(s)-\#1(s)}(-A^2-A^{-2})^{|s|-1}\langle O\rangle,$$

where we use the evaluation $\langle O\rangle = 1$. A non-standard version of the Kauffman bracket (that will be used in Chapter 5) uses the evaluation $\langle\varnothing\rangle = 1$ and can be expressed by the formula:

$$\langle D\rangle = \sum_{s\in\{0,1\}^{\mathcal{X}}} A^{\#0(s)-\#1(s)}(-A^2-A^{-2})^{|s|}\langle\varnothing\rangle.$$

As we have already stated, the Kauffman bracket is an invariant of framed links, since it is invariant only under regular isotopy, i.e. not invariant under R-I [30].

We continue by presenting a recursive characterization of $\langle\cdot\rangle$.

Let $L$, $L_0$, and $L_\infty$ be unoriented links that are identical except inside a small 3-ball, where their projections look like those presented in Figure 2.16. We will refer to such a triple as a **Kauffman triple** and a relation involving this triple as a **Kauffman relation**.

(a) $L$ (b) $L_0$ (c) $L_\infty$

Figure 2.16: The Kauffman triple.

The Kauffman bracket is characterized by the following relations [30]:

$$\langle L\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle, \qquad \text{(Kauffman relation)}$$
$$\langle L\sqcup O\rangle = (-A^2-A^{-2})\langle L\rangle, \qquad \text{(framing relation)}$$
$$\langle O\rangle = 1. \qquad \text{(normalization)}$$

**Proposition 2.2.1.** *For the connected sum of knots $K$ and $K'$, the following equality holds:*

$$\langle K\#K'\rangle = \langle K\rangle\langle K'\rangle.$$

*Proof.* We prove this by induction $n$, where $n$ is the number of crossings of $K'$. For $n = 0$ the equality $\langle K\#O\rangle = \langle K\rangle\langle O\rangle$ holds, since $K\#O = K$ and $\langle O\rangle = 1$. For $n > 0$ we resolve a crossing of $K'$ using the Kauffman relation:

$$\langle K\#K'\rangle = A\langle K\#K'_0\rangle + A^{-1}\langle K\#K'_\infty\rangle.$$

Since $K'_0$ and $K'_\infty$ both have $n-1$ crossings, we use the induction hypothesis:

$$\langle K\#K'\rangle = A\langle K\rangle\langle K'_0\rangle + A^{-1}\langle K\rangle\langle K'_\infty\rangle = \langle K\rangle(A\langle K'_0\rangle + A^{-1}\langle K'_\infty\rangle) = \langle K\rangle\langle K'\rangle,$$

where the last equality holds by the Kauffman relation on the previously resolved crossing. □

If, for a given diagram, we multiply the bracket polynomial by $(-A^3)^{-wr(L)}$, we get the **Kauffman polynomial** $X$:

$$X(L) = (-A^3)^{-wr(L)}\langle L\rangle,$$

which, due to the normalization, becomes an invariant under isotopy and is therefore an invariant of unframed links [31].

The polynomial $X$ equals the Jones polynomial by a change of variable [1]:

$$X(L)(A) = J(L)(A^{-4}).$$

Since it holds that $wr(K\#K') = wr(K) + wr(K')$, the formula $X(K\#K') = X(K)X(K')$ also holds for $X$ and thus for the Jones polynomial:

$$J(K\#K') = J(K)J(K').$$

At this point, we define a variant of the Jones polynomial that will be used in the construction of the Khovanov homology in Section 2.4. Khovanov uses a slightly modified version of the Kauffman bracket that is defined by the following axioms [32]:

$$\langle L\rangle_m = \langle L_0\rangle_m - q\langle L_\infty\rangle_m, \qquad \text{(Kauffman relation)}$$
$$\langle L \sqcup O\rangle_m = (q+q^{-1})\langle L\rangle_m, \qquad \text{(framing relation)}$$
$$\langle \varnothing\rangle_m = 1. \qquad \text{(normalization)}$$

To turn this bracket into a link invariant, we must multiply it by $(-1)^{n_-}q^{n_+-2n_-}$. This leads us to the **unnormalized Jones polynomial** $\hat{J}$:

$$\hat{J}(L) = (-1)^{n_-}q^{n_+-2n_-}\langle L\rangle_m.$$

The Laurent polynomial $\hat{J}$ in variable $q$ is an invariant of oriented links [50].

The **HOMFLYPT**[4] **polynomial** of an oriented knot is a 2-variable Laurent polynomial in variables $v$ and $z$ defined by the following relations[5]:

$$P(O) = 1, \qquad \text{(normalization)}$$
$$v^{-1}P(L_+) - vP(L_-) = zP(L_0). \qquad \text{(skein relation)}$$

[4]The HOMFLYPT is often referred to as the HOMFLY polynomial. It was independently discovered by five different groups: Hoste; Ocneanu; Millett and Lickorish; Freyd and Yetter; Przytycki and Traczyk. Unfortunately, because of the martial law in Poland, the discovery of Przytycki and Traczyk did not reach the United States before D. Yetter coined the original acronym [27].

[5]Due to various authors, about four distinct definitions of the HOMFLYPT polynomial can be found in modern literature, each involving different signs and letters.

To demonstrate the strength of the HOMFLYPT polynomial, we note that all of the previously defined polynomials can be obtained from the HOMFLYPT polynomial [1, 50]:

$$\begin{aligned}
\Delta(t) &= P(1, t^{1/2} - t^{-1/2}),\\
\nabla(z) &= P(1, z),\\
J(t) &= P(t, t^{1/2} - t^{-1/2}),\\
X(A) &= P(A^{-4}, A^{-2} - A^{2}),\\
\langle \cdot \rangle &= (-A^{3})^{w(\cdot)} P(A^{-4}, A^{-2} - A^{2}).
\end{aligned}$$

## 2.3 Skein modules of 3-manifolds

Skein modules have their origin in the observation made by J. W. Alexander that the three polynomials $\Delta(L_+)$, $\Delta(L_-)$, and $\Delta(L_0)$ of links $L_+$, $L_-$, and $L_0$, respectively, are linearly related by the skein relation[6]

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0).$$

J. H. Conway pursued this idea by considering the free $\mathbb{Z}[z]$-module over the set of isotopy classes of links in $S^3$ modulo the $\mathbb{Z}[z]$-module generated by the skein relation of the Alexander-Conway polynomial [42, 46].

By formalizing such a construction and generalizing it for any 3-manifold (not just $S^3$), J. H. Przytycki and V. G Turaev introduced the theory of skein modules in 1987 [55, 44]. In Przytycki's own words, the theory of skein modules is the idea of building an algebraic topology based on knots. The building blocks of such a theory are considered up to isotopy, where instead of formal linear combinations of simplices we use linear combinations of links and instead of boundary relations, we use properly chosen skein relations [42].

In the remaining part of this section we overview the theory of skein modules as described in [46, 44].

Let $M$ be a 3-manifold and let $R$ be a commutative ring with identity. Let $\mathcal{L}(M)$ be the set of isotopy classes of links in $M$ and let $R\mathcal{L}(M)$ the free $R$-module generated by $\mathcal{L}(M)$. Let $\mathcal{S}(M;\mathcal{R})$ be the submodule generated by a collection $\mathcal{R} \subset R\mathcal{L}(M)$ of finite formal expressions

$$r_0 L_0 + r_1 L_1 + \cdots + r L_\infty,$$

where $r, r_0, r_1, r_2, \ldots \in R$ and $L_0, L_1, L_2, \ldots$ being classes of links that are identical except in the parts shown in Figure 2.17.

The **skein module** $\mathcal{S}(M;R,\mathcal{R})$ is $R\mathcal{L}(M)$ modulo $\mathcal{S}(M;\mathcal{R})$:

$$\mathcal{S}(M;R,\mathcal{R}) = R\mathcal{L}(M)/\mathcal{S}(M,\mathcal{R}).$$

**Example 2.3.1.** $\mathcal{S}(M;R;\varnothing) = R\mathcal{L}(M)$.

[6] The Alexander polynomial was originally defined through certain manipulation of diagrams [2].

(a) $L_0$ (b) $L_1$ (c) $L_2$ (d) $L_3$ (e) $L_\infty$

Figure 2.17: The skein tuple.

**Example 2.3.2.** *$\mathcal{S}_\pm(M;R) = \mathcal{S}(M;R,$ $-$ $)$ is a free $R$-module over the homotopy classes of closed curves in $M$ (i.e. two links $L$ and $L'$ are equivalent in $\mathcal{S}_\pm(M)$ if they are homotopic) [46]. The relation* $-$ *means that we identify all links that are identical except inside a 3-ball where one of them looks like* *and the other one looks like* .

We continue by describing two particular skein modules in detail: the Kauffman bracket skein module and the HOMFLYPT skein module.

### 2.3.1 The Kauffman bracket skein module

The Kauffman bracket skein module (KBSM) generalizes the Kauffman bracket and is one of the most extensively studied skein module [39] as it has been calculated for a number of different 3-manifolds [43, 26, 38, 39].

To construct KBSM of a 3-manifold $M$, take a coefficient ring $R$ with $A \in R$ being a unit (an element with a multiplicative inverse). Since, as in the case of the Kauffman bracket, we would like to study framed links, we let $\mathcal{L}_{\mathrm{fr}}(M)$ be the set of isotopy classes of framed links in $M$, including the class of the empty link $[\varnothing]$. As before, let $R\mathcal{L}_{\mathrm{fr}}(M)$ be the free $R$-module spanned by $\mathcal{L}_{\mathrm{fr}}(M)$. We assume the ring $R$ is fixed and omit it in further notations.

We would like to impose the Kauffman relation and the framing relation in $R\mathcal{L}_{\mathrm{fr}}(M)$. We therefore take the submodule $\mathcal{S}_{\mathrm{fr}}(M)$ of $R\mathcal{L}_{\mathrm{fr}}(M)$ generated by

$- A$  $- A^{-1}$ , (Kauffman relator)

$L \sqcup$ $- (-A^2 - A^{-2})L.$ (framing relator)

The **Kauffman bracket skein module** $\mathcal{S}_{2,\infty}(M)$ is $R\mathcal{L}_{\mathrm{fr}}(M)$ modulo these two relations:

$$\mathcal{S}_{2,\infty}(M) = R\mathcal{L}_{\mathrm{fr}}(M)/\mathcal{S}(M).$$

**Example 2.3.3.** *For the 3-sphere, $\mathcal{S}_{2,\infty}(S^3)$ is a free $R$-module with the basis being just the equivalence class of the unknot. Expressing a knot in this basis and evaluating $[O] = 1$, we get exactly the Kauffman bracket. Similarly, we can set the basis to be the empty knot and by evaluating $[\varnothing] = 1$, we get the previously described non-standard version of $\langle \cdot \rangle$ (these two facts can be easily seen as the definition corresponds directly to the definition of the polynomial).*

The Kauffman bracket skein module of the solid torus $T$ has been calculated in [55]:

**Theorem 2.3.4.** *KBSM of the solid torus $T$ is freely generated by an infinite set of generators $\{x^n\}_{n=0}^{\infty}$, where $x^n$, $n > 0$, is a parallel copy of $n$ longitudes of $T$ and $x^0$ is the affine unknot.*

The Kauffman bracket skein module of $L(p,q)$ has been calculated in [24]:

**Theorem 2.3.5.** *KBSM of $L(p,q)$ is freely generated by the set of generators $\{x^n\}_{n=0}^{\lfloor p/2 \rfloor}$, where $x^n$, $n > 0$, is a parallel copy of $n$ longitudes of $T \subset L(p,q)$ and $x^0$ is the affine unknot (see Chapter 3).*

### 2.3.2 The HOMFLYPT skein module

The HOMFLYPT skein module of a 3-manifold $M$ generalizes the HOMFLYPT polynomial. Let the ring $R$ this time have two units $v, z \in R$. Let $\mathcal{L}_{\text{or}}(M)$ be the set of isotopy classes of oriented links in $M$, including the class of the empty link $[\varnothing]$ and let $R\mathcal{L}_{\text{or}}(M)$ be the free $R$-module spanned by $\mathcal{L}_{\text{or}}(M)$. We again assume $R$ is fixed and omit it in the notations.

As in the case of the HOMFLYPT polynomial, we would like to impose the HOMFLYPT skein relation in $R\mathcal{L}_{\text{or}}(M)$. We take the submodule $\mathcal{S}(M)$ of $R\mathcal{L}_{\text{or}}(M)$ generated by the expressions

$v^{-1}$  $- v$  $- z$  . (HOMFLYPT relator)

We also add to $\mathcal{S}(M)$ the HOMFLYPT relation involving the empty knot:

$$v^{-1}\varnothing - v\varnothing - zO. \qquad \text{(HOMFLYPT relator)}$$

The **HOMFLYPT skein module** $\mathcal{S}_3(M)$ of $M$ is $R\mathcal{L}_{\text{or}}(M)$ modulo the above relations:

$$\mathcal{S}_3(M) = R\mathcal{L}(M)/\mathcal{S}(M).$$

**Example 2.3.6.** *For the 3-sphere $S^3$, $\mathcal{S}_3(S^3)$ is freely generated by $[O]$. By expressing a knot $K$ in this basis and evaluating $[O] = 1$, we get the HOMFLYPT polynomial of $K$.*

**Example 2.3.7.** *The HOMFLYPT skein module of the solid torus $T$ is a free $R$-module, generated by an infinite set of generators (see Chapter 4 for details on the basis of this module) [23, 55].*

*Remark.* The integer subscript in the skein module indicates what types of links are used in the relations of the module. A skein module denoted by $\mathcal{S}_k(M)$ indicates the relations are generated by skein expressions $r_0L_0 + r_1L_1 + \cdots r_{k-1}L_{k-1}$, with $r_0, r_1, \ldots, r_{k-1} \in R$ and $L_0, L_1, \ldots, L_{k-1}$ being classes of links from Figure 2.17. Similarly, $\mathcal{S}_{k,\infty}(M)$ indicates that we are also taking into account $L_\infty$, i.e. the module is generated by expressions $r_0L_0 + r_1L_1 + \cdots r_{k-1}L_{k-1} + rL_\infty$, with $r, r_0, r_1, \ldots, r_{k-1} \in R$.

## 2.4 Khovanov homology

A major breakthrough in the study of knots in $S^3$ appeared in the late 1990s by a series of lectures by M. Khovanov, who managed to categorify the Jones polynomial by constructing a chain complex of graded vector spaces with the property that the homology of this chain complex, the Khovanov homology, is a link invariant. Moreover, the graded Euler characteristic of this complex is the Jones polynomial [32]. Perhaps the most outstanding consequence of this theory is the $s$-invariant of Rasmussen, which gives a bound on a knot's slice genus and is sufficient to prove the Milnor conjecture [47]. More recently, Kronheimer and Mrowka showed that the Khovanov homology detects the unknot [33].

Khovanov's categorification is the idea to replace polynomials with graded vector spaces of appropriate graded dimensions to turn the Jones polynomial into a homological object. The Kauffman bracket $\langle \cdot \rangle$ is replaced with the Khovanov bracket $[\![ \cdot ]\!]$, which is a chain complex of graded vector spaces whose graded Euler characteristic, as we have mentioned before, is the Jones polynomial [4].

Since we will use the construction of the Khovanov chain complex (although substantially modified) in Chapter 5, we at this point repeat the classical construction of the Khovanov homology as it is constructed in [4] verbatim. The Khovanov chain complex and the Khovanov homology will be constructed using graded vector spaces (as it is done in [4]), but the construction works just as well, with suitable modifications, over $\mathbb{Z}$-modules – such a construction is used to categorify KBSM of $\mathbb{R}\mathrm{P}^3$ in Chapter 5.

### 2.4.1 The Khovanov chain complex

Let $W = \bigoplus_m W_m$ be a **graded vector space** with homogeneous components $\{W_m\}_m$. The **graded dimension** of $W$ is the power series

$$q\text{-}\dim W = \sum_m q^m \dim W_m.$$

Let $\cdot\{l\}$ be the **degree shift operation** on graded vector spaces. That is, if $W = \bigoplus_m W_m$ is a graded vector space we set $W\{l\}_m = W_{m-l}$, so that $q\text{-}\dim W\{l\} = q^l q\text{-}\dim W$.

Likewise, let $\cdot[s]$ be the **height shift operation** on chain complexes. That is, if $\overline{\mathcal{C}}$ is a chain complex

$$\cdots \longrightarrow \overline{\mathcal{C}}^{r-1} \longrightarrow \overline{\mathcal{C}}^{r} \longrightarrow \overline{\mathcal{C}}^{r+1} \longrightarrow \cdots$$

of graded vector spaces and if $\mathcal{C} = \overline{\mathcal{C}}[s]$, then $\mathcal{C}^r = \overline{\mathcal{C}}^{r-s}$ (with differentials shifted accordingly).

Let $\mathcal{X}$, $n$ and $n_\pm$ be as section 2.2. Let $V$ be the graded vector space with two basis elements $v_\pm$ whose degrees are $\pm 1$, respectively. Note that $q\text{-}\dim V = q + q^{-1}$. With every state $s \in \{0,1\}^{\mathcal{X}}$ of the discrete cube $\{0,1\}^{\mathcal{X}}$ we associate the graded vector space $V(L) = V^{\otimes |s|}\{r\}$, with $|s|$ again being the number of circles in $s$. We then set the $r$-th chain group $[\![L]\!]^r$ (for $0 \le r \le n$) to be the

direct sum of all the vector spaces associated with the states with the number of 1-smoothenings being $r$:

$$[\![L]\!]^r = \bigoplus_{s:r=\#1(s)} V_s(L).$$

Ignoring that $[\![L]\!]$ is not yet a complex, for we have not yet endowed it with a differential, we set $\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$.

The graded Euler characteristic $\chi_q(\mathcal{C})$ of a chain complex $\mathcal{C}$ is defined to be the alternating sum of the graded dimensions of its homology groups, and, if the degree of the differential $d$ is 0 and all chain groups are finite dimensional, it is also equal to the alternating sum of the graded dimensions of the chain groups. A few paragraphs below we will endow $\mathcal{C}(L)$ with a degree 0 differential. This granted and given that the chains of $\mathcal{C}(L)$ are already defined, we can state the following theorem:

**Theorem 2.4.1** ([4]). *The graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalized Jones polynomial of $L$:*

$$\chi_q \overline{\mathcal{C}}(L) = \hat{J}(L).$$

Next, we wish to turn the sequence of spaces $\mathcal{C}(L)$ into a chain complex.

### 2.4.2 The differential

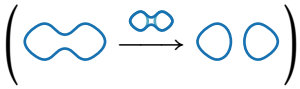
The edges of the discrete cube $\{0,1\}^{\mathcal{X}}$ can be labeled by sequences in $\{0,1,\star\}^{\mathcal{X}}$ with just one $\star$ (so the tail of such an edge is found by setting $\star \to 0$ and the head by setting $\star \to 1$). The number $\#1(\xi)$ of an edge $\xi$ is defined to be the number of 1's in its tail, and hence if the maps on the edges are called $d_\xi$, then we collapse the maps with the same number of 1's:

$$d^r = \sum_{\#1(\xi)=r} (-1)^\xi d_\xi.$$

It remains to explain the signs $(-1)^\xi$ and to define the per-edge maps $d_\xi$. The former is easy. To get the differential $d$ to satisfy $d \circ d = 0$, it is enough that all square faces of the cube would anti-commute. But it is easier to arrange the $d_\xi$'s so that these faces would (positively) commute; so we do that and then change signs to make the faces anti-commutative. This is done by multiplying $d_\xi$ by $(-1)^\xi = (-1)^{\sum_{i<j} \xi_i}$, where $j$ is the location of the $\star$ in $\xi$.

It remains to find maps $d_\xi$ that make the cube commutative (when taken with no signs) and that are of degree 0. The space $V_s$ on each state $s$ has as many tensor factors as there are circles in the smoothing of corresponding to $s$. Thus we put these tensor factors in $V_s$ and circles in $s$ in bijective correspondence. Now for any edge $\xi$, the smoothing at the tail of $\xi$ differs from the smoothing at the head of $\xi$ by just a little: either two of the circles merge into one or one of the circle splits in two. So for any $\xi$, we set $d_\xi$ to be the identity on the tensor factors corresponding to the circle that do not participate, and then we complete the definition of $d_\xi$ using two linear maps $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ as follows:

$$\longrightarrow (V \otimes V \xrightarrow{m} V) \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_- \mapsto 0 \end{cases}$$



$$\longrightarrow (V \xrightarrow{\Delta} V \otimes V) \qquad \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

Note that because of the degree shifts in the definition of $V_s$ and because we want $d_\xi$ to be of degree 0, the maps $m$ and $\Delta$ must be of degree $-1$. Also, as there is no canonical order on the circles in $s$ (and hence on the tensor factors of $V_s$), $m$ and $\Delta$ must be commutative and co-commutative respectively. These requirements force the equality $m(v_- \otimes v_-) = m(v_- \otimes v_+)$ and force the values of $m$ and $\Delta$ to be as shown above up to scalars.

### 2.4.3 The homology

We conclude with the following proposition and theorem.

**Proposition 2.4.1** (Khovanov [32]). *The sequences $[\![L]\!]$ and $\mathcal{C}(L)$ are chain complexes.*

Let $\mathcal{H}^r(L)$ denote the $r$-th cohomology of the complex $\mathcal{C}(L)$. It is a graded vector space depending on the link projection $L$. Let $\mathcal{P}(L)$ denote the graded Poincaré polynomial of the complex $\mathcal{C}(L)$ in the variable $t$:

$$\mathcal{P}(L) = \sum_r t^r q\text{-}\dim \mathcal{H}^r(L).$$

**Proposition 2.4.2** (Khovanov [32]). *The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are link invariants, and hence $\mathcal{P}(L)$, a polynomial in the variables $t$ and $q$, is a link invariant that specializes to the unnormalized Jones polynomial at $t = -1$.*

# 3

# Knots in $L(p,q)$ and their diagrams

In 1884 W. Dyck introduced the concept of constructing a lens space, but unfortunately did not publish more than just ideas before withdrawing from topology completely. The first intensive study of lens spaces was done by H. Tieze in 1908, who laid down the importance of such spaces since they are simple on the one hand, but have intriguing properties and diverse applicability on the other hand [54]. The term "linsenräume" (lens space) itself was coined by Seifert and Threlfall in 1930 [21].

In her seminal work, Yu. V. Drobotukhina introduced the study of knots in the real projective space. She constructed a version of the Jones polynomial for $\mathbb{R}\mathrm{P}^3$ [16] and managed to classify links in $\mathbb{R}\mathrm{P}^3$ up to 6 crossings [17]. J. Hoste and J. H. Przytycki furthermore generalized the Jones polynomial by calculating KBSM of $L(p,q)$[24].

In order to have a working theory of links in $L(p,q)$, we start off by defining a lens space and continue by defining a diagram of a link in it.

## 3.1 Lens spaces

At least five more or less distinct definitions of the 3-dimensional lens space can be found in modern topology textbooks [57]. Out of these five, we overview three models, out of which the second and third model will be used throughout the rest of the thesis. For all three constructions we assume $0 < p < q$ are two coprime integers.

**First model of $L(p,q)$.** Let $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$ be the 3-sphere. The **lens space** is defined to be the orbit space of the free action of the cyclic group $\mathbb{Z}/p$ on $S^3$ given by

$$n \cdot (z_0, z_1) \longmapsto (z_0 \cdot e^{2\pi i n/p}, z_1 \cdot e^{2\pi i q n/p}).$$

**Second model of $L(p,q)$.** Consider the 3-ball $B^3 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ with $S_+ = \partial B^3 \cap \{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$ and $S_- = \partial B^3 \cap \{(x,y,z) \in \mathbb{R}^3 : z \leq 0\}$ being the upper and lower closed hemispheres of $\partial B^3$, respectively. Let $r_{p,q} : S_+ \to S_+$ be the rotation of $S_+$ by $2\pi q/p$ around the $z$-axis and let $m : S_+ \to S_-$ be the reflection with respect to the $xy$-plane. We identify each point $w \in S_+$ on the upper hemisphere with the point $m \circ r_{p,q}(w)$ on the lower hemisphere

(Figure 3.1). The lens space $L(p,q)$ is the quotient of $B^3$ by this equivalence relation:

$$L(p,q) = B^3/\sim .$$

Note that on the equator $\{(x,y,0) \in \mathbb{R}^3, x^2+y^2 = 1\}$ each equivalence class contains $p$ points.

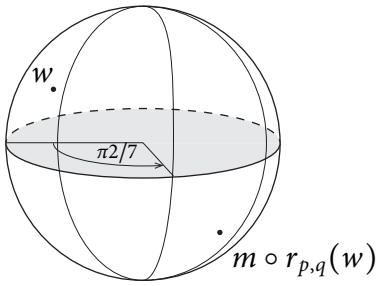



Figure 3.1: Representation of $L(7,2)$.

**Third model of $L(p,q)$.** Take a solid torus $T = S^1 \times D^2$ and fix a point $x_0 \in \partial T$ on its boundary. Let $l$ and $m$ be the generic longitude and meridian of $T$ as shown in Figure 3.2. Note that $l$ and $m$ generate the fundamental group $\pi_1(\partial T, x_0)$.

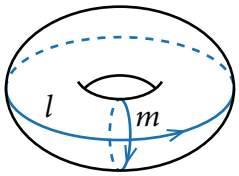



Figure 3.2: The generic longitude $l$ and meridian $m$ of $T$.

If a curve on $\partial T$ represents the class $p[l]+q[m] \in \pi_1(T, x_0)$, we call such a curve a $(p,q)$-**curve.** The lens space $L(p,q)$ is the result of gluing to solid tori $T_1$ and $T_2$ along their boundaries via the homeomorphism $h_{p,q} : \partial T_1 \to \partial T_2$ that takes a meridian $m \subset \partial T_1$ to a $(p,q)$-curve in $\partial T_2$ (see Figure 3.3 for an example of $p = 3, q = 1$).

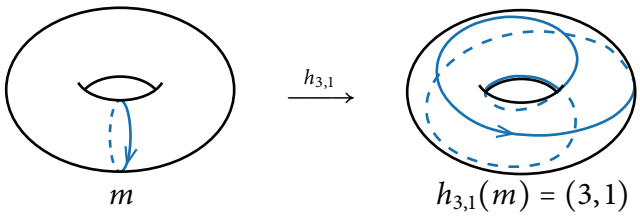



Figure 3.3: The boundary homeomorphism $h_{3,1} : \partial T_1 \to \partial T_2$.

**Proposition 3.1.1.** *All three models agree.*

For this folklore fact, a comprehensive proof can be found in [57].

The **projective space**[1] $\mathbb{R}P^3$ is the lens space $L(2,1)$.

[1]The 3-dimensional real projective space is defined to be the set of lines in $\mathbb{R}^3$ through the origin. Using homogeneous coordinates it is not hard to show the validity of the statement.

The classification problem of 3-dimensional lens spaces up to homeomorphism was solved by Reidemeister in 1935 [48]:

**Theorem 3.1.1.** *Three dimensional lens spaces $L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $p = p'$ and $q \equiv \pm q'^{\pm 1} (mod\ p)$.*

The solution of the classification problem up to homotopy equivalence is due to Whitehead [58]:

**Theorem 3.1.2.** *Three dimensional lens spaces $L(p,q)$ and $L(p',q')$ are homotopy equivalent if and only if $p = p'$ and $qq' = \pm n^2 (mod\ p)$ for some $n \in \mathbb{N}$.*

In the sections that follow, we present the disk diagram of a link presented in the second model of $L(p,q)$ and continue by presenting two types of diagrams for links in the solid torus: the punctured disk diagram and the arrow diagram. Taking the (standard) inclusion $T \hookrightarrow L(p,q)$ by the third model of $L(p,q)$, a diagram of a link in the solid torus, equipped with additional Reidemeister moves, becomes a diagram of a link in $L(p,q)$.

## 3.2 Disk diagrams

We overview the construction of the disk diagram as it is constructed in [41]. To get the disk diagram of a link in $L(p,q)$, we take the second model of $L(p,q)$ and mark with $N$ and $S$ respectively the north pole $N = (0,0,1)$ and south pole $S = (0,0,-1)$ of $B^3$; we label the equatorial disk lying in the $xy$-plane by $D$. Let $L$ be a link in $L(p,q)$ and by potentially making a small isotopy on $L$, we assume that:

- $N \notin L$, $S \notin L$,
- $L \cap \partial B^3$ consist of a finite set of points,
- $L$ is not tangent to $\partial B^3$,
- $L \cap \partial D = \varnothing$.

Let $p : B^3 \setminus \{N,S\} \to D$ be the projection defined by $p(w) = c(w) \cap D$, where $c(w)$ is the circle (possibly line) through $N$, $w$, and $S$ (Figure 3.4).

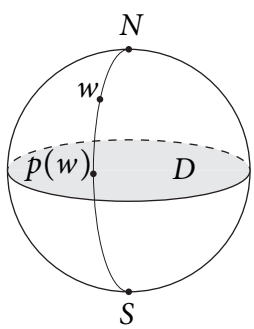
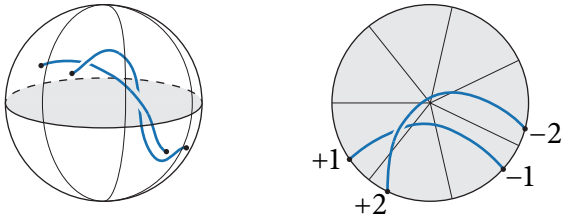



Figure 3.4: The projection $p : B^3 \setminus \{N,S\} \to D$.

Making this projection to a knot diagram, we mark over- and undercrossings, i.e. consider a double point $P \in p(L)$ and take the preimage $p^{-1}(P) = \{P_1, P_2\}$ of this point. Suppose $P_1$ is closer to $N$ than $P_2$ is. The image of a small arc $a_1 \ni P_1$ on $L$ represents an overpass and the image of $a_2 \ni P_2$ represents an underpass.

Since we would like to be able to reconstruct a link $L \subset L(p,q) = B^3/\sim$ from the diagram up to isotopy, it is essential that we have knowledge of what hemisphere the endpoints $L \cap \partial B^3$ belong to. We label by $+1, +2, \ldots, +n$ the endpoints in the upper hemisphere and by $-1, -2, \ldots, -n$ the points in the lower hemisphere, respecting the rule $+i \sim -i$. An example is presented in Figure 3.5.

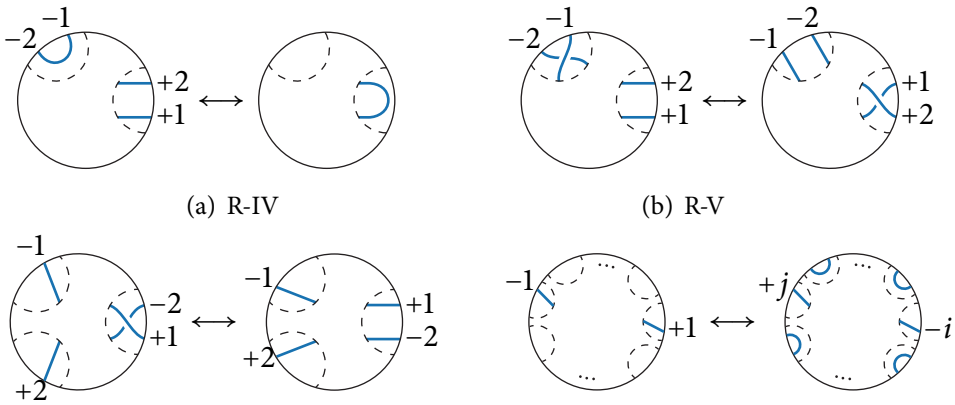



(a) a link $L$ (b) the diagram of $L$

Figure 3.5: A link in $L(p,q)$ and its diagram.

### 3.2.1 Reidemeister moves of the disk diagram

There are seven Reidemeister moves in a disk diagram of $L(p,q)$: three classical Reidemeister moves (R-I, R-II, and R-III) and four additional Reidemeister moves (R-IV, R-V, R-VI, and R-VII) acting across the boundary of $D$ as viewed in Figure 3.6 [41].

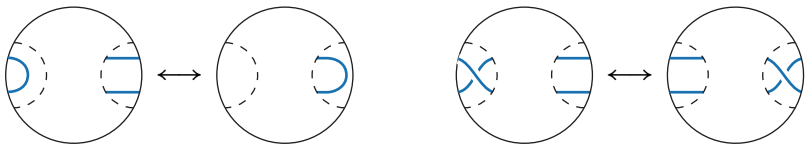



(a) R-IV (b) R-V

(c) R-VI (d) R-VII

Figure 3.6: Additional Reidemeister moves in the disk diagram of $L(p,q)$.

In the case of $\mathbb{R}P^3$, the move R-VII enables us to switch between positive and negative endpoints, so there is no need to include the labels. Also, the move R-VI does not appear, since it involves the case $p > 2$. The additional Reidemeister moves in the case of $\mathbb{R}P^3$ thus simplify to moves R-IV and R-V presented in Figure 3.7 [16].

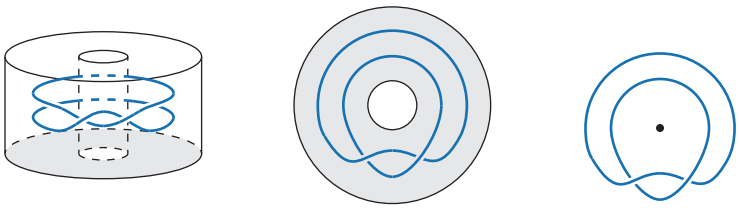


(a) R-IV (b) R-IV

Figure 3.7: Additional Reidemeister moves in the disk diagram of $\mathbb{R}P^3$.

## 3.3 Punctured disk diagrams of the solid torus

Imagine a knot $K$ in the solid torus $T = A \times I$, with $A$ being an annulus (Figure 3.8(a)). A **punctured disk diagram** of a knot $K$ is the regular projection of $K$ on $A$, keeping the information of over- and undercrossings (Figure 3.8(b)). We resolve the inconvenience of drawing the annulus by making a dot in the region of $\mathbb{R}^2 \supset A$ where the inner component of $\partial A$ lies on and assume the outer component of $\partial A$ lies in the unbounded region of $D$ (Figure 3.8(c)).



(a) (b) (c)

Figure 3.8: Construction of a punctured disk diagram of a link in the solid torus.

The Reidemeister moves of a punctured disk diagram of a knot in the solid torus correspond to the classical ones (R-I, R-II, and R-III), except that we cannot perform any move through the puncture.

## 3.4 Punctured disk diagrams of $L(p,q)$

Taking the third model of $L(p,q)$, we move a link $L \in L(p,q)$ into the first component $T_1$ of $T_1 \cup_{h_{p,q}} T_2$ by isotopy. Since $L$ now lies entirely in $T_1$, we project $L$ on the annulus $A$ of $T_1 = A \times I$, such a diagram corresponds to the punctured disk diagram of a link in $T_1$.

### 3.4.1 Reidemeister moves of the punctured disk diagram of $L(p,q)$

We equip a punctured disk diagram with an additional Reidemeister move SL also known as the **slide move**[2]. This move arises from gluing the solid torus $T_2$ to $T_1$ via the boundary homeo-

[2] We use SL instead of R-IV, since we will often use this move in the chapters that follow and would like to, for the ease of reading, make it distinct from the other Reidemeister moves.

morphism $h_{p,q}$. The move is presented in Figure 3.9 [24]. One can visualize this move by sliding an arc of the link over the meridional disk of the solid torus $T_2$ glued to $T_1$.

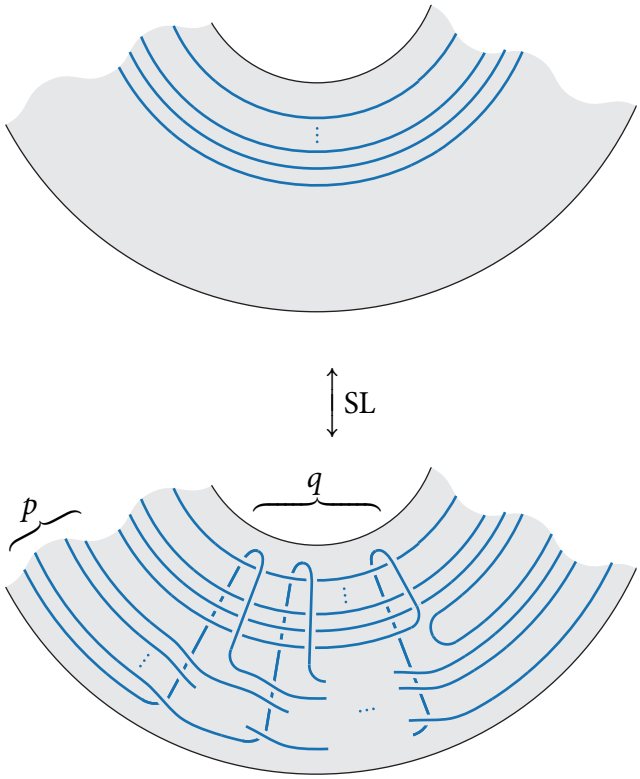
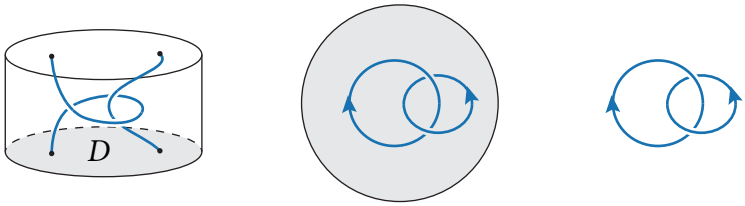



Figure 3.9: The slide move for $L(p,q)$.

## 3.5 Arrow diagrams

An **arrow diagram** [39] is obtained from a link $L \subset T$ by cutting $T$ along a meridional disk $D$, projecting the resulting vertical cylinder $D \times I$ onto the disk $D \times \{0\}$ while keeping information of over- and undercrossings of the projection (Figure 3.10(a)). The intersection points $L \cap (D \times \{0\})$ project onto arrows in the diagram, pointing to the part of $L$ that is close to $D \times \{0\}$ in the cylinder and away from the part of $L$ that is close to $D \times \{1\}$ (Figure 3.10(b)). We assume, by making a small isotopy on $L$, that there are no singularities near the arrows. If we do not draw the bounding disk $D$ in the diagram, but place the diagram on the plane, we assume $\partial D$ lies in the unbounded region of the plane (Figure 3.10(c)).

By convention we denote a set of consecutive arrows lying on an arc by a single arrow with an integer above it representing the number of arrows – a negative integer indicating the directions of the arrows are reversed (Figure 3.11).




(a) (b) (c)

Figure 3.10: Construction of an arrow diagram.

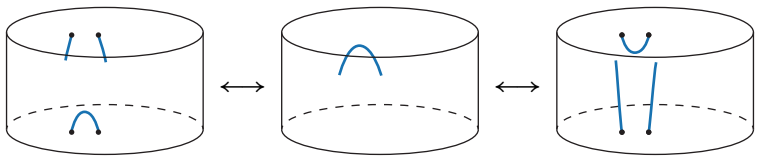
Figure 3.11: Joining arrows in a diagram.

### 3.5.1 Reidemeister moves of the arrow diagram of the torus

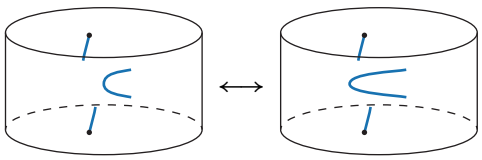
Along with the three classical Reidemeister moves, there are two additional moves that act across the disk $D$ of the projection. These two Reidemeister moves, "cancellation of arrows" R-IV and "pushing an arrow over an arc" R-V are presented in Figure 3.12. The geometric interpretations of these moves are evident from Figure 3.13 [39].

(a) R-IV (b) R-V

Figure 3.12: Additional Reidemeister moves in the arrow diagram.



(a) interpretation of R-IV



(b) interpretation of R-V

Figure 3.13: Geometrical interpretations of moves R-IV and R-V in the arrow diagram.

## 3.6 Arrow diagrams of $L(p,q)$

As in the case of the punctured disk diagram, we can represent a link in $L(p,q)$ with a diagram of a link in the solid torus. Such diagrams are again accompanied by the extra slide move SL that arises from the boundary-gluing homeomorphism of the tori.

### 3.6.1 Reidemeister moves of the arrow diagram of $L(p,q)$

A $(p,q)$-curve inside $D \times S^1$ (Figure 3.14) can be viewed as a $(q,1)$-curve in the disk $D$ of an arrow diagram with an arrow on every $2q\pi/p$ angle. The result is a diagram with $\lfloor p/q \rfloor + 1$ arrows on the outer arcs and $\lfloor p/q \rfloor$ arrows on the inner arcs in such a way that there are $p$ arrows in total (i.e. $(p \bmod q)$ outer arcs with $\lfloor p/q \rfloor + 1$ arrows and $q - (p \bmod q)$ inner arcs with $\lfloor p/q \rfloor$ arrows), see Figure 3.15 .

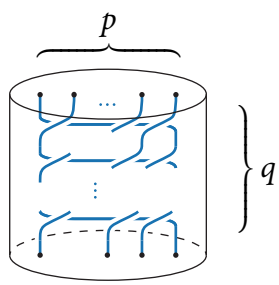



Figure 3.14: A $(p,q)$-curve inside $D \times S^1$.

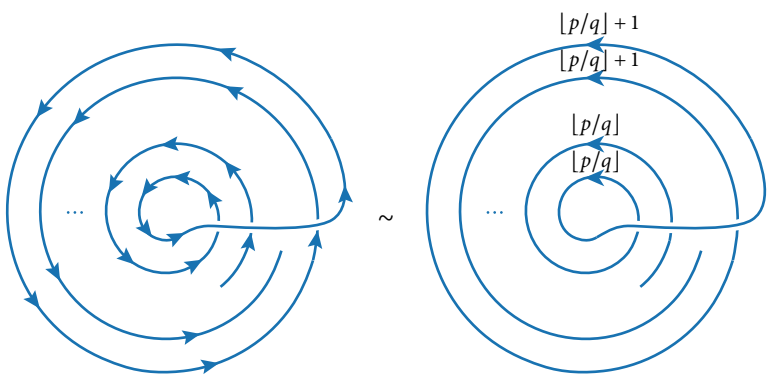



Figure 3.15: A $(p,q)$-curve viewed as an arrow diagram.

The SL move in the punctured disk diagram of Figure 3.9 can be viewed in $D \times S^1$ as the move in Figure 3.16. Note that SL is constructed by attaching (or detaching) a $(p,q)$-curve to $K$.

Passing this construction to the arrow diagram, the slide move SL presents (de)attaching a $(p,q)$-curve as presented in Figure 3.17.

For $q = 1$ the move SL specializes to the move presented in Figure 3.18.

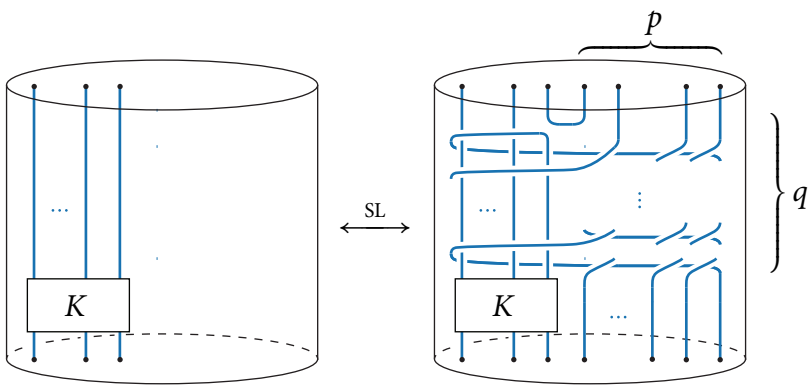



Figure 3.16: The slide move in $D \times S^1$.

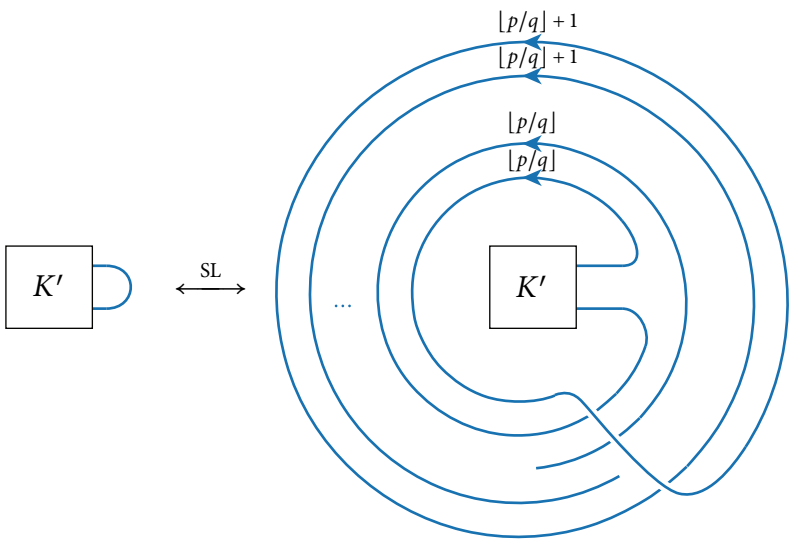



Figure 3.17: The slide move in the arrow diagram of $L(p,q)$.

We take a closer look on how the punctured disk diagrams and the arrow diagrams are related.

### 3.6.2 Transition from punctured disk diagrams to arrow diagrams

A punctured disk diagam of a link $L$ can be viewed as a closed tangle in the punctured plane such as the one presented in Figure 3.19(a). The link $L$ can therefore be presented as a closed tangle inside the solid torus as pictured in Figure 3.19(b). By rotating this tangle inside the cylinder $D \times I$ used in the construction of the arrow diagram (Figure 3.19(c)), the tangle can be projected to $D$ with each strand, used to close the tangle, containing an arrow (since they pass through the meridional disk $D$) as pictured in Figure 3.19(d). Applying R-V moves, all arrows can be moved to the upper endpoints of the tangle (Figure 3.19(f)). Notice that we can isotope the strands, so they go from the upper to the lower endpoints of $T$ making a full negative twist in between (Figure 3.19(g)).

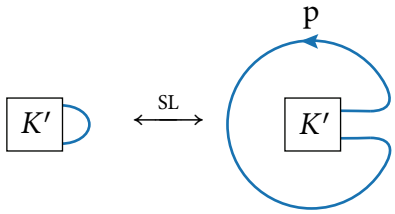



Figure 3.18: The slide move in an arrow diagram of $L(p,1)$.

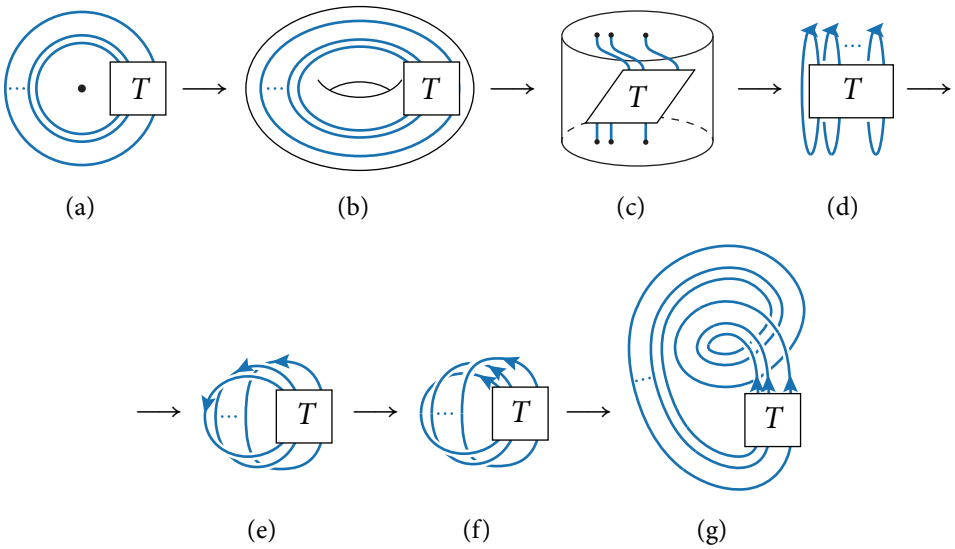



Figure 3.19: Passing from the punctured disk diagram to an arrow diagram.

A similar construction can be used by making a full positive twist and close the tangle on the right side.

### 3.6.3 Transition from arrow diagrams to punctured disk diagrams

An arrow diagram of a link $L$ (Figure 3.20(a)) can be viewed as an almost flat diagram outside small neighborhoods of the arrows, i.e. it lies in a thickened $D\times\{1\}$ in $D\times S^1$. The neighborhoods of arrows correspond to vertical strands parallel to $\{*\}\times S^1$, $*\in D$. By rotating the diagram and closing the vertical strands in the solid torus (Figure 3.20(b)), we can project the $L$ to a the annulus $I\times S^1$. The result is presented in Figure 3.20(c).

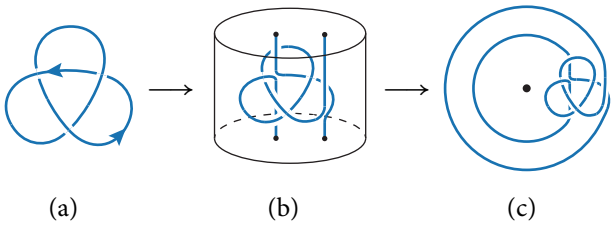



Figure 3.20: Passing from the arrow diagram to the punctured disk diagram.

# 4

# The HOMFLYPT skein module of $L(p,1)$

In this chapter we use arrow diagrams of links in $L(p,1)$ to compute the HOMFLYPT skein module of $L(p,1)$: we show that it is a free $R$-module and we exhibit the sets of generators for each $p \geq 2$ [14]. We also conjecture about the HOMFLYPT skein module of $L(p,q)$, with $q > 1$.

## 4.1 The HOMFLYPT skein module of the solid torus

We translate the free basis of the HOMFLYPT skein module $\mathcal{S}_3(T)$ of the solid torus $T$ (computed in [23, 55]) into diagrams with arrows and establish some properties of these diagrams in the skein module (see Section 2.3.2 for the definition of the HOMFLYPT skein module).

**Theorem 4.1.1.** *$\mathcal{S}_3(T)$ is a free $R$-module, generated by $\mathcal{B} = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, i_1, \dots, i_s \in \mathbb{N}\} \cup \{\varnothing\}$, where $\varnothing$ is the empty knot. For $k > 0$, $t_k$ is the oriented knot in $T$ representing $k$ in $\pi_1(T) \cong \mathbb{Z}$ that has a diagram such as the one pictured in Figure 4.1(a). For $k < 0$, $t_k$ is $t_{|k|}$ with reversed orientation.*

For example, $t_3$ and $t_3 t_{-1}$ are presented in Figures 4.1: in the punctured disk diagrams on the left and in the arrow diagram on the right. Recall that the number above the arrow indicates the number of consecutive arrows.

Let $L$ be a link in the solid torus $T$. We denote by $H_T(L)$ the expression of $L$ in $\mathcal{S}_3(T)$. Thus $H_T(L)$ is a linear expression in elements of $\mathcal{B}$ with coefficients in the ring $R$.

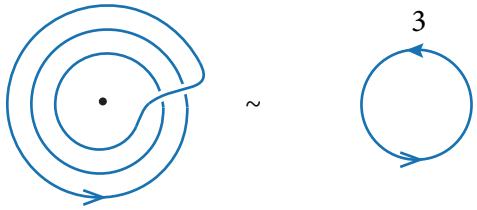
In an arrow diagram, we will refer to a circle with possible arrows on it as an **oval**.

We conclude this section with some properties of $\mathcal{S}_3(T)$ that will be needed later.

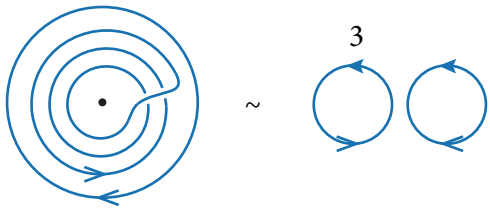
Denote by $\bar{t}_n$, $n > 0$, an oval with clockwise orientation and $n$ clockwise arrows. For $n < 0$, let $\bar{t}_n$ be $\bar{t}_{-n}$ with reversed orientation.

**Lemma 4.1.1.** *In $\mathcal{S}_3(T)$ we can revert clockwise arrows on an oval in the sense that for $n > 0$*

$$\bar{t}_n = \sum_i A_i T_i \quad \text{or} \quad \bar{t}_{-n} = \sum_i A'_i T'_i,$$




(a) $t_3$

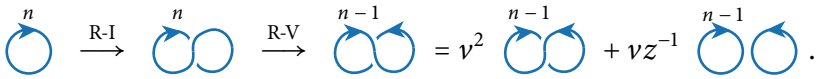



(b) $t_3 t_{-1}$

Figure 4.1: Two generators of $\mathcal{S}_3(T)$.

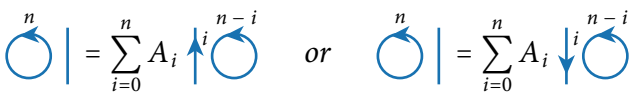
*for some $A_i, A'_i \in R$. Any $T_i$ has the form $t_{k_1}^{i_1} \dots t_{k_s}^{i_s}$, where $k_1 > 0, \dots, k_s > 0$ and $i_1 k_1 + \dots + i_s k_s = n$. Similarly, any $T'_i$ has the form $t_{k_1}^{i_1} \dots t_{k_s}^{i_s}$, where $k_1 < 0, \dots, k_s < 0$ and $i_1 k_1 + \dots + i_s k_s = -n$.*

*Proof.* The proof is by induction on $n$. For $n = 1$, $\overline{t}_1$ is transformed into $t_1$ with a sequence of R-I, R-V, and R-I moves. Similarly, $\overline{t}_{-1}$ is transformed into $t_{-1}$. Suppose that the statement of the lemma is true for $k < n$. Let $c$ be an oval with $n$ clockwise arrows on it. We can perform a R-I move on $c$, then push one arrow on the kink by R-V and then use the HOMFLYPT relation:




We repeat this process until all arrows are pushed to the right side of the kink, which leaves us with terms with either counter-clockwise arrows or at most $n-1$ clockwise arrows. We apply induction on the latter. □

**Lemma 4.1.2.** *Let $D$ be a diagram of a link $L$ with an oval $c$ containing $n$ arrows, $n \in \mathbb{Z}$, and a strand $s$ adjacent to $c$, that may contain arrows outside the drawn region (Figure 4.2). The oval $c$ can be pushed over $s$ in the sense that either*




*holds in $\mathcal{S}_3(T)$ for some $A_i \in R$, depending on the orientation of the components $c$ and $s$.*

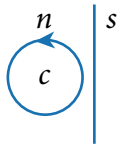
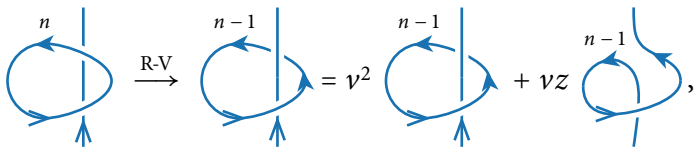
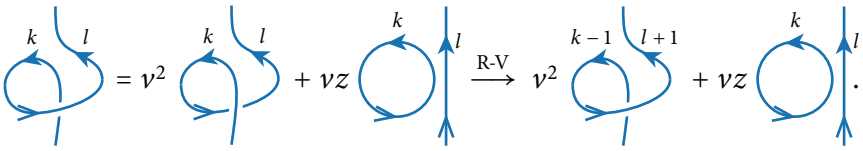



Figure 4.2: A strand adjacent to an oval.

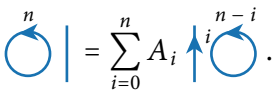
*Proof.* We consider the case where $c$ is oriented counter-clockwise and $s$ is oriented "upwards". For $n = 0$ the proof is trivial. For $n > 0$ we first push an arc of $c$ over $s$ by R-II and successively apply Reidemesiter moves and HOMFLYPT relations until there are no arrows left on the left side of $s$:



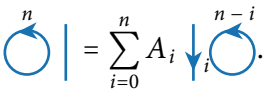

where we resolve the last term by successively applying the following Reidemeister moves and HOMFLYPT relations:




It follows that the link can be expressed as




A similar calculation shows that the above formula holds for any $n < 0$. The formula also holds if we reverse both orientations of $c$ and $s$. If $c$ is oriented clockwise and $s$ is oriented "upwards" or if $c$ is oriented counter-clockwise and $s$ is oriented "downwards", then one gets similarly for any $n \in \mathbb{Z}$:




□

**Lemma 4.1.3** (1- and 2-gon lemma)**.** *Let $D$ be a diagram of a link in $T$ and suppose that either a strand of $D$ forms a 1-gon or two strands of $D$ form a 2-gon (Figure 4.3). Then $D$ can be expressed in $\mathcal{S}_3(T)$ with diagrams which have less crossings than $D$ and no more arrows than $D$.*

*Proof.* Let $d$ be the disk bounded by a 1-gon or a 2-gon. We may assume that there are no 1-gons or 2-gons inside $d$ (in the sense that the discs they bound would be included in $d$). Otherwise, we consider the most nested 1-gon or 2-gon instead (with no 1-gons or 2 gons inside).

(a) 1-gon (b) 2-gon

Figure 4.3: $n$-gons.

**Case 1.** Suppose $d$ bounds a 1-gon. By assumption, there may be only ovals inside $d$, which we push outside of $d$ using Lemma 4.1.2. We continue by pushing the arrows lying on the strand of the 1-gon outside of $d$ using HOMFLYPT relations and R-V. Finally, we remove the crossing of the 1-gon with R-I.

**Case 2.** Suppose $d$ bounds a 2-gon. If there are ovals inside $d$, we push them outside of $d$ using Lemma 4.1.2. We continue by pushing the arrows lying on the strands of the 2-gon outside of $d$ using HOMFLYPT relations and R-V. If the interior of $d$ is not empty, there must be a braid going from one strand of the 2-gon to the other. In this case we push the arrows on the braid outside of $d$ and continue by pushing the crossings of the braid outside of $d$ using HOMFLYPT relations and R-III. We are left with a trivial braid whose strands can be pushed outside of $d$ through the double points of the 2-gon using HOMFLYPT relations and R-III. Once there are no strands inside of $d$, we can perform an R-II move (using possibly a HOMFLYPT relation before) and remove the two crossings of the 2-gon.

□

## 4.2 The construction of $H$ and the main theorem

Let $\mathcal{B}_p = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, -\frac{p}{2} < k_1 < \dots < k_s \leq \frac{p}{2}, i_1, \dots, i_s \in \mathbb{N}\} \cup \{\varnothing\}$ and let $\mathcal{D}$ be the set of arrow diagrams of links in $L(p,1)$.

We will construct a function $H : \mathcal{D} \to R\mathcal{B}_p$. It will be shown that $H$ induces an isomorphism from $\mathcal{S}_3(L(p,1))$ to $R\mathcal{B}_p$.

The function $H$ is defined on standard diagrams of elements of $\mathcal{B}$, such as the one in Figure 4.1, and extended linearly to all diagrams of links in $T$ by the formula $H(D) = H(H_T(D))$, where $H_T(D)$ is to be understood as a linear expression in standard diagrams of elements of $\mathcal{B}$ (and not just a linear expression in these elements).

Let $g$ be a standard diagram of $t_{k_1}^{i_1} \dots t_{k_s}^{i_s} \in \mathcal{B}$. Let $m$ be the maximum of $|k_i|$. If $m < p/2$, or if $m = p/2$ and no $k_i$ equals $-m$, we define $H(g) = g$. Otherwise, let $c$ be a component $t_{-m}$ of $g$ or, if there are no such components, let $c$ be a component $t_m$ of $g$. Let $g'$ be obtained from $g$ by performing an SL move on the component $c$.

We say, that an arrow is **bad** if its orientation is opposite to the orientation of the component on which it lies and **good** otherwise. Notice, that there are less arrows in $g'$ than in $g$ or, if this number is unchanged, there are less bad arrows in $g'$ than in $g$ (this can occur only when $m = p/2$). We recursively define $H(g) = H(H_T(g'))$. From Lemma 4.1.1 and Lemma 4.1.2 it follows that no diagram of $H_T(g')$ has more arrows than $g'$. Note that $H_T(g')$ is computed by

pushing ovals out of the oval $c$ transformed by SL, then possibly changing clockwise arrows into counter-clockwise arrows. Following the proof of the lemmas, it is also clear that no additional bad or good arrows appear in the computation of $H_T(g')$. Thus, all terms in $H_T(g')$ have less arrows than $g$ or the same number of arrows but less bad arrows, and the recursive process in the definition of $H$ must end in finitely many steps.

**Theorem 4.2.1.** *$H$ induces an isomorphism from $\mathcal{S}_3(L(p,1))$ to $R\mathcal{B}_p$. Thus, $\mathcal{S}_3(L(p,1))$ is a free $R$-module with the basis $\mathcal{B}_p$.*

*Proof.* $H_T$ is invariant under all Reidemeister moves except SL, so $H$ is also invariant under these moves. Also, $H_T$ satisfies the HOMFLYPT relation, so $H$ satisfies it as well. $H$ is invariant under *some* SL moves, namely the ones between pairs of diagrams $g$ and $g'$ used in the definition of $H$. Thus, $H$ will be an isomorphism from $\mathcal{S}_3(L(p,1))$ to $R\mathcal{B}_p$ provided it is invariant under *all* SL moves. This fact will be proven in the next section. □

## 4.3 Invariance of $H$ under SL moves

Let $D$ be a diagram consisting of an element of $\mathcal{B}$ to which an oval $e$ is added in such a way that there are no nested ovals in $D$. The oval $e$ has $b \in \mathbb{Z}$ arrows and it may have any orientation. Let $D'$ be obtained from $D$ by an SL move performed on oval $e$. We call such a move from $D$ to $D'$ **standard** (see Figure 4.4).

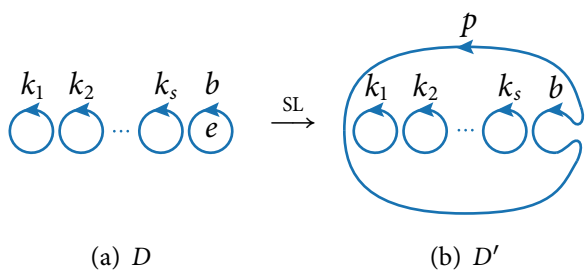



Figure 4.4: A standard SL move.

**Proposition 4.3.1.** *$H$ is invariant under all* SL *moves if it is invariant under all standard* SL *moves.*

*Proof.* Let $D$ be a diagram on which we mark by $s$ a small part on one of its arcs where SL can be performed on; $s$ is chosen in such a manner that no arrows lie on it. Let $D'$ be obtained from $D$ by performing a SL on $s$. It is enough to prove that by leaving the small arc $s$ stationary, we can express $D$ as a linear expression of elements of $\mathcal{B}$ with an added oval $e$ on which $s$ lies.

If $D$ has at least one crossing, then it must contain a 1-gon or a 2-gon. If $s$ lies on it, then it is easy to check, that there must be another 1-gon or 2-gon on which $s$ does not lie ($s$ also does not lie in the disc this 1-gon or 2-gon bounds, otherwise one could not perform an SL move on it). One may therefore apply lemma 4.1.3 to this 1-gon or 2-gon not containing $s$. In this way we will eventually remove all crossings in $D$. Using Lemma 4.1.2, we push all ovals out of the oval on which $s$ lies. Using Lemma 4.1.1, we revert the ovals with clockwise arrows on them, except for the oval on which $s$ lies. All these operations (removal of crossings, pushing and reverting

of ovals) can be performed simultaneously on $D$ and $D'$. In this way the SL move from $D$ to $D'$ can be expressed with several standard SL moves. □

**Proposition 4.3.2.** *$H$ is invariant under all standard* SL *moves.*

*Proof.* We denote by $\mathrm{SL}_n$ a standard SL move from $D$ to $D'$ (such as in Figure 4.4), where one of the diagrams has $n$ arrows and the other one has at most $n$ arrows. Here, we are counting all the arrows as positive, even clockwise arrows (which may only lie on the component on which the move is performed).

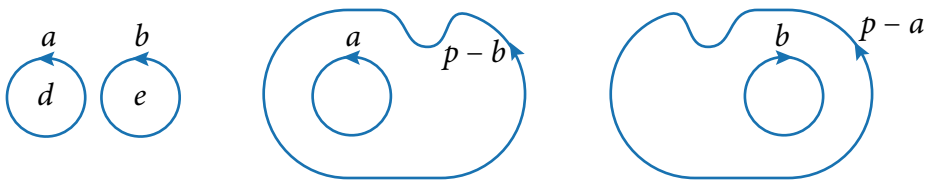
Let $\mathrm{SL}_n$ be a standard move from $D$ to $D'$ and let $e$ be the component of $D$ on which the move is performed. The proof that $H(D) = H(D')$ is by a double induction, on $n$ and $l$, where $l$ is the number of arrows on the components of $D$ except for $e$.

For $n = 0$ there are no SL moves, so the hypothesis is true. Suppose that $n > 0$ and the hypothesis holds for all $k < n$. Notice, that if $l = 0$ there are no arrows on the components outside of $e$, so these components are mapped by $H_T$ to elements in the ring $R$, namely $(v^{-1} - v)z^{-1}$, and $H(D) = H(D')$ holds by definition.

There are 3 cases to consider.

**Case 1.** $D$ has $n$ arrows and $D'$ has less than $n$. If $e$ is a component of $D$ with the maximal number of arrows and, moreover, bad arrows if there is any other component of $D$ the maximal number of bad arrows on it, then $H(D') = H(D)$ by definition (see Figure 4.5(a) and 4.5(b), where only the ovals in question are pictured).

Otherwise, there is a component $d$ of $D$ such that $H(D) = H(D'')$ by definition, where $D''$ is obtained from $D$ by performing an SL move on $d$ (see Figure 4.5(c)). In this case, either $d$ has more arrows than $e$, or it has the same number of arrows as $e$, but the arrows on $d$ are bad and the arrows on $e$ are good.

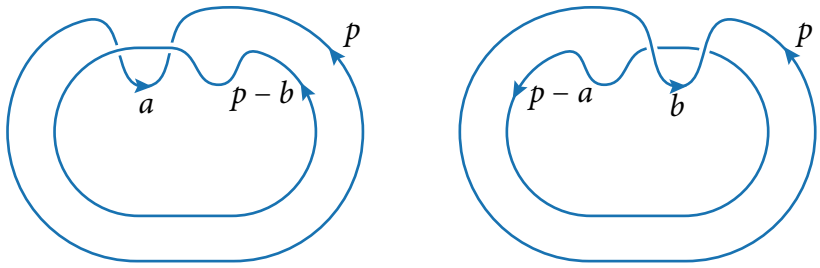



(a) $D$ (b) $D'$ (c) $D''$

Figure 4.5: Diagrams of $D$, $D'$, and $D''$.

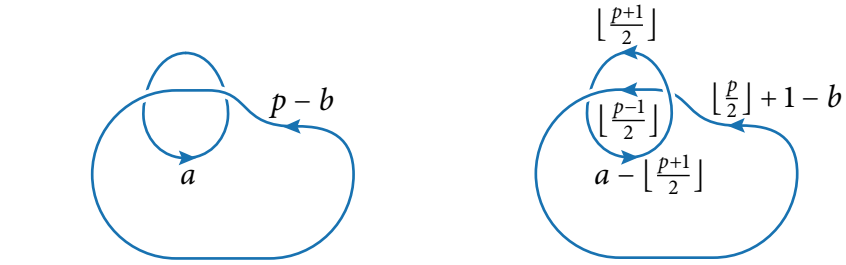
Let $a$ and $b$ be the numbers of arrows on $d$ and $e$, respectively. Let $E'$ be the diagram obtained from $D'$ by applying R-II and SL moves on $d$ (see left side of Figure 4.6). Similarly, let $E''$ be the diagram obtained from $D''$ by applying R-II and SL moves on $e$ (see right side of Figure 4.6). Now, $E'$ and $E''$ are diagrams of the same link in the torus $T$: one gets $E''$ from $E'$ by applying $2p$ times an R-V move on the right crossing of $E'$, thus pushing $p$ arrows next to the $a$ arrows and $p$ arrows away from the $b$ arrows. It holds that $H(E') = H(E'')$.

To show that $H(D') = H(E')$, we first transform $D'$ and $E'$ with R-II and R-V moves into $D'_t$ and $E'_t$ as shown in Figure 4.7.

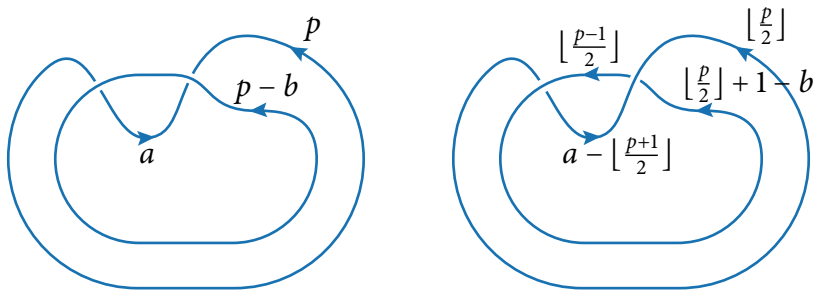



(a) $E'$ (b) $E''$

Figure 4.6: Diagrams of $E$ and $E''$.




(a) $D'$ (b) $D'_t$




(c) $E'$ (d) $E'_t$

Figure 4.7: Diagrams of $D'_t$ and $E'_t$.

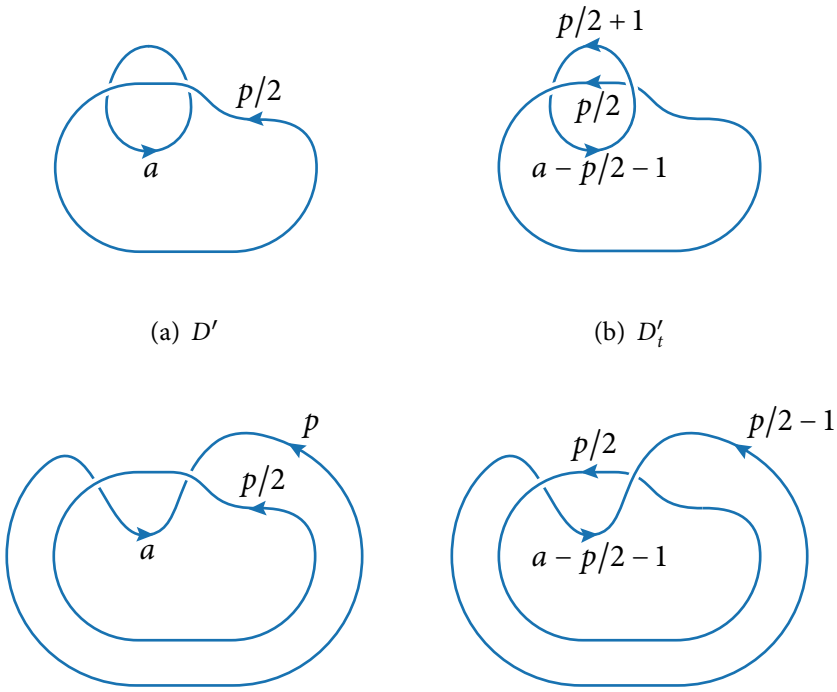
As $D''$ and $D'$ have less arrows than $D$, we have $a \geq b > p/2$. On $D'_t$ the number of arrows on the component $e$ is $\lfloor\frac{p-1}{2}\rfloor + |\lfloor\frac{p}{2}\rfloor + 1 - b| = \lfloor\frac{p-1}{2}\rfloor + b - 1 - \lfloor\frac{p}{2}\rfloor$, which equals $b-1$ if $p$ is odd and $b-2$ if $p$ is even. On component $d$ there are $a$ arrows. Thus the number of arrows on $D'_t$ is $n-1$ or $n-2$. Similarly, on $E'_t$ there are less than $n$ arrows: $b-1$ or $b-2$ on $e$ and $a$ or $a-1$ on $d$.

We claim that $H(E'_t) = H(D'_t)$. Indeed, using the same methods as in Lemma 4.1.2, one can push out the $a-\lfloor\frac{p+1}{2}\rfloor$ arrows on $d$, push any ovals out of $e$ and, using Lemma 4.1.1, revert all ovals with clockwise arrows except for $d$, all this being done without increasing the number of arrows. Thus, $D'_t$ and $E'_t$ are simultaneously expressed with diagrams with no crossings, $D'_t = \sum_i A_i D'_i$ and $E'_t = \sum_i A_i E'_i$, where $A_i \in R$ and $E'_i$ is obtained from $D'_i$ with a standard SL move. As the number of arrows in all $D'_i$ and $E'_i$ is less than $n$, we have $H(D'_i) = H(E'_i)$ by induction on $n$. Thus $H(E') = H(E'_t) = H(D'_t) = H(D')$.

A similar argument shows that $H(E'') = H(D'')$. Indeed, by exchanging the roles of $a$ and $b$ and switching both crossings (which can be done by going around a component with an arrow and applying R-V twice), $E''$ is obtained from $E'$.

Combining preceding equalities, one gets $H(D') = H(D'') = H(D)$.

**Case 2.** $D$ has less than $n$ arrows and $D'$ has $n$ arrows. Recall that $l$ is the number of arrows in $D$ outside $e$. If $l > 0$, there are ovals with arrows other than $e$. We push these ovals through $e$ at the same time in $D$ and $D'$ (i.e. in $D$ we push them into $e$ and in $D'$ we push them out of $e$). From Lemma 4.1.2, in the resulting linear expressions for $D$ and $D'$ two situations can occur: all ovals are pushed through with their arrows, and we can apply Case 1 by reverting the roles of $D$ and $D'$; some arrows are transferred onto $e$, which decreases $l$ and we apply the induction on $l$. In the second case it may happen that the total number of arrows is decreased and the global induction on $n$ can be applied.




(a) $D'$ (b) $D'_t$

(c) $E'$ (d) $E'_t$

Figure 4.8: Diagrams of $D'$, $D'_t$, $E'$, and $E'_t$.

**Case 3.** Both $D$ and $D'$ have $n$ arrows (this occurs when $p$ is even and the number of arrows on $e$ is $b = p/2$). Two subcases are possible:

**Case 3.1.** There is a component $d$ in $D$ with $a$ arrows, $a > b = p/2$, such that $H(D) = H(D'')$ by definition, where $D''$ is obtained from $D$ by performing an SL move on $d$. Using the same notations as in Case 1, one gets again $H(E') = H(E'')$ (see Figure 4.6).

Showing that $H(D'') = H(E'')$ is done exactly as in Case 1: in both cases in order to apply induction on $n$ we only use the fact that $a > p/2$ ($b$ can be equal to $p/2$).

Using several R-V moves $D'$ and $E'$ can be transformed simultaneously into diagrams $D'_t$ and $E'_t$

shown in Figure 4.8. Notice that the number of arrows in $D'_t$ is $n$ (because $a > p/2$ by assumption) and the number of arrows in $E'_t$ is less than $n$. As in Case 1, $D'_t$ and $E'_t$ are simultaneously expressed with diagrams with no crossings, $D'_t = \sum_i A_i D'_i$ and $E'_t = \sum_i A_i E'_i$, where $A_i \in R$ and $E'_i$ is obtained from $D'_i$ with a standard SL move. As the number of arrows in all $E'_i$ is less than $n$, we have $H(D'_i) = H(E'_i)$ by Case 1, if there are $n$ arrows in $D'_i$, or by induction on $n$ otherwise. Thus $H(D'_t) = H(E'_t)$ and $H(D') = H(D'_t) = H(E'_t) = H(E')$.

From preceding equalities it follows that $H(D') = H(D'') = H(D)$.

**Case 3.2.** There are no components in $D$ with more than $p/2$ arrows. If the number of bad arrows is decreased from $D$ to $D'$, then $H(D) = H(D')$ by definition. Otherwise, we use again induction on $l$. If $l > 0$, we push all the ovals into $e$ in $D$ (and out of $e$ in $D'$), as it was done in Case 2, obtaining again two situations: all ovals with their arrows are pushed and $H(D') = H(D)$ by definition; some arrows are transferred onto $e$. In the last case, if the number of arrows on $e$ is still $p/2$, induction on $l$ is applied and if this number is changed, we get Case 1 or Case 2. □

## 4.4 The case of $L(p,q)$

For the case of $L(p,q)$, we construct a function $H : \mathcal{D} \to R\mathcal{B}_{p,q}$, where

$$\mathcal{B}_{p,q} = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, -p/2 < k_1 < \dots < k_s \leq p/2, i_1..i_s \in \mathbb{N}\} \cup \{\varnothing\},$$

and $H$ is defined analogously to that of the case of $L(p,1)$. In short, if $H_T$ is the linear expression in standard diagrams of elements of the basis $\mathcal{B}$ of $\mathcal{S}_3(T)$ and if $g$ is a diagram of $t_{k_1}^{i_1} \dots t_{k_s}^{i_s} \in \mathcal{B}$ and $m = \max\{|k_i|\}_i$, then $H$ is defined recursively as:

$$H(g) = \begin{cases} g, & g \in \mathcal{B}_{p,q} \\ H(H_T(g')), & \text{otherwise,} \end{cases}$$

where $g'$ is the diagram obtained by performing a SL move on the component $t_{-m}$ of $g$, or, if no such components exist, SL is performed on the component $t_m$.

With similar arguments than in the $L(p,1)$ case, it can be shown that the recursive process ends in finitely many steps, so the only thing left to show is that $H$ is invariant under SL moves. Unfortunately the SL moves produce diagrams that are too complicated to apply the methods described in the previous section.

**Conjecture 4.4.1.** *The module $\mathcal{S}_3(L(p,q))$, $q > 1$, is a free $R$-module with the basis:*

$$\mathcal{B}_{p,q} = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, -p/2 < k_1 < \dots < k_s \leq p/2, i_1, \dots, i_s \in \mathbb{N}\} \cup \{\varnothing\}.$$

# 5

# The categorification of the Kauffman bracket skein module of $\mathbb{R}\mathrm{P}^3$

Various generalizations of the Khovanov homology have been constructed so far. One generalization was done by Asaeda, Przytycki, and Sikora [3], who generalized the construction by defining a double graded homology theory that categorifies the Kauffman bracket skein module of links in $I$-bundles over surfaces. Their construction did not work for the twisted bundle $\mathbb{R}\mathrm{P}^2\tilde{\times}I$, the problem being in essence the strange behavior of links projected to the non-orientable surface $\mathbb{R}\mathrm{P}^2$. Another generalization was done by Manturov, who managed to categorify the Jones polynomial of virtual links [37], which, as a special case, includes the categorification of the Jones polynomial of links in $\mathbb{R}\mathrm{P}^3$.

This section provides details and explicitly shows how to categorify KBSM of links in $\mathbb{R}\mathrm{P}^3$, which is equivalent to categorifying KBSM of the twisted $I$-bundle over $\mathbb{R}\mathrm{P}^2$ ($\mathbb{R}\mathrm{P}^2\tilde{\times}I \approx \mathbb{R}\mathrm{P}^3 \setminus \{*\}$), the missing piece of the puzzle in [3]. At the same time, this categorification will also provide a strong invariant of links in $\mathbb{R}\mathrm{P}^3$.

Disk diagrams are used throughout this section.

## 5.1 The Kauffman bracket skein module of $\mathbb{R}\mathrm{P}^3$

It is shown in [24] that the Kauffman bracket skein module of the lens space $L(p,q)$ is a free $R$-module with $\lfloor p/2 \rfloor + 1$ generators, in particular, $\mathcal{S}_{2,\infty}(\mathbb{R}\mathrm{P}^3)$ has two generators, the isotopy class of the empty set $[\varnothing]$ and the class of the orientation reversing curve $[x]$ of $\mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^3$, shown in Figure 5.1 [24][1].

Using notation as in Section 2.3.1, the state-sum formula for expressing $[L] \in \mathcal{S}_{2,\infty}(\mathbb{R}\mathrm{P}^3)$ in the above basis is given by the formula

$$\mathcal{S}_{2,\infty}(\mathbb{R}\mathrm{P}^3)([L]) = \sum_{s\in\{0,1\}^{\mathcal{X}}} A^{\#0(s)-\#1(s)}(-A^2-A^{-2})^{|s|_T}[x]^{|s|_P}[\varnothing]^{1-|s|_P},$$

[1] In [24] they use an equivalent basis $\{[O],[x]\}$.

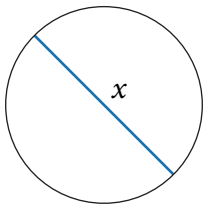



Figure 5.1: A generator of KBSM of $\mathbb{R}\mathrm{P}^3$.

where $|s|_T$ denotes the number of affine trivial circles in $s$ and $|s|_P$ denotes the number of **projective circles** (i.e. non-affine trivial circles, such as the one in Figure 5.1) in $s$. In the above formula we often use the evaluation $[\varnothing] = 1$ and replace $[x]$ by a variable $x$.

## 5.2 The chain complex

As we have already mentioned, Asaeda et al. managed to construct a chain complex of bigraded vector spaces that categorify KBSM of $I$-bundles over surfaces [3], where their construction fails for $\mathbb{R}\mathrm{P}^2\widetilde{\times}I$ due to strange behavior of links projected to $\mathbb{R}\mathrm{P}^2$ (i.e. the existence of $1 \to 1$ bifurcations which will be described latter). Manturov managed to overcome the problem of $1 \to 1$ bifurcations when he categorified the Jones polynomial of virtual links [37].

In this section we define the Khovanov chain complex with gradings similar to those defined by Asaeda et al., but in the next section, we will redefine their differential using techniques of Manturov to control the odd behavior of circles mentioned above.

Since, in this chapter, we will construct the chain complex as a chain complex of $\mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{Z}$-modules, we adjust most of the vocabulary made in section 2.4 accordingly.

Let $W = \bigoplus_{j,k} W_{j,k}$ be a $\mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{Z}$-module. The **Poincaré polynomial** of a $\mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{Z}$-module in variables $A$ and $z$ is defined as

$$\mathcal{P}(W) = \sum_{j,k} A^j z^k \operatorname{rank} W_{j,k}.$$

The standard construction of the Khovanov homology, as well as the construction in [3], uses tensor products to form chain complexes, we, however, use the wedge product (which is in some sense an ordered version of the tensor product).

The **wedge product** $W \wedge W'$ of the modules $W = \bigoplus_{j,k} W_{j,k}$ and $W' = \bigoplus_{j,k} W'_{j,k}$ is defined as

$$W \wedge W' = \bigoplus_{\substack{j=j_1+j_2 \\ k=k_1+k_2}} W_{j_1,k_1} \wedge W'_{j_2,k_2}.$$

The wedge product is associative and anticommutative.

For a permutation $\sigma \in S_n$, the wedge product of $n$ modules $W_1, W_2, \ldots, W_n$ is subject to the **permutation rule**:

$$W_{\sigma(1)} \wedge W_{\sigma(1)} \wedge \cdots \wedge W_{\sigma(n)} = \operatorname{sign}(\sigma)\, W_1 \wedge W_2 \wedge \cdots \wedge W_n.$$

The **degree shift** operator $\cdot\{l,m\}$ shifts the gradings of $W = \bigoplus_{j,k} W_{j,k}$ by $(l,m)$:

$$W\{l,m\} = \bigoplus_{j,k} W_{j-l,k-m}.$$

The degree shift $\cdot\{l,0\}$ is abbreviated to $\cdot\{l\}$, such a shift corresponds to multiplication by $A^l$ in the Poincaré polynomial: $\mathcal{P}(W\{l\}) = A^l\mathcal{P}(W)$.

Let $V = \langle 1, X\rangle$[2] be the bigraded $\mathbb{Z}$-module freely generated by elements 1 and $X$ with bigradings $\deg 1 = (-2,0)$ and $\deg X = (2,0)$, and let $\overline{V} = \langle \overline{1}, \overline{X}\rangle$ be the module generated by $\overline{1}$ and $\overline{X}$ with $\deg \overline{1} = (0,1)$ and $\deg \overline{X} = (0,-1)$. Note that $\mathcal{P}(V) = A^2 + A^{-2}$ and $\mathcal{P}(\overline{V}) = z + z^{-1}$, which is exactly what we assign a circle (trivial or projective, respectively) in the state-sum formula of KBSM.

Recall from Section 2.2 that for the set of crossings $\mathcal{X} = \{1,2,\dots,n\}$ of the diagram $D$ of a link $L$, a state $s$ is an element of the discrete cube $\{0,1\}^{\mathcal{X}}$ and corresponds to a complete smoothening of $D$. The number of 0 and 1 factors in $s$ is denoted by $\#0(s)$ and $\#1(s)$, respectively. Assuming the circles are enumerated in a way that the possible projective circle is at the end, $C_s$ will represent the module associated with the state $s$. A circle in $s$, enumerated by $i$, contributes a factor $V_i$ if the circle is trivial or a factor $\overline{V}_i$ if the circle is projective. The module $C_s$ is therefore either equal to

$$C_s = (V_1 \wedge V_2 \wedge \cdots \wedge V_n)\{\#0(s) - \#1(s)\}$$

or is equal to

$$C_s = (V_1 \wedge V_2 \wedge \cdots \wedge V_{n-1} \wedge \overline{V}_n)\{\#0(s) - \#1(s)\},$$

depending on whether $s$ has a projective circle or not.

The module of $i$-chains is now defined to be the sum of all $C_s$'s where the difference between the number of 0 and 1 factors in $s$ is $i$:

$$\mathcal{C}_i = \bigoplus_{\substack{s\in\{0,1\}^{\mathcal{X}} \\ \#0(s)-\#1(s)=i}} C_s.$$

Forming such direct sums of modules is sometimes called a **flattening** of $\{C_s\}_s$. We claim that

$$0 \longrightarrow \mathcal{C}_n \longrightarrow \mathcal{C}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-n+2} \longrightarrow \mathcal{C}_{-n} \longrightarrow 0$$

with suitable differentials yet to be defined, forms a Khovanov chain complex.

## 5.3 The differential

Let $\xi \in \{0,1,\star\}^{\mathcal{X}}$ be a sequence of length $n$ with the property that $\star$ appears only once in $\xi$. By replacing $\star$ by 0 we get a vertex of the discrete cube $\{0,1\}^{\mathcal{X}}$ denoted by $\xi_{\star\to 0}$ and by replacing $\star$ by 1 we get an adjacent vertex $\xi_{\star\to 1}$. We call such a $\xi$ an **edge** of $\{0,1\}^{\mathcal{X}}$. Recall that each vertex corresponds to a state of the diagram, that being so, each edge corresponds to a local "change"

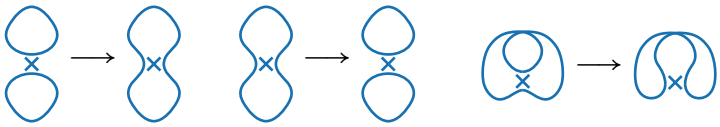
[2] The symbols 1 and $X$ play a similar role than $v_-$ and $v_+$ in Section 2.4. The choice of using 1 and $X$ instead of $v_\pm$ is just a personal preference of the author and is the notation used in several papers.

of two adjacent states. Each $\xi$ is associated with a linear map $d_\xi : C_{\xi_{\star\to 0}} \to C_{\xi_{\star\to 1}}$ called a **partial differential**, which will be defined in the next few paragraphs.

For a diagram $D$, we call the collection of modules $\{C_s\}_{s\in\{0,1\}^{\mathcal{X}}}$, together with their partial differentials, the **cube** $[\![D]\!]$ of $D$. Now, $[\![D]\!]$ needs to form an anticommutative diagram, since this forces the flattened $[\![D]\!]$ to form a well-defined chain complex $(\mathcal{C}_\bullet, \partial_\bullet)$. As seen in [3], there is no obvious way to define the partial differential for links in $\mathbb{R}\mathrm{P}^3$, but as shown in [37] this is possible to achieve for the Jones polynomial, if the links and the circles in the states are oriented and certain signs are applied to the partial differentials.

Hence, we orient both the diagram $D$ and the circles of the states in an arbitrary manner. In the neighborhood of a crossing there are either one or two circles. If we change the smoothening in a crossing from type 0 to type 1, we call the change of the involved circles a **bifurcation**. It is evident that the circles involved in the states $\xi_{\star\to 0}$ and $\xi_{\star\to 1}$ are in a bijective correspondence, with one of these exceptions (Figure 5.2):

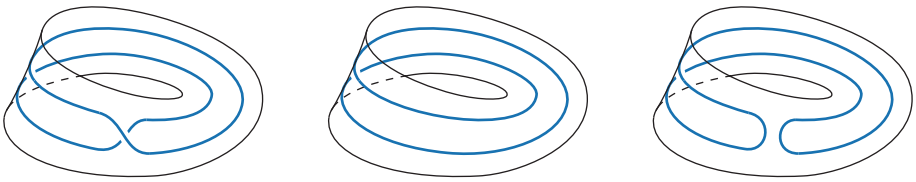
1. two circles in $\xi_{\star\to 0}$ join into one circle in $\xi_{\star\to 1}$ (type $2 \to 1$ bifurcation),
2. a circle in $\xi_{\star\to 0}$ splits into two circles in $\xi_{\star\to 1}$ (type $1 \to 2$ bifurcation),
3. a circle in $\xi_{\star\to 0}$ twists into a circle in $\xi_{\star\to 1}$ (type $1 \to 1$ bifurcation).



(a) type $2 \to 1$ (b) type $1 \to 2$ (c) type $1 \to 1$

Figure 5.2: Three types of bifurcations.

Note that the type $1 \to 1$ bifurcation can only appear when the circle lies within a Möbius band and can therefore only appear in the case of projecting to a non-orientable surface, such as $\mathbb{R}\mathrm{P}^2$ (see Figure 5.3).



(a) $L$ (b) state 0 of $L$ (c) state 1 of $L$

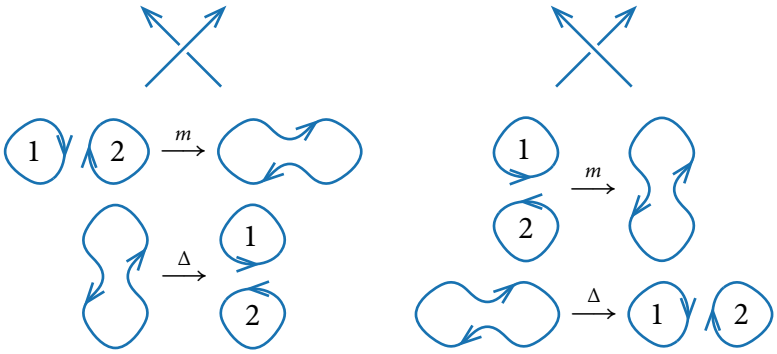
Figure 5.3: The appearance of a $1 \to 1$ bifurcation on a Möbius band.

Looking at a crossing $c$ of $D$ in such a way that the two outgoing arcs are facing northwest and northeast and that the ingoing arcs are facing southwest and southeast, we call a circle of some state **locally consistently oriented** at $c$ if its orientation agrees with the orientation of the northeast arc or disagrees with the orientation of the southwest arc (with respect to the arc before the smoothening); vice versa, a circle is **locally inconsistently oriented** at $c$ if its orientation disagrees with the northeast arc or agrees with the southwest arc (see Figure 5.4). It

can happen that both arcs in the state belong to the same circle and the consistency cannot be uniquely defined, in this case the local consistency is **indetermined** at $c$.

Figure 5.4: Locally consistent orientations.

In order to define the differential, we rearrange the $\wedge$-factors of $C_{\xi_{\star\to 0}}$ by a permutation $\sigma \in S_{|\xi_{\star\to 0}|}$ in such a way that the factors involved in the bifurcation are at the beginning of the wedge product. Furthermore, if the bifurcation is of type $2 \to 1$, assuming that the arcs of the diagram are facing northwest and northeast, we wish that the first factor in the domain is represented by a "left" circle if the bifurcation site is at a positive crossing and is represented by a "top" circle if the bifurcation site is at a negative crossing (see Figure 5.5).




(a) positive crossing (b) negative crossing

Figure 5.5: Order of (locally consistently oriented) circles of a bifurcation.

If the bifurcation is of type $2 \to 1$, we **multiply** the first two factors; if the bifurcation is of type $1 \to 2$, we **comultiply** the first factor; and if the bifurcation is of type $1 \to 1$ we apply the 0 map to the first factor. On the remaining factors we apply the semi-identity $\mathcal{I} = I \wedge I \wedge \cdots \wedge I\,(\wedge \overline{I})$, where the map $I : V \to V$ (resp. $\overline{I} : \overline{V} \to \overline{V}$) is defined by $I(1) = 1, I(X) = \pm X$ (resp. $\overline{I}(\overline{1}) = \overline{1}, \overline{I}(\overline{X}) = \pm\overline{X}$), with a plus sign on the generator $X$ if the circles that two $V$'s (resp. $\overline{V}$'s) represent have the same orientations and with a minus sign if the orientations are opposite.

Let $m$ be the **multiplication operator** and $\Delta$ the **comultiplication operator**, they are both linear maps subject to the rules in Table 5.1.

As suggested in Figure 5.5, the order of the factors in the codomain of $\Delta$ depend on the position of the circles they represent. At a positive crossing the first factor is presented by the "top" circle and at a negative crossing the first factor is represented by a "left" circle.

The sign of each factor $X$ (resp. $\overline{X}$) in Table 5.1 depends on the local consistency of the circle that $X$ (resp. $\overline{X}$) represents at the crossing where the bifurcation appears. If the circle is locally consistently oriented at the crossing, the sign is positive and if the circle is locally inconsistently oriented, the sign is negative. Note that the consistency is indetermined only at bifurcations of type $1 \to 1$.

Table 5.1: Multiplication and comultiplication (shifts omitted).

| | |
|---|---|
| $m: \begin{array}{ccccc} V_1 & \wedge & V_2 & \longrightarrow & V \\ 1_1 & \wedge & 1_2 & \longmapsto & 1 \\ 1_1 & \wedge & (\pm X_2) & \longmapsto & (\pm X) \\ (\pm X_1) & \wedge & 1_2 & \longmapsto & (\pm X) \\ (\pm X_1) & \wedge & (\pm X_2) & \longmapsto & 0 \end{array}$ | $\Delta: \begin{array}{ccccc} V & \longrightarrow & V_1 & \wedge & V_2 \\ 1 & \longmapsto & 1_1 & \wedge & (\pm X_2) + (\pm X_1) \wedge 1_2 \\ (\pm X) & \longmapsto & (\pm X_1) & \wedge & (\pm X_2) \end{array}$ |
| $m: \begin{array}{ccccc} \overline{V}_1 & \wedge & V_2 & \longrightarrow & \overline{V} \\ \bar{1}_1 & \wedge & 1_2 & \longmapsto & \bar{1} \\ \bar{1}_1 & \wedge & (\pm X_2) & \longmapsto & 0 \\ (\pm \overline{X}_1) & \wedge & 1_2 & \longmapsto & (\pm \overline{X}) \\ (\pm \overline{X}_1) & \wedge & (\pm X_2) & \longmapsto & 0 \end{array}$ | $\Delta: \begin{array}{ccccc} \overline{V} & \longrightarrow & \overline{V}_1 & \wedge & V_2 \\ \bar{1} & \longmapsto & \bar{1}_1 & \wedge & (\pm X_2) \\ (\pm \overline{X}) & \longmapsto & (\pm \overline{X}_1) & \wedge & (\pm X_2) \end{array}$ |
| $m: \begin{array}{ccccc} V_1 & \wedge & \overline{V}_2 & \longrightarrow & \overline{V} \\ 1_1 & \wedge & \bar{1}_2 & \longmapsto & \bar{1} \\ 1_1 & \wedge & (\pm \overline{X}_2) & \longmapsto & (\pm \overline{X}) \\ (\pm X_1) & \wedge & \bar{1}_2 & \longmapsto & 0 \\ (\pm X_1) & \wedge & (\pm \overline{X}_2) & \longmapsto & 0 \end{array}$ | $\Delta: \begin{array}{ccccc} \overline{V} & \longrightarrow & V_1 & \wedge & \overline{V}_2 \\ \bar{1} & \longmapsto & (\pm X_1) & \wedge & \bar{1}_2 \\ (\pm \overline{X}) & \longmapsto & (\pm X_1) & \wedge & (\pm \overline{X}_2) \end{array}$ |

After (co)multiplying and applying the identity, we rearrange the factors in the result by the permutation $\rho \in S_{|\xi_{\star\to 1}|}$, so that they agree with the order of factors in the codomain $C_{\xi_{\star\to 1}}$.

The above steps describe the partial differential $d_\xi : C_{\xi_{\star\to 0}} \to C_{\xi_{\star\to 1}}$. In detail, if $P_\sigma$ is a map that rearranges factors in $C_{\xi_{\star\to 0}}$ by $\sigma$ and $P_\rho$ is the map that rearranges factors in $C_{\xi_{\star\to 1}}$ by $\rho$, then

$$d_\xi = \begin{cases} P_\rho \circ (m \wedge \mathcal{I}) \circ P_\sigma, & \text{if } |\xi_{\star\to 0}| = |\xi_{\star\to 1}| + 1, \\ P_\rho \circ (\Delta \wedge \mathcal{I}) \circ P_\sigma, & \text{if } |\xi_{\star\to 0}| + 1 = |\xi_{\star\to 1}|, \\ P_\rho \circ (0 \wedge \mathcal{I}) \circ P_\sigma, & \text{if } |\xi_{\star\to 0}| = |\xi_{\star\to 1}|. \end{cases} \tag{5.1}$$

The **total differential** $\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-2}$ is the sum of partial differentials:

$$\partial_i = \bigoplus_{\substack{\xi \in \{0,1,\star\}^{\mathcal{X}} \\ \#0(\xi) - \#1(\xi) = i-1}} d_\xi.$$

We shall call such a sum a **flattening** of partial differentials.

As an illustration of how the partial and total differentials work, a detailed example is presented at the end of the next section.

## 5.4 The homology

The $i$-**th Khovanov homology group** is

$$\mathcal{H}_i = \frac{\operatorname{Ker} \partial_i}{\operatorname{Im} \partial_{(i+2)}}.$$

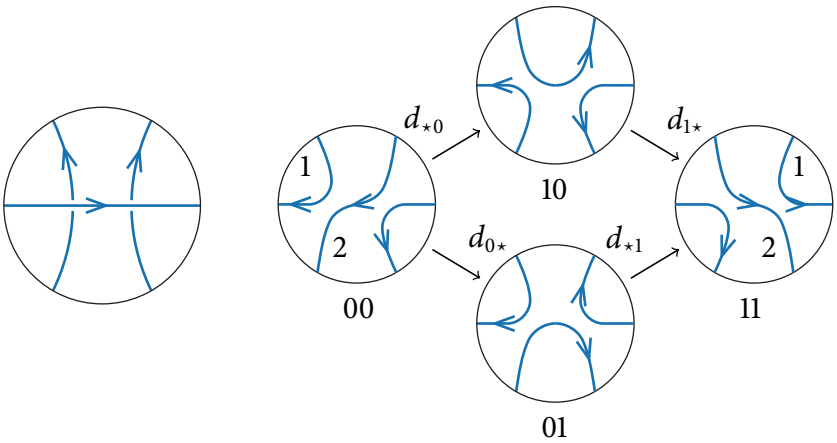
**Theorem 5.4.1** (proof in section 5.5)**.** *Let $L$ be a link in $\mathbb{R}P^3$ and $D$ a diagram of $L$. Then $\partial \circ \partial = 0$ so that $(\mathcal{C}_\bullet, \partial_\bullet)$ is a chain complex.*

**Theorem 5.4.2** (proof in section 5.5)**.** *For a diagram $D$ of the link $L \subset \mathbb{R}P^3$, $\mathcal{H}_\bullet(D)$ is preserved under* R-II, R-III, R-IV, *and* R-V. *Therefore $\mathcal{H}_\bullet$ is an invariant of framed links in $\mathbb{R}P^3$.*

The **Euler characteristic** of a $\mathbb{Z}^3$-graded $\mathbb{Z}$-module $W = \bigoplus_{i,j,k}(W_i)_{j,k}$ is for the purpose of this thesis defined to be

$$\chi(W) = \sum_{i,j,k} (-1)^{\frac{j-i}{2}} A^j z^k \operatorname{rank}(W_i)_{j,k}.$$

Since we have adjusted our gradings so that they agree with the state-sum formula $L$ in $\mathcal{S}_{2,\infty}(\mathbb{R}\mathrm{P}^3)$, it follows from the construction that $\chi(\mathcal{C}_\bullet(D)) = \chi(\mathcal{H}_\bullet(D)) = \mathcal{S}_{2,\infty}(\mathbb{R}\mathrm{P}^3)(L)$, where the last equation holds by substituting $z + z^{-1}$ with $[x]$.




(a) link $L$ (b) cube of $L$

Figure 5.6: The link $L$ and its cube.

**Example 5.4.3.** *In the link in Figure 5.6 we get:*

$$C_{00} = V_1 \wedge \overline{V}_2\{2\}, \quad C_{01} = \overline{V}, \quad C_{10} = \overline{V}, \quad C_{11} = V_1 \wedge \overline{V}_2\{-2\}.$$

*Omitting shifts, partial differentials work in the following manner:*

$$\begin{array}{llll}
d_{0\star}(1_1 \wedge \overline{1}_2) = \overline{1}, & d_{0\star}(-1_1 \wedge \overline{X}_2) = \overline{X}, & d_{0\star}(X_1 \wedge \overline{1}_2) = 0, & d_{0\star}(-X_1 \wedge \overline{X}_2) = 0; \\
d_{\star 0}(\overline{1}_2 \wedge 1_1) = \overline{1}, & d_{\star 0}(-\overline{1}_2 \wedge X_1) = 0, & d_{\star 0}(\overline{X}_2 \wedge 1_1) = -\overline{X}, & d_{\star 0}(-\overline{X}_2 \wedge X_1) = 0; \\
d_{\star 1}(\overline{1}) = X_1 \wedge \overline{1}_2, & d_{\star 1}(-\overline{X}) = -X_1 \wedge \overline{X}_2; & d_{1\star 0}(\overline{1}) = -\overline{1}_2 \wedge X_1 v, & d_{1\star 0}(\overline{X}) = -\overline{X}_2 \wedge X_1.
\end{array}$$

*The total differential is:*

$$\begin{aligned}
&\partial_2(1_1 \wedge \bar{1}_2) = \bar{1} \oplus (-\bar{1}), && \partial_2(1_1 \wedge \overline{X}_2) = (-\overline{X}) \oplus \overline{X}, && \partial_2(X_1 \wedge \bar{1}_2) = 0 \oplus 0,\\
&\partial_2(X_1 \wedge \overline{X}_2) = 0 \oplus 0; && \partial_0(\bar{1} \oplus 0) = X_1 \wedge \bar{1}_2, && \partial_0(\overline{X} \oplus 0) = X_1 \wedge \overline{X}_2,\\
&\partial_0(0 \oplus \bar{1}) = X_1 \wedge \bar{1}_2, && \partial_0(0 \oplus \overline{X}) = X_1 \wedge \overline{X}_2; && \partial_{-2} = 0.
\end{aligned}$$

*It clearly holds that $\partial \circ \partial = 0$. The homology groups are:*

$$\mathcal{H}_2 \cong \langle X_1 \wedge \bar{1}_2, X_1 \wedge \overline{X}_2 \rangle \{2\}, \quad \mathcal{H}_0 \cong 0, \quad \mathcal{H}_{-2} \cong \langle 1_1 \wedge \bar{1}_2, 1_1 \wedge \overline{X}_2 \rangle \{-2\},$$

*i.e. the only non-trivial dimensions of $\mathcal{H}_\bullet$ are*

$$(\mathcal{H}_2)_{4,1} \cong (\mathcal{H}_2)_{4,-1} \cong (\mathcal{H}_{-2})_{-4,1} \cong (\mathcal{H}_{-2})_{-4,-1} \cong \mathbb{Z}.$$

*The Euler characteristic is*

$$\chi(\mathcal{H}_\bullet) = (-A^4 - A^{-4})(z + z^{-1}) = (-A^4 - A^{-4})[x].$$

## 5.5 Proofs

Lemmas 5.5.1 – 5.5.3 will show that the homology is independent of the free choices we made on choosing the orderings of circles, the orientations of the link, and the orientations of the circles in the states, respectfully.

**Lemma 5.5.1.** *The homology is invariant under the ordering of circles.*

*Proof.* For each state $s$ of $D$, let $o_s$ and $o'_s$ be two different orderings of circles in $s$. Let $o_s$ and $o'_s$ differ by a permutation $\pi_s$. Permutations $\pi_s$ induce isomorphisms $P_{\pi_s} : C_s \to C_s$ on the associated modules. Let, for two adjacent states $q = \xi_{\star\to 0}$ and $r = \xi_{\star\to 1}$, the partial differential $d_\xi : C_q \to C_r$ be defined in terms of the first ordering as $P_{\rho_r} \circ d \circ P_{\sigma_q}$, where in place of $d$'s we have either $m \wedge \mathcal{I}$, $\Delta \wedge \mathcal{I}$, or 0. It follows from the definition that the differential in terms of the second ordering equals $P_{\pi_r \rho_r} \circ d \circ P_{\sigma_q \pi_q^{-1}}$. Since $P_{\pi_r \rho_r} \circ d \circ P_{\sigma_q \pi_q^{-1}} \circ P_{\pi_q} = P_{\pi_r} \circ P_{\rho_r} \circ d \circ P_{\sigma_q}$, it follows that the maps $\{P_{\pi_s}\}_s$ form a chain map. □

**Lemma 5.5.2.** *The homology is invariant under the change of link orientation.*

*Proof.* It is enough to prove invariance under changing the orientation of one component. Let $c$ be a crossing of the changed component. If $c$ is a self-crossing, changing the orientation of both strands of $c$ has the same effect as transposing the orderings of the circles that split at $c$ (see Figure 5.7), which is invariant by Lemma 5.5.1. If $c$ is not a self-crossing, changing the orientation of one component leads to inverting the locally consistent orientation on all participating circles and transposing the ordering of the two circles if the bifurcation site is of type $1 \to 2$ (see Figure 5.8); by writing down the partial differentials before and after the orientation change, one has to check that they both agree.

We check the differential at a $1 \to 2$ bifurcation in the case of a negative crossing, other cases are easier to check and are treated similarly.

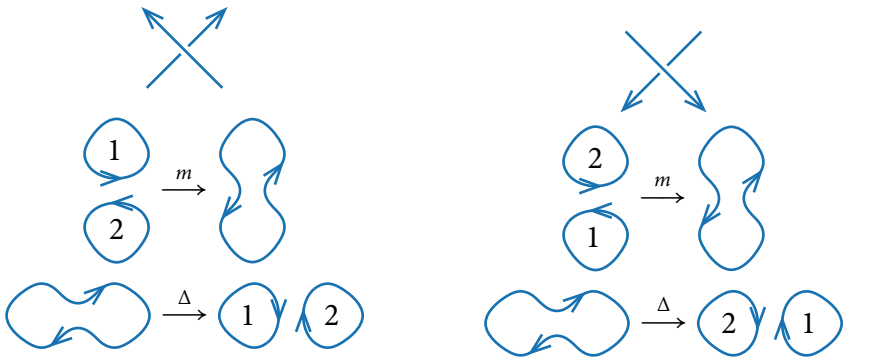
The partial differential before the change is the map:

$$\begin{aligned} \Delta : \ 1 &\longmapsto 1_1 \wedge (\pm X_2) + (\pm X_1) \wedge 1_2, \\ \pm X &\longmapsto \pm X_1 \wedge \pm X_2, \end{aligned}$$

after the change of orientation the map agrees with the above (see Figure 5.8):

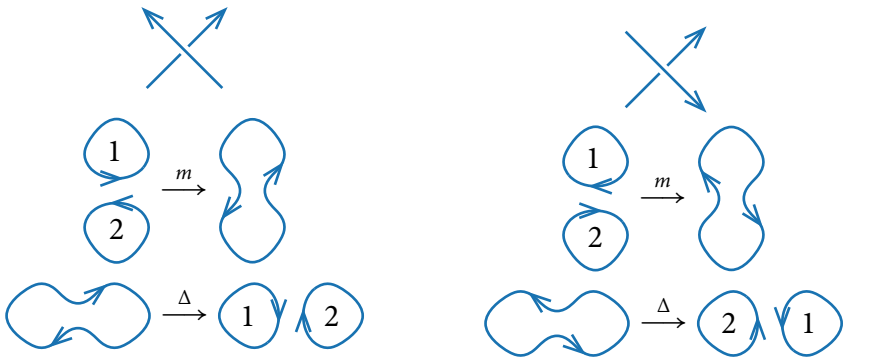
$$\begin{aligned} \Delta : 1 &\longmapsto 1_2 \wedge (\mp X_1) + (\mp X_2) \wedge 1_1 = 1_1 \wedge (\pm X_2) + (\pm X_1) \wedge 1_2, \\ \pm X &\longmapsto -(\mp X_2 \wedge \mp X_1) = \mp X_1 \wedge \mp X_2 = \pm X_1 \wedge \pm X_2. \end{aligned}$$

□




(a) possible positions of circles at $c$ (b) crossing $c$ with changed orientation

Figure 5.7: Order of locally consistently oriented circles before and after a change of orientation at a self-crossing.




(a) possible positions of circles at $c$ (b) crossing $c$ with a changed orientation

Figure 5.8: Order of locally consistently oriented circles before an after a change of orientation of a component at a crossing.

**Lemma 5.5.3.** *The homology is invariant under the change of orientations of the circles in the states.*

*Proof.* Let $f_s : C_s \to C_s$ be the semi-identity that sends $X$ to $-X$ and 1 to 1 (resp. $\overline{X}$ to $-\overline{X}$ and $\overline{1}$ to $\overline{1}$) on the $V$ (resp. $\overline{V}$) components of $C_s$ that represent circles with changed orientations. We observe that changing the orientation of a circle changes the local consistency in the definition of the partial differential $d$. But this is exactly the opposite of what $f$ does, so $f \circ d \circ f^{-1} = d$. □

*Proof of theorem 5.4.1.* To show that $\partial \circ \partial = 0$ holds for a projective link diagram $D$, it is sufficient to prove that every 2-dimensional face of $[\![D]\!]$ is anticommutative. Every 2-dimensional face corresponds to a smoothening of $n-2$ crossings, since the remaining two crossings are resolved in four different ways. Since the circles that are non-adjacent to either of the two crossings are mapped by $\mathcal{I}$, they can be omitted in the proofs. It therefore suffices to prove anticommutativity for all possible 2-crossing projective diagrams. For constructing all such diagrams, we take a similar approach as in [3, 37]. Here is an outline.

Take all 4-valent graphs in $\mathbb{R}\mathrm{P}^2$ with 2 rigid vertices and replace each vertex with either an overcrossing or an undercrossing. This produces all possible 2-crossing diagrams. Certain graphs can be omitted due to symmetries that leave circles in a natural bijective correspondence preserved under partial differentials. Take vertices $v$ and $u$ and enumerate their edges by $v_0, v_1, v_2, v_3$, and $u_0, u_1, u_2, u_3$ (Figure 5.9).

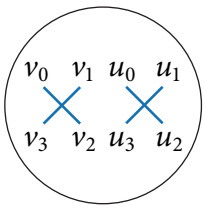



Figure 5.9: Vertices $v$ and $u$.

We disregard non-connected graphs, since it clearly holds that in this case either $d_{0\star} = -d_{1\star}$, $d_{\star 0} = d_{\star 1}$ or $d_{0\star} = d_{1\star}$, $d_{\star 0} = -d_{\star 1}$. Without loss of generality, we may assume that $v_0$ is connected to $v_1$. Two graphs are **symmetric** if they are related by the following operations [3]:

1. Exchanging $v$ and $u$ produces the same graph.
2. A cyclic permutation $v_i \to v_{i-k \bmod 4}$ and $u_i \to u_{i-k \bmod 4}$ for $i \in \{0,1,2,3\}$ if $v_k$ and $u_k$ are connected for some $k \in \{0,1,2,3\}$.
3. A flip: either $v_i \leftrightarrow v_{i+2 \bmod 4}$ or $u_i \leftrightarrow u_{i+2 \bmod 4}$ are exchanged for some $i \in \{0,1,2,3\}$.

The last relation follows from the fact that if link diagrams $D_1$ and $D_2$ are represented by graphs $\Gamma_1$ and $\Gamma_2$ related by a flip, then there is a natural bijective correspondence between the states of $D_1$ and the states of $D_2$. This correspondence is preserved by changing of smoothing and partial differentials.

**Lemma 5.5.4.** *There are exactly 6 non-symmetric graphs in $\mathbb{R}P^2$. The graphs are represented in Figure 5.10.*

To list all the possible 2-crossing links, we place all 4 possible combinations of crossing on each vertex of the above graphs. We can omit links where the $1 \to 1$ bifurcation appears in each of the composites of the partial differentials, since in this case, it clearly holds that $\partial \circ \partial = 0$.

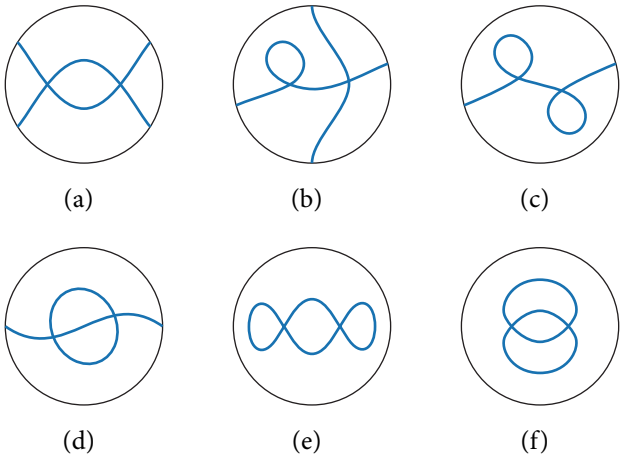



Figure 5.10: Non-symmetric 4-valent connected graphs with 2 vertices in $\mathbb{R}P^2$.

**Lemma 5.5.5.** *Up to symmetry and disregarding non-essential cases with two* $1 \to 1$ *bifurcations appearing on opposite sides of the composites, there are 11 projective links with two crossings that are represented in Figure 5.11.*

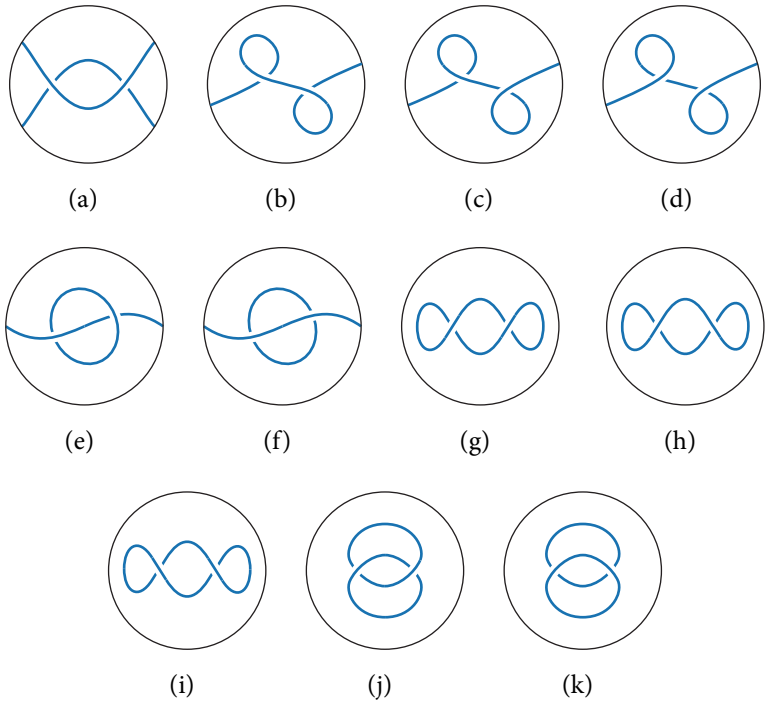
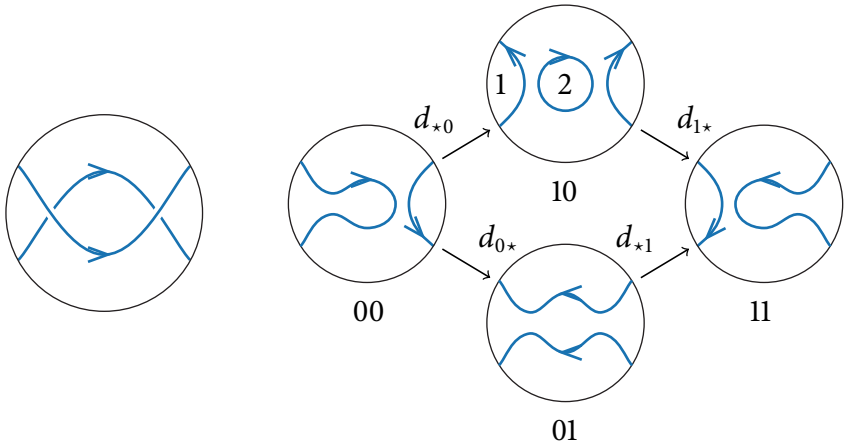



Figure 5.11: Non-symmetric essential links with 2 crossings in $\mathbb{R}P^2$.

The proofs of Lemma 5.5.4 and Lemma 5.5.5 rely on an extensive case-by-case checking and are left to the reader.




(a) link $L$ (b) cube of $L$

Figure 5.12: The link $L$ of Figure 5.11(a) and its cube.

Due to Lemma 5.5.1, Lemma 5.5.2 and Lemma 5.5.3, we can orient the above 11 links and their states arbitrarily, order the circles arbitrarily and calculate the differentials to show that $\partial \circ \partial = 0$. We check anticommutativity only for the link in Figure 5.11(a), the rest can be checked similarly.

From Figure 5.12 we get

$$\begin{aligned} d_{\star 0}(1) &= 1_2 \wedge X_1 - X_2 \wedge 1_1, \\ d_{\star 0}(-X) &= -X_2 \wedge X_1; \\ d_{1\star}(1_1 \wedge 1_2) &= 1, \\ d_{1\star}(-1_1 \wedge X_2) &= X, \\ d_{1\star}(-X_1 \wedge 1_2) &= X, \\ d_{1\star}(X_1 \wedge X_2) &= 0; \\ d_{0\star} = d_{\star 1} &= 0, \end{aligned}$$

hence $d_{\star 0} d_{1\star} = d_{0\star} d_{\star 1} = 0$. □

*Proof of theorem 5.4.2.* The proofs of invariance under moves R-II and R-III almost entirely coincide with those of the classical case [4]. The main tool used for showing invariance is the following cancellation principle:

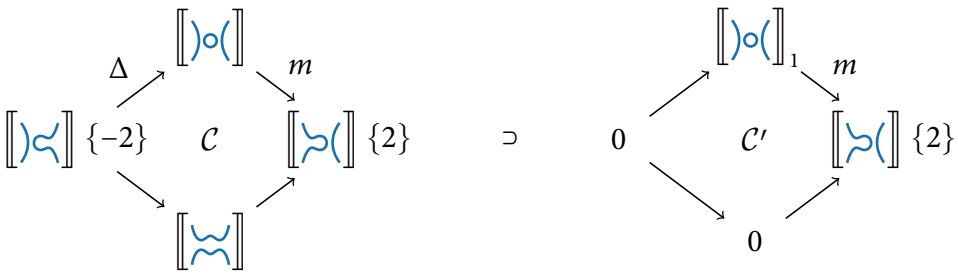
**Lemma 5.5.6.** *Let $C$ be a chain complex and let $C' \subset C$ be a sub-chain complex. Then, if $C'$ is acyclic it holds that $H(C) = H(C/C')$, on the other hand, if $C/C'$ is acyclic it holds that $H(C) = H(C')$.*

*Proof.* Both equalities follow trivially from the long exact homology sequence

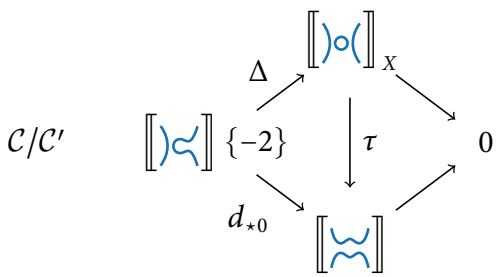
$$\cdots \longrightarrow H_n(C') \longrightarrow H_n(C) \longrightarrow H_n(C/C') \longrightarrow H_{n+1}(C) \longrightarrow \cdots$$

associated with the short exact sequence $0 \longrightarrow C' \longrightarrow C \longrightarrow C/C' \longrightarrow 0$. □

**Invariance under R-II.** The cube of $[\![\,]\!]$ can be expressed in terms of subcubes as indicated on the left diagram below.

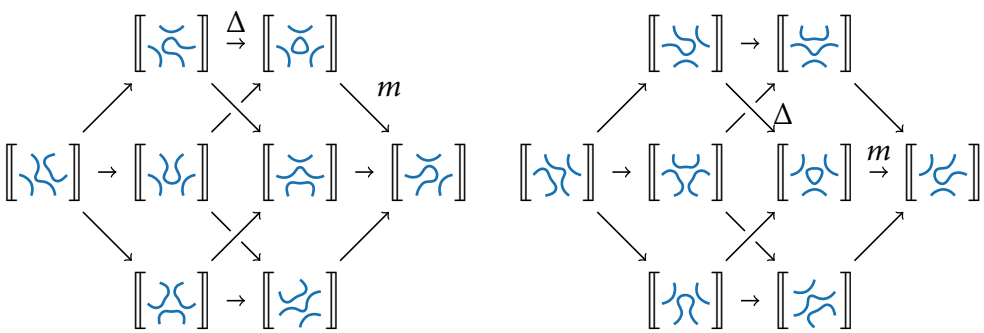



Let $\mathcal{C}$ be the flattening of this cube. The chain complex $\mathcal{C}$ contains a subcomplex $\mathcal{C}'$ in which the subscript 1 of $[\![\,]\!]_1$ denotes the cube of submodules where the trivial "middle" circle is not assigned the module $V$, but instead the free module $\langle 1\rangle$. Since $m$ is an isomorphism in $\mathcal{C}'$, $\mathcal{C}'$ is acyclic and $\mathcal{H}(\mathcal{C}) \cong \mathcal{H}(\mathcal{C}/\mathcal{C}')$ by Lemma 5.5.6.



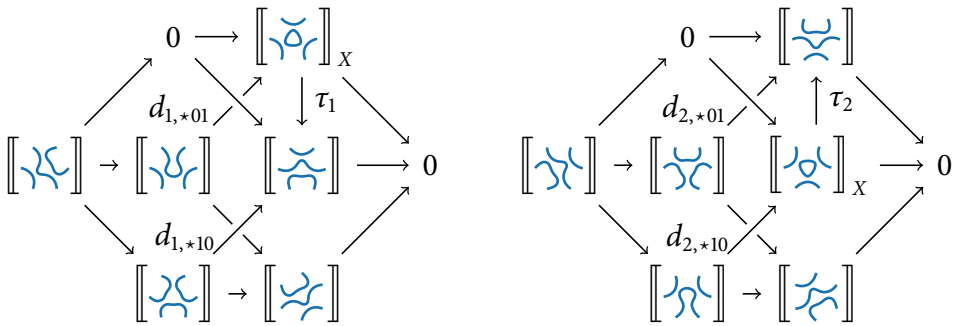

The map $\Delta$ is an isomorphism in $\mathcal{C}/\mathcal{C}'$, so we can define a map $\tau$ that is the composition $\tau = d_{\star 0}\Delta^{-1}$. Let $\mathcal{C}''$ be the subcomplex of $\mathcal{C}/\mathcal{C}'$ consisting of all elements $\xi \in [\![\,]\!]$ and all elements of the form $\beta \oplus \tau\beta \in [\![\,]\!]_X \oplus [\![\,]\!]$. Taking $\mathcal{C}/\mathcal{C}'$ mod $\mathcal{C}''$, we kill $[\![\,]\!]$ and impose the relation $\beta \oplus 0 = 0 \oplus \tau\beta$ in $[\![\,]\!]_X \oplus [\![\,]\!]$, but we have an arbitrary choice of $\gamma \in [\![\,]\!]$. Hence $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ is isomorphic to $[\![\,]\!]$.

**Invariance under R-III.** In order to prove invariance under the third Reidemeister move, we expand both cubes $[\![\,]\!]$ and $[\![\,]\!]$ to the left and right diagrams below, respectfully.



In these diagrams we omit degree shifts. For simplicity, we also keep the states of both cubes coherently oriented.

We repeat the definitions of modding out acyclic complexes $\mathcal{C}'$ and $\mathcal{C}''$ from the proof of R-II with the top-right parallelograms, as the result we obtain:




Let $T$ be the homomorphism that sends the bottom-left parallelogram of the left cube to the bottom-left parallelogram of the right cube, but transposes the top-right parallelogram by sending $[\![\ ]\!]_X \oplus [\![\ ]\!]$ to $[\![\ ]\!]_X \oplus [\![\ ]\!]$ via $T(\beta_1 \oplus \gamma_1) = \beta_2 \oplus \gamma_2$ for $\beta_1 \in [\![\ ]\!]_X$, $\gamma_1 \in [\![\ ]\!]$, $\beta_2 \in [\![\ ]\!]_X$ and $\gamma_2 \in [\![\ ]\!]$. Carefully writing down the definitions of the partial differentials, it easily follows that $\tau_1 \circ d_{1,\star 01} = d_{2,\star 01}$ and $d_{1,\star 10} = \tau_2 \circ d_{2,\star 10}$, hence, $T$ is an isomorphism.

**Invariance under R-IV and R-V.** Invariance under the Reidemeister moves R-IV and R-V follows trivially, since there is a natural bijective correspondence that preserves the differentials between states of the diagram before and after these moves. □

# 6

# Classification

In this chapter we describe the computer algorithm used to classify non-affine prime knots in the solid torus $T$ and in the lens spaces $L(p,q)$. We provide a full classification of knots up to four crossings in the above spaces. We also provide a classification up to five crossings with a few exceptions. We establish which of the knots are amphichiral in $T$. A knot in $T$ or $L(p,q)$ is considered prime if it cannot be written as a connected sum of two non-trivial knots, with one of the knots in the composition being affine. The results are presented at the end of the chapter with details available in the Appendices. The computer algorithm is available online at [10].

## 6.1 Knot notation

Knot notations have been developed to describe knots in a way that the knot can be reconstructed from the notation up to isotopy. Out of various notations, the Gauss code is perhaps the most uninvolved one and is therefore the most suitable notation for storing and processing knots by computer.

A **Gauss word** of length $n$ is a word on the alphabet $\{\pm 1, \pm 2, \ldots \pm n\}$, where each letter appears exactly once.

Let $D$ be a diagram of a knot in $S^3$ with $n$ crossings. The Gauss word associated with the diagram $D$ is the Gauss word of length $n$ obtained by the following steps [20, 49]:

1. Enumerate the crossings from 1 to $n$.
2. Orient the diagram (if not already oriented) and choose an initial point on it.
3. Starting with an empty word, travel from the initial point back to it according to the orientation, appending the letter $\pm k$ to the end of the word when passing through a crossing $k$ with a + sign if it is an overcrossing and a − sign if it is an undercrossing.

We use the convention that we place the initial point right before an overcrossing and number the crossings in such a way that the first encountered crossing is +1 and for each crossing encountered for the first time, the next available integer is used.

In general, we can reconstruct a knot from its associated Gauss word only up to its mirror image, but having the knowledge of the knot's crossings signs enables us to reconstruct the knot completely.

A **Gauss code** is a Gauss word followed by a sequence of $n$ signs.

The Gauss code of a diagram $D$ is the Gauss word of $D$ followed by the sequence of crossings signs (in the order of the enumeration).

**Example 6.1.1.** *The Gauss code of a diagram of the figure eight knot pictured in Figure 6.1 is equal to*

```
1 -2 3 -4 2 -1 4 -3, --++.
```

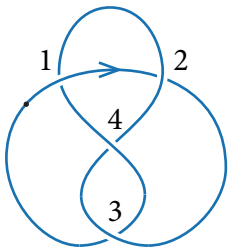



Figure 6.1: The figure eight knot with labeled crossings.

As the reader has probably surmised, not all Gauss codes realize knots [36]. For example, the code

```
1 2 -1 -2, ++
```

cannot belong to a knot, so we call such a code **nonrealizable**. In fact, a necessary but not sufficient condition for realizability is that every number and its opposite are odd distances apart in the Gauss word [35].

We modify the Gauss code to a code of a knot lying in the solid torus $T = A \times I$. Let $D$ be a punctured disk diagram. We specify a region by a subset of arcs enumerated in the direction of the orientation, starting with the arc 0 positioned right after the first crossing we encounter from the initial point in the direction of the orientation. An **extended Gauss code** is a Gauss code followed by two sequences: the arcs of the region in which the dot (i.e. "inner" component of $\partial A$) of $D$ lies on, followed by the arcs of the $\infty$ region (the unbounded region or the "outer" component of $\partial A$), see Example 6.1.2. We will often omit the word "extended" if the variant of the Gauss code is evident from the context.

**Example 6.1.2.** *The extended Gauss code of the diagram pictured in figure 6.2 equals*

```
1 -2 3 -4 2 -1 4 -3, --++, 0 3 5, 1 4 7.
```

We introduce a **total order** on the set of (extended) Gauss codes. The order is the lexicographical order obtained by considering in turn:

1. the length of the Gauss code $n$,
2. the lexicographical ordering $1 < -1 < 2 < -2 < \ldots < n < -n$ of the Gauss word,

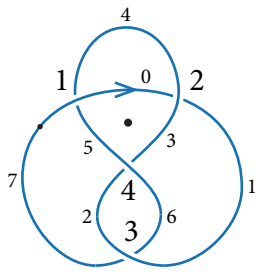



Figure 6.2: A punctured disk diagram.

3. the lexicographical ordering $+ < -$ of signs,
4. the lexicographical ordering $0 < 1 < 2 < \ldots < 2n-1$ of the arcs of the punctured region,
5. the lexicographical ordering $0 < 1 < 2 < \ldots < 2n-1$ of the arcs of the $\infty$ region.

If $W' < W$ and the Gauss codes $W$ and $W'$ represent the same knot, we say that $W'$ is a **reduction** of $W$. A Gauss code $W$ that allows a reduction is called **reducible**.

**Example 6.1.3.** *The Gauss code calculated in Example 6.1.2 is reducible, since, if we choose our initial point on arc* 1*, the Gauss code of the diagram becomes*

```
1 -2 3 -4 2 -1 4 -3, ++--, 1 3 6, 2 5 7.
```

*This Gauss code is indeed a reduction of the Gauss code in Example 6.1.2, since* $+ < -$ *in the lexicographical ordering of signs.*

## 6.2 Reidemeister moves

The classification algorithm uses Reidemeister moves to determine the equivalences of knots. We overview how a Reidemeister move is performed by computer, that is, how such a transformation changes the Gauss word. All equivalences below are considered up to cyclic shifts. Gauss subwords are denoted by $A$, $B$, and $C$, individual letters are denoted by $i$, $j$, and $k$.

An R-I move is a creation or removal of a kink:

$$\pm i \mp i\ A \longleftrightarrow A.$$

An R-II move creates or removes two adjacent crossings and has either one of the following forms:

$$\pm i \pm j\ A \mp i \mp j\ B \longleftrightarrow A\ B,$$
$$\pm i \pm j\ A \mp j \mp i\ B \longleftrightarrow A\ B.$$

An R-III move on the Gauss word permutes the crossings involved in the move and has one of

the following forms:

$$\begin{aligned}
i\ j\ A\ {-i}\ k\ B\ {-j}\ {-k}\ C &\longleftrightarrow j\ i\ A\ k\ {-i}\ B\ {-k}\ {-j}\ C,\\
i\ j\ A\ {-i}\ k\ B\ {-k}\ {-j}\ C &\longleftrightarrow j\ i\ A\ k\ {-i}\ B\ {-j}\ {-k}\ C,\\
i\ j\ A\ k\ {-i}\ B\ {-j}\ {-k}\ C &\longleftrightarrow j\ i\ A\ {-i}\ k\ B\ {-k}\ {-j}\ C,\\
i\ j\ A\ k\ {-i}\ B\ {-k}\ {-j}\ C &\longleftrightarrow j\ i\ A\ {-i}\ k\ B\ {-j}\ {-k}\ C.
\end{aligned}$$

In order to optimize the number of moves needed to find equivalent knots, we also add the flype in our collection of moves (since a flype is relatively easy to perform on the Gauss code, but otherwise involves a large sequence of Reidemeister moves). We only consider certain types of flypes – the ones where the strand enters and exits the same side of the 2-tangle $T$ involved in the flype (see Figure 2.6 in Chapter 2). Such a flype has the following effect on the Gauss word:

$$\pm i\ W\ \mp i\ A\ W'\ B \longleftrightarrow \overline{W}\ A\ \mp i\overline{W'}\pm i\ B,$$

where $W$ and $W'$ are subwords consisting of letters of two types: for each letter $l \in W$ (resp. $l \in W'$) either $-l \in W$ (resp. $-l \in W'$) or $-l \in W'$ (resp. $-l \in W$) holds, i.e. the word $WW'$ consists only of double letters (up to signs) that correspond to crossings inside the flipped tangle $T$. The word $\overline{W}$ (resp. $\overline{W'}$) is obtained from $W$ (resp. $W'$) by multiplying each letter by $-1$ (this corresponds to the flip of the tangle $T$).

We call the isotopy that is obtained by rotating each meridional disk in the solid torus by $\pi$, a **meridional rotation**. We also add this operation to our collection of moves. Applying a meridional rotation on a a diagram has the effect of multiplying each letter in the Gauss word by $-1$:

$$W \longleftrightarrow \overline{W},$$

with $W$ being the Gauss word of the diagram.

## 6.3 The classification algorithm

First, we describe the algorithm that enables us to classify knots in the solid torus and continue by describing the steps needed to classify knots in $L(p,q)$.

### 6.3.1 Classifying knots in the solid torus

In order to classify knots in the solid torus $T$ with at most $n$ crossings, we use the following steps:

1. Find all possible realizable Gauss codes having length at most $n$.
2. Partition the codes by a common, strong enough knot invariant (the HOMFLYPT skein module).

3. Remove Gauss codes that realize connected sums.
4. Consider the first knot in each partition as prime.
5. In each partition systematically perform Reidemeister moves, flypes and meridional rotations on the knots not marked as prime, until they are reduced.

We describe the steps in more detail.

**Step 1. Finding all realizable Gauss codes.**

Since there is a large number of Gauss codes of knot diagrams in $T$ with at most five crossings (see Table 6.1), we need to optimize this step by generating only certain types of Gauss codes that will produce all prime knots.

The first optimization step is done by generating all possible Gauss codes in the lexicographical order. If, in this process, a Gauss code $W$ is obviously reducible (see the next paragraph), we can eliminate $W$ as a Gauss code of a prime knot candidate, since the reduction of $W$ already exists in the list of candidates generated beforehand.

In this first step of the algorithm, a Gauss code of a diagram $D$ is considered as reducible if any combination of the following operations reduces its code:

- renaming the letters of the Gauss code (corresponds to renaming the crossings in $D$),
- a cyclic shift of the Gauss code (corresponds to choosing an initial point in $D$),
- reversing the Gauss code (corresponds to reversing the orientation of $D$),

A Gauss code is in its **canonical form** if it cannot be reduced using the above operations. Note that a knot diagram can be uniquely represented by the canonical form of its Gauss code.

The number of Gauss codes up to a length of 7 is presented in Table 6.1: the first row of the table is given by the formula $N = (2n)! \cdot 2^n$, other rows are determined experimentally.

Table 6.1: Number of Gauss codes up to length 7.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Gauss codes | 4 | 96 | 5760 | 645120 | $1.16 \cdot 10^8$ | $3.07 \cdot 10^{10}$ | $1.12 \cdot 10^{13}$ |
| Realizable | 4 | 24 | 432 | 13344 | $7.06 \cdot 10^5$ | $5.18 \cdot 10^7$ | $4.88 \cdot 10^9$ |
| Canonical realizable | 2 | 6 | 36 | 278 | $2.94 \cdot 10^3$ | $3.59 \cdot 10^4$ | $4.84 \cdot 10^5$ |
| Extended realizable | 36 | 384 | 10800 | 480384 | $3.46 \cdot 10^7$ | $3.31 \cdot 10^9$ | $3.95 \cdot 10^{11}$ |
| Canon. Ext. realizable | 10 | 68 | 714 | 9392 | $1.41 \cdot 10^5$ | $2.29 \cdot 10^6$ | $3.91 \cdot 10^7$ |

At this step we also eliminate diagrams that:

- the puncured region and the $\infty$ region coincide (affine knot),
- the punctured region and the $\infty$ region are adjacent (the sum of an affine knot and the non-affine unknot).

**Step 2. Partitioning the knots that share the common HOMFLYPT skein module (HSM).** Let $\mathcal{W}$ be the set of all Gauss codes (i.e. diagrams of knot candidates) calculated in Step 1.

For each $W \in \mathcal{W}$, we calculate HSM and partition $\mathcal{W}$ into partitions $P_0, P_1, \ldots, P_m$, so that all codes in $P_i$ share the same HSM (i.e. are in the same equivalence class in $\mathcal{S}_3(T)$). HSM is calculated recursively using the HOMFLYPT relation described in Section 2.3.2. The recursive algorithm also reduces the knot in each step using the breath-first search algorithm described in Step 5.

**Step 3. Removing Gauss codes that represent connected sums.** In this step we remove all Gauss codes that represent non-trivial connected sums. Note that KBSM detects connected sums in the sense that $\mathcal{S}_{2,\infty}(K\#K') = \mathcal{S}_{2,\infty}(K)\mathcal{S}_{2,\infty}(K')$, the proof being identical to that of Proposition 2.2.1. So, if we wish to guess if a knot $K$ is a connected sum with, say, the affine trefoil $\clubsuit$, we check if $\mathcal{S}_{2,\infty}(K)$ is divisible by $\mathcal{S}_{2,\infty}(\clubsuit)$.

First, we calculate KBSM of all $W \in \mathcal{W}$. If KBSM of $W \in \mathcal{W}$ is divisible by KBSM of an affine knot, $W$ is a candidate for a connected sum. It turns out, in our case, that all such "divisible" Gauss codes represented connected sums – this is verified by printing out all of the connected sum candidates and verifying that they are indeed composites by hand.

Since Step 4 is trivial, we continue with Step 5.

**Step 5. Finding isotopic knot diagrams.** A brute-force algorithm for finding all possible Reidemeister moves and search for isotopic knot diagrams would need a lot of computer power to finish, thus we describe the steps made to optimize this part of the algorithm.

The key observation was already made in Step 1: we can eliminate Gauss codes that allow a reduction. Take each Gauss code in $\mathcal{W}$ not marked as prime by Step 4 and systematically perform Reidemeister moves, flypes and meridional rotations up to a depth $d \in \mathbb{N}$. We do this using a breadth-first search algorithm using $d+1$ **binary search trees** (BSTs) $B, B_0, B_1, \ldots, B_d$, with $B$ storing all knots after Reidemeister moves and $B_i$ storing knots after performing a Reidemeister moves at a depth $i \leq d$. A BST is a computer data structure that realizes an ordered set. For a given Gauss code $W \in \mathcal{W}$, the steps of the algorithm for detecting reducible codes are the following:

1. Insert $W$ into $B$ and $B_0$ as the root node.
2. For each $i = 1, \ldots, d$ take all codes $W' \in B_{i-1}$, systematically perform all Reidemeister moves, and put them into their canonical form. Insert all these new Gauss codes into $B$, if an insertion of a code is successful, insert it also into $B_i$. A successful insertion of an element into a BST means that it is a new element in the set and has not yet previously been in it.
3. If $W \neq \min(B)$, $W$ has been reduced and is therefore not considered as prime.

The advantage of using a breadth-fist algorithm is obvious: if less than $d$ Reidemeister moves are required to reduce a code, the algorithm will finish before reaching the depth of $d$ (as opposed to a depth-first algorithm that could perform $d$ Reidemeister moves even if less than $d$ moves are necessary). We use binary search trees for two reasons:

1. Binary search trees realize sets, so two equal codes will never be processed simultaneously. This is a major step in the optimization, since, without the set structure, we could create a kink by R-I at depth 1, remove it at depth 2, then search for the reduction of the original

knot again at depth 3. This is the reason for inserting the Gauss code into $B$. The BST $B$ stores all the codes appearing during the process and by checking if an insertion is successful, we are in fact checking if the knot has been encountered before.

2. We only use the operations of inserting elements, retrieving elements in order and retrieving the minimal element of a BST. The time complexities of these operations lie in our favor: insertion of an element into s BST is logarithmic time $O(\ln n)$, retrieving elements in order and finding the minimal element are both constant time $O(1)$ [9].

The other observation that improves the performance of the algorithm lies in the fact that most of the moves R-I and R-II are redundant. For example, creating $d$ kinks in a row would be pointless. Also, for a given diagram, there are in general considerably more crossing-increasing R-I and R-II moves than there are crossing-decreasing R-I and R-II moves, R-III moves, flypes and meridional rotations, together – for any given arc, there are four different crossing-increasing R-I moves and for any two arcs bounding a common region, there are two different crossing-increasing R-II moves (even more if the region contains a dot). An experimentation process on a small portion of the table was required for determining the best ratio of non-increasing Reidemeister moves and (increasing or non-increasing) Reidemeister moves. Fortunately this ratio turns out to be 2:1 – the best results (speed vs. successful reductions) was achieved by making an arbitrary Reidemeister move only every third step[1], at depth $d \equiv 1 \pmod 3$.

Detecting amphichirality is simple: we mirror the knot and look in which partition the mirror belongs: if it is in the same partition of the original knot, the knot is amphichiral, otherwise it is chiral (note that this only works if the knots lie in partitions of size 1, otherwise, we would have to determine amphichirality by other means).

### 6.3.2 Classifying knots in $L(p,q)$

As seen in section 3.4.1, a punctured disk diagram of a knot in $L(p,q)$ is a punctured disk diagram in the solid torus $T$ accompanied by the additional slide move SL.

The algorithm for classifying knots in $L(p,q)$ is therefore similar to the one in $T$, with two differences: we only search for prime knots in $L(p,q)$ within the set of prime knots of $T$ and we use an $L(p,q)$ invariant for partitioning the knots.

Thus, we take the following steps in the classification algorithm for $L(p,q)$:

1. Partition all codes representing prime knots in $T$ by a common invariant (the HOMFLY skein module for $L(p,1)$ and KBSM for $L(p,q)$, $q > 1$).
2. Remove affine knots and knots that are connected sums with non-trivial affine knots.
3. Consider the first knot in each partition as prime.
4. In each partition systematically perform moves (Reidemeister moves, the SL move, flypes and meridional rotations) on the knots not marked as prime, until they are reduced (if

[1]Perhaps it would be interesting to investigate why this is so. It may come from the observation that moving a string through several crossings involves such a ratio of Reidemeister moves.

any partition is left with more than one knot, we have to treat those knots with other means).

All of the above steps are performed in an analogous fashion to those in the case of the solid torus.

Unfortunately the Khovanov homology for KBSM is too cumbersome to calculate, for it being a convenient invariant of $L(2,1)$. As seen in Example 5.4.3, even a 2-crossing knot requires complex calculations.

## 6.4 The results

### 6.4.1 Solid torus

The number of non-affine prime knots in the solid torus up to $n \leq 5$ crossings is presented in Table 6.2. The knots are tabulated in Appendix A, the HOMFLYPT skein modules are presented in Appendix C and the Kauffman bracket skein modules are presented in Appendix D. Since there is a well-defined mirror operation in the solid torus $T = A \times I$, i.e. reflection of the knot through the annulus $A \times \{½\}$, Appendix A contains only one of the possible two pairs of each knot, the mirror of the knot is obtained by switching all undercrossings to overcrossings and vice-versa.

Table 6.2: The number of non-affine prime knots in the solid torus.

| $n$ | Number of prime knots | |
|---|---|---|
| **0** | 1 | (0 mirror pairs + 1 amphichiral) |
| **1** | 2 | (1 mirror pair + 0 amphichiral) |
| **2** | 5 | (2 mirror pairs + 1 amphichiral) |
| **3** | 16 | (8 mirror pairs + 0 amphichiral) |
| **4** | 50 | (23 mirror pairs + 4 amphichiral) |
| **5** | 190* | (95* mirror pairs + 0 amphichiral) |

* Assuming the Conjecture 6.4.1 is true.

Up to five crossings there are six non-affine prime amphichiral knots: $0_1$, $2_2$, $4_{13}$, $4_{14}$, $4_{21}$, and $4_{27}$ (see Appendix A). There are five pairs of knots that are not distinguished by KBSM: $4_{23}$, $\overline{5_1}$; $5_1$, $\overline{4_{23}}$; $5_{26}$, $5_{27}$; $\overline{5_{26}}$, $\overline{5_{27}}$; and $5_{76}$, $\overline{5_{76}}$. The first two pairs are distinguished by HSM and for the last pair, KBSM suggests amphichirality, but HSM shows otherwise.

The pair $5_{26}$, $5_{27}$ (and their mirrors $\overline{5_{26}}$, $\overline{5_{27}}$) are not distinguished neither by KBSM, neither by HSM. Several approached have been tried to determine (in)equality of these two knots:

- a breadth-first search with up to 50 non-crossing-increasing Reidemeister moves, with up to six crossing-increasing Reidemeister moves,
- a breadth-first search with up to 70 Reidemeister moves, with making a crossing-increasing Reidmeister move only when no new non-crossing-increasing Reidemeister moves can

be found,

- a two-direction breadth-first search of the above, i.e. making the Reidemister moves of $5_{27}$ and $5_{26}$ simultaneously and checking if there is a non-empty intersection,
- making up to 30.000 random Reidemeister moves of the diagram $5_{26}$ (and $5_{27}$), reducing the resulting diagram to a diagram with 5 crossings; this process was repeated several thousand times with the knots not turning out to be equal,
- using several approaches, the (in)equality of the fundamental groups $\pi_1(T \setminus 5_{26})$ and $\pi_1(T \setminus 5_{27})$ could not be determined using the computational tools GAP and Magma [15, 6].

We point out that the diagrams of knots $5_{26}$ and $5_{27}$ (as well as their mirrors) differ by switching the punctured region with the unbounded region, which corresponds to the flip operation in [12].

An oriented knot $K$ in a solid torus naturally corresponds to a two-component oriented link $L(K)$ in $S^3$ with one trivial component – the extra trivial component being the meridian of the torus (see Figure 6.3 for the two-component link $L(5_{26})$ corresponding to the knot $5_{26}$) [23]. Since knots $5_{26}$ and $5_{27}$ are so-called flips of each other, the two corresponding links $L(5_{26})$ and $L(5_{27})$ differ only by reversing orientations of both components of the link, i.e. it holds that $L(5_{26}) = -L(5_{27})$. A link with the property that reversing both components yields the same link is called an invertible link. We therefore transfer the question of inequivalence of links $5_{26}$ and $5_{27}$ (and their mirrors $\overline{5_{26}}$ and $\overline{5_{27}}$) to the question of non-invertibility of the link $L(5_{26})$. Non-invertibility is a very difficult question in knot theory and only few examples of such links have been found [59, 60].

(a) $5_{26}$ (b) $L(5_{26})$

Figure 6.3: Diagrams of the knot $5_{26}$ and the corresponding two-component link $L(5_{26})$.

Considering the arguments above, we state the following conjecture:

**Conjecture 6.4.1.** *Knots $5_{26}$ and $5_{27}$ are not equivalent.*

### 6.4.2 Lens spaces

We classify non-affine prime knots in the lens spaces $L(p,q)$, $p \leq 12$. By theorem 3.1.1, these are exactly the lens spaces $L(2,1)$, $L(3,1)$, $L(4,1)$, $L(5,1)$, $L(5,2)$, $L(6,1)$, $L(7,1)$, $L(7,2)$, $L(8,1)$, $L(8,3)$, $L(9,1)$, $L(9,2)$, $L(10,1)$, $L(10,3)$, $L(11,1)$, $L(11,2)$, $L(11,3)$, $L(12,1)$, and $L(12,5)$.

The **winding number** $wind(L)$ of a knot in $L \subset T$ is the integer $[L] \in \pi(T) \cong \mathbb{Z}$, or equivalently, the integer $[D] \in \pi(\mathbb{R}^2 \setminus \{\cdot\}) \cong \mathbb{Z}$ if $D$ is a diagram of $L$ lying in the punctured plane $\mathbb{R}^2 \setminus \{\cdot\}$.

**Proposition 6.4.1.** *Let $i : T \hookrightarrow L(p,q)$ be the standard inclusion of the solid torus $T$ in the lens space $L(p,q)$. If two knots $K_1$ and $K_2$ with at most $n$ crossings have distinct KBSM in the solid torus, then the knots $i(K_1)$ and $i(K_2)$ have distinct KBSM in $L(p,q)$, $p \geq 2(n+1)$.*

*Proof.* Since a knot $K \subset T$ that allows a diagram $D$ with $n$ crossings has the winding number bounded by $|wind(K)| \leq n+1$ ($D$ can go up to $(n+1)$ times around the puncture and thus needs $n$ crossings to complete the circle), it can be expressed in the standard generators of KBSM of the solid torus as an expression $\mathcal{S}_{2,\infty}(T)(K) = \sum_{i=0}^{n+1} A_i x^i$, for some $A_i \in R$. Since $p \geq 2(n+1) \Rightarrow \lfloor p/2 \rfloor \geq n+1$ and the generators $\{x'^n\}_{n=0}^{\lfloor p/2 \rfloor}$ of KBSM of $L(p,q)$ are induced by the inclusion, $i_*(x^i) = x'^i, 0 \leq i \leq \lfloor p/2 \rfloor$, it holds that $\mathcal{S}_{2,\infty}(L(p,q))(i(K)) = \sum_{i=0}^{n+1} A_i x'^i$, $A_i \in R$. That is, the knots $K$ and $i(K)$, although lying in different spaces, have an equal KBSM expression. □

A similar proposition can be made for the HOMFLYPT skein module.

**Proposition 6.4.2.** *Let $i : T \hookrightarrow L(p,1)$ be the standard inclusion of the solid torus $T$ in the lens space $L(p,1)$. If two knots $K_1$ and $K_2$ with at most $n$ crossings have distinct HSM in the solid torus, then the knots $i(K_1)$ and $i(K_2)$ have distinct HSM in $L(p,1)$, $p > 2(n+1)$.*

*Proof.* Since the generators $\{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < .. < k_s \in \mathbb{Z} \setminus \{0\}, -\frac{p}{2} < k_1 < .. < k_s \leq \frac{p}{2}, i_1..i_s \in \mathbb{N}\} \cup \{\varnothing\}$ of HSM of $L(p,1)$ are induced by the inclusion of the generators of HSM of $T$, by the same arguments that we use in Proposition 6.4.1 and by the fact that $p > 2(n+1) \Rightarrow \frac{p}{2} > n$, we conclude that the expression $\mathcal{S}_3(T)(K)$ in the standard generators is equal to the expression of $\mathcal{S}_3(L(p,1))(i(K))$. □

From Conjecture 4.4.1 we have the following corollary:

**Conjecture 6.4.2.** *Let $i : T \hookrightarrow L(p,q)$ be the standard inclusion of the solid torus $T$ in the lens space $L(p,q)$, $q > 1$. If two knots $K_1$ and $K_2$ with at most $n$ crossings have distinct HSM in the solid torus, then the knots $i(K_1)$ and $i(K_2)$ have distinct HSM in $L(p,q)$, $p > 2(n+1)$, $q > 1$.*

The number of non-affine prime knots with up to five crossings in $L(p,q)$ is presented in Table 6.3. The set of knots that are either reducible, affine or composites in $L(p,q)$ are tabulated in Appendix B, i.e. the classification is given by the set of prime knots in the solid torus minus the set of knots in Appendix B. Note that we also tabulate the mirrors (in terms of mirrors in the solid torus), since there is no well-defined mirror operation in the punctured disk diagram of $L(p,q)$, e.g. the knot $1_1$ is the unknot in $L(2,1)$, but the knot $\overline{1_1}$ is not.

Since KBSM is unique for all knots up to four crossings in the solid torus and the lens spaces $L(p,q)$, $p < 10$, we conclude by specializing Proposition 6.4.1 to the case $n = 4$, that KBSM distinguishes these knots in all lens spaces. A similar argument can be made for the HOMFLYPT skein module: all knots up to four crossings are unique in HSM of the solid torus and $L(p,1)$, $p \leq 10$, so they are unique in HSM of all $L(p,1)$.

There are several irreducible knots that are excluded from the tables:

- In $L(2,1)$: the knot $5_{77}$ is a connected sum with the trefoil, the knot $5_{24}$ is the affine trefoil, and the knot $1_1$ is the affine unknot.
- In $L(3,1)$: the knot $2_3$ is the affine unknot.

Table 6.3: Number of non-affine prime knots in $L(p,q)$ with up ot 5 crossings.

| $n$ | $L(2,1)$ | $L(3,1)$ | $L(4,1)$ | $L(5,1)$ | $L(5,2)$ | $L(6,1)$ | $L(7,1)$ |
|---|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 4 | 5 | 4 | 4 | 5 | 5 |
| **3** | 12 | 13 | 14 | 16 | 16 | 15 | 15 |
| **4** | 40 | 42 | 45 | 46 | 49 | 49 | 49 |
| **5** | $155^\star$ | $163^\star$ | $176^\star$ | $183^\star$ | $189^{\star\diamond}$ | $184^\star$ | $187^\star$ |

| $n$ | $L(7,2)$ | $L(8,1)$ | $L(8,3)$ | $L(9,1)$ | $L(9,2)$ | $L(10,1)$ | $L(10,3)$ |
|---|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 15 | 16 | 16 | 16 | 16 | 16 | 16 |
| **4** | 50 | 49 | 50 | 49 | 49 | 50 | 50 |
| **5** | $189^{\star\diamond}$ | $188^\star$ | $190^{\star\diamond}$ | $189^\star$ | $190^{\star\diamond}$ | $189^\star$ | $190^{\star\diamond}$ |

| $n$ | $L(11,1)$ | $L(11,2)$ | $L(11,3)$ | $L(12,1)$ | $L(12,5)$ | $L(p,q),\ p \geq 13$ |
|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 16 | 16 | 16 | 16 | 16 | 16 |
| **4** | 50 | 50 | 50 | 50 | 50 | 50 |
| **5** | $189^\star$ | $189^{\star\diamond}$ | $190^{\star\diamond}$ | $190^\star$ | $190^{\star\diamond}$ | $190^{\star\diamond}$ |

$^\star$ Assuming Conjecture 6.4.3 is true.
$^\diamond$ Assuming Conjecture 6.4.4 is true.

- In $L(4,1)$: the knot $3_6$ is the affine unknot.
- In $L(5,1)$: the knot $4_{24}$ is the affine unknot.
- In $L(6,1)$: the knot $5_{85}$ is the affine unknot.

In Table 6.4 we present inequivalent knots that are not distinguished by KBSM, for $p = 1$ all but the pair $5_{26}$, $5_{27}$ (and their mirrors) are distinguishable by HSM. For $p > 1$ we have to conjecture about the knots in the solid torus that are distinguished only by HSM. Recall that these are exactly the pairs $4_{23}$, $\overline{5_1}$; $5_{26}$, $5_{27}$; and $5_{76}$, $\overline{5_{76}}$ (and the mirrors of the first two pairs).

A high depth breadth-first searched has been performed to check that these knots are (plausibly) not equivalent in $L(p,q)$, $p \leq 12$.

**Conjecture 6.4.3.** *Knots $5_{26}$ and $5_{27}$ and their solid torus mirrors are not equivalent in any $L(p,q)$.*

**Conjecture 6.4.4.** *The pair $4_{23}$, $\overline{5_1}$ (and their mirrors); and the pair $5_{76}$, $\overline{5_{76}}$ are not equivalent in $L(p,q)$, $q > 1$.*

We point out that the Conjecture 6.4.4 is a corollary of Conjecture 4.4.1.

We also point out that in Table 6.4 the conjecture above also applies to the case of $L(5,2)$ since the knot $3_6$ is a reduction of the knot $4_{23}$ and the knot $5_{75}$ is a reduction of the knot $5_{76}$ (that is, it holds that $3_6 = 4_{23}$ and $5_{75} = 5_{76}$ in $L(5,2)$, see Appendix B).

Table 6.4: Ineqivalent prime knots in $L(p,q)$ that share KBSM.

| **Space** | **Knots that share KBSM** | | | | |
|---|---|---|---|---|---|
| $L(2,1)$ | $2_3, \overline{5_1}, \overline{5_5}, \overline{5_{76}}$; | | $\overline{4_1}, \overline{4_{10}}$; | $\overline{4_3}, 4_{11}$; | $4_8, \overline{4_{16}}$; |
| | $\overline{4_{23}}, 5_1$; | $\overline{5_{10}}, 5_{32}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{31}, 5_{60}$; |
| | $5_{36}, \overline{5_{60}}$; | $\overline{5_{39}}, 5_{52}$ | | | |
| $L(3,1)$ | $\overline{1_1}, \overline{5_1}$; | $\overline{2_1}, \overline{5_{76}}$; | $3_5, 4_9$; | $\overline{4_{23}}, 5_1$; | $5_{17}, 5_{28}$; |
| | $\overline{5_{11}}, 5_{59}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{49}, 5_{56}$ | |
| $L(4,1)$ | $2_2, \overline{5_1}$; | $\overline{3_4}, \overline{5_{76}}$; | $\overline{4_6}, \overline{5_{77}}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; |
| | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{49}, \overline{5_{95}}$ | | | |
| $L(5,1)$ | $\overline{3_2}, \overline{5_1}$; | $\overline{4_{17}}, \overline{5_{76}}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$ |
| $L(5,2)$ | $3_6, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{75}, \overline{5_{76}}^{\diamond}$ |
| $L(6,1)$ | $\overline{4_2}, \overline{5_1}$; | $4_{18}, 5_3$; | $\overline{4_{23}}, 5_1, 5_{72}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; |
| | $\overline{5_{68}}, \overline{5_{76}}$ | | | | |
| $L(7,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(7,2)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(8,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(8,3)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(9,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(9,2)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(10,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(10,3)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(11,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(11,2)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(11,3)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(12,1)$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(12,5)$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |
| $L(p,1)^{\dagger}$ | $4_{23}, \overline{5_1}$; | $\overline{4_{23}}, 5_1$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}$ |
| $L(p,q)^{\ddagger}$ | $4_{23}, \overline{5_1}^{\diamond}$; | $\overline{4_{23}}, 5_1^{\diamond}$; | $5_{26}, 5_{27}^{\star}$; | $\overline{5_{26}}, \overline{5_{27}}^{\star}$; | $5_{76}, \overline{5_{76}}^{\diamond}$ |

$^{\star}$ Assuming Conjecture 6.4.3 is true.
$^{\diamond}$ Assuming Conjecture 6.4.4 is true.
$^{\dagger}$ $p > 12$.
$^{\ddagger}$ $p > 12$, $q > 1$.

# 7

# Conclusion and open questions

The HOMFLYPT skein modules of $L(p,1)$ have been calculated. We have shown that they are free $R$-modules generated by an infinite set of generators.

Clearly the open question from here on lies in calculating the HOMFLYPT skein modules of $L(p,q)$, $q > 1$. We have given a conjecture that they are free $R$-modules with an infinite set of generators. The next open question, which, by the author's opinion, can be calculated by similar methods of those used in this thesis, is the calculation of the 2-variable Kauffman skein module of $L(p,q)$.

We have categorified the Kauffman bracket skein module (KBSM) of the projective space.

With this result, we closed the chapter of categorifying the Kauffman bracket skein module of links in $I$-bundles over surfaces (since this was the only remaining case left in [3]) and perhaps opened a new chapter on categorifying Jones-like invariants of other 3-manifolds, the most obvious candidates being the general lens space $L(p,q)$ and $S^1$-bundles over surfaces. For the case of $L(p,q)$, the main difficulty of categorifying KBSM lies in the question of developing a non-recursive state-sum formula for describing a link in the basis of the skein module. The author's opinion is, that no easy-to-describe formula exists for the KBSM of $L(p,q)$ with $(p,q) \neq (2,1)$.

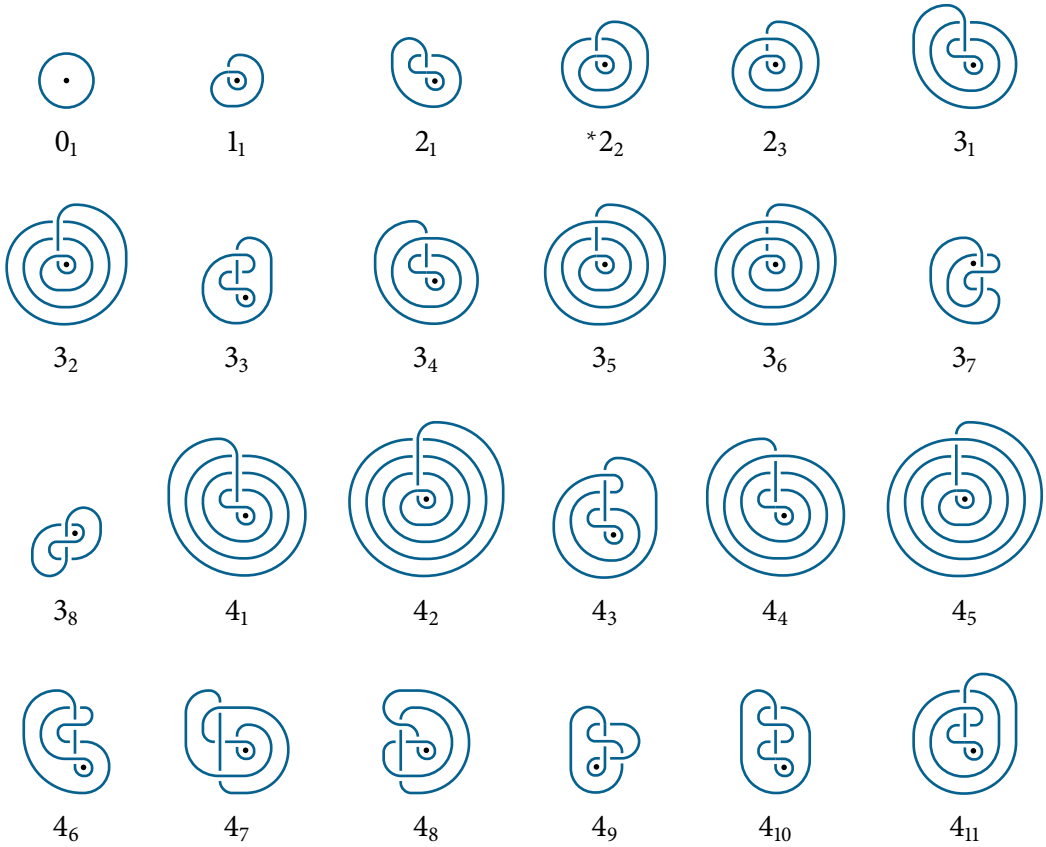
Up to now, knots have been classified only in the 3-dimensional Euclidean space, the projective space, and, up to a so called "flip", in the solid torus. We have refined the classification in the solid torus by giving a full classification of knots up to four crossings. We have also given a classification up to five crossings with one exception: we have not been able to prove that the knots $5_{26}$ and $5_{27}$ are inequivalent, but have provided evidence that this is probably the case. We have given a full classification of knots in all lens spaces $L(p,q)$ up to four crossings and a classification up to five crossings with a few exceptions.

Preliminary calculations show that, using the same techniques that we have used, a few dozen of similar exceptions appear in the classification up to six crossings, both in the solid torus and in $L(p,q)$. Developing techniques that will resolve these difficulties and a classification of knots in $L(p,q)$ with a higher number of crossings is certainly a result the author wishes to achieve in the near future.

# Appendices

# A

# Table of knots in the solid torus



$4_{12}$

$^*4_{13}$

$^*4_{14}$

$4_{15}$

$4_{16}$

$4_{17}$

$4_{18}$

$4_{19}$

$4_{20}$

$^*4_{21}$

$4_{22}$

$4_{23}$

$4_{24}$

$4_{25}$

$4_{26}$

$^*4_{27}$

$5_1$

$5_2$

$5_3$

$5_4$

$5_5$

$5_6$

$5_7$

$5_8$

$5_9$

$5_{10}$

$5_{11}$

$5_{12}$

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$5_{75}$

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$5_{78}$

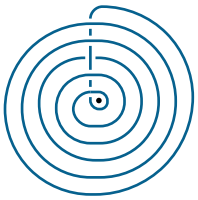
$5_{79}$

$5_{80}$

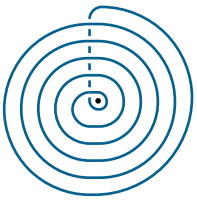
$5_{81}$

$5_{82}$

$5_{83}$



$5_{84}$



$5_{85}$

$5_{86}$

$5_{87}$

$5_{88}$

$5_{89}$

$5_{90}$

$5_{91}$

$5_{92}$

$5_{93}$

$5_{94}$

$5_{95}$

Legend: $*$ – amphichiral knot

# B

# Non-prime knots in lens spaces

Table B.1: Equivalences of knots in lens spaces.

| Space | Equivalences | | | |
|---|---|---|---|---|
| $\mathbf{L(2,1)}$ | $1_1, 5_{74} \sim O$ | $4_{16} \sim \overline{3_4}$ | $\overline{5_2} \sim \overline{4_6}$ | $5_{24} \sim \clubsuit$ |
| | $3_6, \overline{3_8} \sim \overline{1_1}$ | $\overline{5_{41}}, \overline{5_{90}} \sim \overline{3_3}$ | $4_{27} \sim \overline{4_7}$ | $5_{54} \sim 5_{32}$ |
| | $3_7 \sim 2_1$ | $5_{85} \sim \overline{3_6}$ | $5_{87} \sim 4_{10}$ | $\overline{5_{51}}, \overline{5_{92}} \sim \overline{5_{32}}$ |
| | $4_{26}, 5_{75}, 5_{81} \sim \overline{2_1}$ | $4_1, 5_{88} \sim 3_8$ | $5_{39} \sim \overline{4_{11}}$ | $\overline{5_{87}} \sim \overline{5_{43}}$ |
| | $4_{22}, 4_{23}, 5_{76}, 5_{78} \sim 2_3$ | $\overline{4_{22}} \sim \overline{4_2}$ | $\overline{4_{25}} \sim \overline{4_{17}}$ | $\overline{5_{78}} \sim \overline{5_{68}}$ |
| | $4_{24} \sim \overline{2_3}$ | $5_{93} \sim 4_6$ | $5_{89} \sim 4_{20}$ | $5_{77} \sim 0_1\#\clubsuit$ |
| | $\overline{3_7}, \overline{5_{94}} \sim \overline{3_2}$ | $5_{42} \sim 4_7$ | $5_5 \sim \overline{4_{23}}$ | $5_{91} \sim 5_{79}$ |
| | $4_{25}, 5_{71} \sim 3_3$ | $\overline{5_4}, \overline{5_{91}} \sim 4_9$ | $\overline{5_{38}}, 5_{67} \sim \overline{4_{26}}$ | $\overline{5_{93}} \sim 5_{80}$ |
| | $\overline{5_{95}} \sim 3_5$ | $5_3, 5_{23} \sim \overline{4_8}$ | $\overline{5_{74}} \sim \overline{5_6}$ | |
| $\mathbf{L(3,1)}$ | $3_6, 5_{74}, 5_{75} \sim 1_1$ | $5_{78}, 5_{81} \sim 3_3$ | $\overline{5_2}, 5_{77} \sim \overline{4_7}$ | $\overline{5_4} \sim 5_3$ |
| | $4_{23}, 4_{24}, 5_{24} \sim \overline{1_1}$ | $4_9 \sim 3_5$ | $5_{29} \sim 4_{15}$ | $5_{33}, 5_{82} \sim 5_{22}$ |
| | $4_{22} \sim 2_1$ | $4_{26} \sim \overline{3_4}$ | $\overline{5_{90}} \sim \overline{4_{11}}$ | $\overline{5_{41}} \sim 5_{23}$ |
| | $\overline{3_8} \sim 2_2$ | $5_5 \sim 3_8$ | $4_{25} \sim 4_{16}$ | $5_{55} \sim 5_{34}$ |
| | $5_{76} \sim \overline{2_1}$ | $\overline{4_2} \sim \overline{3_7}$ | $5_{67} \sim \overline{4_{17}}$ | $\overline{5_{91}} \sim 5_{39}$ |
| | $2_3, 5_{71} \sim O$ | $4_{20} \sim 4_6$ | $\overline{5_6} \sim \overline{4_{22}}$ | $5_{93} \sim 5_{40}$ |
| | $5_{85} \sim \overline{2_3}$ | $5_{88} \sim 4_7$ | $\overline{5_{68}} \sim \overline{4_{25}}$ | $5_{87} \sim 5_{42}$ |
| | $3_7 \sim 3_1$ | $\overline{5_{94}} \sim 4_8$ | $\overline{5_{38}} \sim 4_{27}$ | |
| $\mathbf{L(4,1)}$ | $5_{75}, 5_{85} \sim \overline{1_1}$ | $\overline{3_8} \sim \overline{3_2}$ | $5_{78} \sim 4_{16}$ | $5_{77} \sim \overline{5_{38}}$ |
| | $5_{74} \sim 2_1$ | $5_{76} \sim \overline{3_4}$ | $5_{39} \sim 4_{18}$ | $5_{72} \sim 5_{40}$ |
| | $4_{23} \sim 2_2$ | $3_6 \sim O$ | $4_{26} \sim \overline{4_{17}}$ | $\overline{5_{90}} \sim \overline{5_{43}}$ |
| | $4_{24} \sim 2_3$ | $4_1 \sim 3_7$ | $5_{67} \sim 4_{25}$ | $5_{88} \sim 5_{79}$ |
| | $4_{22} \sim 3_1$ | $\overline{5_6} \sim \overline{3_7}$ | $5_{71} \sim 5_{24}$ | $\overline{5_{94}} \sim 5_{80}$ |

| Space | Equivalences | | | |
|---|---|---|---|---|
| $\mathbf{L(5,1)}$ | $2_3 \sim 1_1$<br>$5_{75} \sim 2_2$<br>$5_{74} \sim 3_1$ | $4_{23} \sim \overline{3_2}$<br>$5_{85} \sim 3_6$<br>$5_5 \sim 3_7$ | $\overline{4_2} \sim \overline{3_8}$<br>$4_{22} \sim 4_1$<br>$5_{76} \sim \overline{4_{17}}$ | $4_{24} \sim O$<br>$\overline{5_{68}} \sim 4_{26}$<br>$5_{78} \sim 5_{67}$ |
| $\mathbf{L(5,2)}$ | $2_3 \sim 1_1$ | $4_{23} \sim 3_6$ | $5_{76} \sim 5_{75}$ | |
| $\mathbf{L(6,1)}$ | $3_6 \sim 1_1$<br>$5_{75} \sim \overline{3_2}$ | $\overline{5_6} \sim \overline{3_8}$<br>$5_{74} \sim 4_1$ | $4_{23} \sim \overline{4_2}$<br>$5_5 \sim 4_{22}$ | $5_{76} \sim \overline{5_{68}}$<br>$5_{85} \sim O$ |
| $\mathbf{L(7,1)}$ | $4_{24} \sim 1_1$<br>$3_6 \sim 2_3$ | $5_{75} \sim \overline{4_2}$<br>$\overline{5_6} \sim 4_{23}$ | $5_{74} \sim 5_5$ | |
| $\mathbf{L(7,2)}$ | $3_6 \sim 2_3$ | $5_{75} \sim 4_{24}$ | | |
| $\mathbf{L(8,1)}$ | $5_{85} \sim 1_1$ | $4_{24} \sim 2_3$ | $5_{75} \sim \overline{5_6}$ | |
| $\mathbf{L(8,3)}$ | / | | | |
| $\mathbf{L(9,1)}$ | $5_{85} \sim 2_3$ | $4_{24} \sim 3_6$ | | |
| $\mathbf{L(9,2)}$ | $4_{24} \sim 3_6$ | | | |
| $\mathbf{L(10,1)}$ | $5_{85} \sim 3_6$ | | | |
| $\mathbf{L(10,3)}$ | / | | | |
| $\mathbf{L(11,1)}$ | $5_{85} \sim 4_{24}$ | | | |
| $\mathbf{L(11,2)}$ | $5_{85} \sim 4_{24}$ | | | |
| $\mathbf{L(11,3)}$ | / | | | |
| $\mathbf{L(12,1)}$ | / | | | |
| $\mathbf{L(12,5)}$ | / | | | |

Legend: $O$ – affine unknot
♣ – affine trefoil

# C

# The HOMFLYPT skein modules

The HOMFLYPT skein modules of the non-affine prime knots are expressed in the standard basis using the evaluation $\langle \varnothing \rangle = 1$, see Theorem 4.1.1 and Theorem 4.2.1. For knots in $L(p,1)$ that are not listed below (and all knots in $L(12,1)$) the HSM expressions equal the HSM expressions of the corresponding knots in the solid torus.

## Knots in the solid torus

$\mathcal{S}_3(0_1) = t_1$
$\mathcal{S}_3(1_1) = t_2$
$\mathcal{S}_3(\overline{1_1}) = v^{-2}t_2 - v^{-1}zt_1^2$
$\mathcal{S}_3(2_1) = vzt_{-1}t_1 - v^3z^{-1} + vz^{-1}$
$\mathcal{S}_3(\overline{2_1}) = -v^{-1}zt_{-1}t_1 - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(2_2) = -v^{-1}zt_1t_2 + v^{-2}t_3$
$\mathcal{S}_3(2_3) = t_3$
$\mathcal{S}_3(\overline{2_3}) = v^{-2}z^2t_1^3 - 2v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(3_1) = vzt_{-1}t_2 + v^2t_1$
$\mathcal{S}_3(\overline{3_1}) = v^{-2}z^2t_{-1}t_1^2 - v^{-3}zt_{-1}t_2 + v^{-2}t_1$
$\mathcal{S}_3(3_2) = v^{-2}z^2t_1^2t_2 - v^{-3}zt_1t_3 - v^{-3}zt_2^2 + v^{-4}t_4$
$\mathcal{S}_3(\overline{3_2}) = -v^{-1}zt_1t_3 + v^{-2}t_4$
$\mathcal{S}_3(3_3) = vzt_1^2 + v^2t_2$
$\mathcal{S}_3(\overline{3_3}) = -v^{-1}zt_1^2 - v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(3_4) = -z^2t_{-1}t_1^2 + v^{-1}zt_{-1}t_2 + v^2t_1$
$\mathcal{S}_3(\overline{3_4}) = -v^{-1}zt_{-1}t_2 + v^{-2}t_1$
$\mathcal{S}_3(3_5) = -v^{-1}zt_2^2 + v^{-2}t_4$
$\mathcal{S}_3(\overline{3_5}) = v^{-2}z^2t_1^2t_2 - 2v^{-3}zt_1t_3 + v^{-4}t_4$
$\mathcal{S}_3(3_6) = t_4$
$\mathcal{S}_3(\overline{3_6}) = -v^{-3}z^3t_1^4 + 3v^{-4}z^2t_1^2t_2 - 2v^{-5}zt_1t_3 - v^{-5}zt_2^2 + v^{-6}t_4$
$\mathcal{S}_3(3_7) = v^3zt_1^2 + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{3_7}) = -v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(3_8) = v^3zt_{-1}t_1 - v^3z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{3_8}) = -v^{-3}zt_{-1}t_1 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(4_1) = vzt_{-1}t_3 + v^2t_2$
$\mathcal{S}_3(\overline{4_1}) = -v^{-3}z^3t_{-1}t_1^3 + 2v^{-4}z^2t_{-1}t_1t_2 - v^{-5}zt_{-1}t_3 - v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(4_2) = -v^{-3}z^3t_1^3t_2 + v^{-4}z^2t_1^2t_3 + 2v^{-4}z^2t_1t_2^2 - v^{-5}zt_1t_4 - 2v^{-5}zt_2t_3 + v^{-6}t_5$
$\mathcal{S}_3(\overline{4_2}) = -v^{-1}zt_1t_4 + v^{-2}t_5$
$\mathcal{S}_3(4_3) = v^2z^2t_{-1}t_1^2 - v^4t_1 + 2v^2t_1$
$\mathcal{S}_3(\overline{4_3}) = v^{-2}z^2t_{-1}t_1^2 + 2v^{-2}t_1 - v^{-4}t_1$
$\mathcal{S}_3(4_4) = -z^2t_{-1}t_1t_2 + v^{-1}zt_{-1}t_3 - vzt_1^2 + t_2$
$\mathcal{S}_3(\overline{4_4}) = v^{-2}z^2t_{-1}t_1t_2 - v^{-3}zt_{-1}t_3 + v^{-2}t_2$
$\mathcal{S}_3(4_5) = v^{-2}z^2t_1t_2^2 - 2v^{-3}zt_2t_3 + v^{-4}t_5$
$\mathcal{S}_3(\overline{4_5}) = v^{-2}z^2t_1^2t_3 - 2v^{-3}zt_1t_4 + v^{-4}t_5$
$\mathcal{S}_3(4_6) = v^3z^3t_{-1}t_1 + 2v^3zt_{-1}t_1 - v^5z - v^5z^{-1} + v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{4_6}) = -v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(4_7) = v^3zt_{-1}t_2 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{4_7}) = v^{-4}z^2t_{-1}t_1^2 - v^{-5}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(4_8) = -v^{-3}z^3t_1t_2 - 2v^{-3}zt_1t_2 + v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(\overline{4_8}) = -v^2z^2t_1^3 - vz^3t_1t_2 + z^2t_3 + t_3$
$\mathcal{S}_3(4_9) = -v^{-3}zt_1^2 + v^{-2}z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(\overline{4_9}) = v^3zt_1^2 - vz^3t_1^2 - vzt_1^2 + z^2t_2 + t_2$
$\mathcal{S}_3(4_{10}) = v^3zt_{-1}t_1 + vzt_{-1}t_1 - v^5z^{-1} + v^3z^{-1}$
$\mathcal{S}_3(\overline{4_{10}}) = -v^{-1}zt_{-1}t_1 - v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{11}) = -z^2t_1^3 - vzt_1t_2 + v^{-1}zt_1t_2 + t_3$
$\mathcal{S}_3(\overline{4_{11}}) = -v^{-1}zt_1t_2 - v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(4_{12}) = -z^2t_{-1}t_1t_2 + v^{-1}zt_{-1}t_3 + v^2t_2$
$\mathcal{S}_3(\overline{4_{12}}) = v^{-2}z^2t_{-1}t_1t_2 - v^{-3}zt_{-1}t_3 - v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(4_{13}) = -z^2t_{-1}t_1^2 + v^2t_1 - t_1 + v^{-2}t_1$
$\mathcal{S}_3(4_{14}) = v^{-2}z^2t_1t_2^2 - v^{-3}zt_1t_4 - v^{-3}zt_2t_3 + v^{-4}t_5$
$\mathcal{S}_3(4_{15}) = -v^{-1}zt_{-2}t_2 + v^{-1}zt_{-1}t_1 - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_{15}}) = -v^{-2}z^2t_{-2}t_1^2 + v^{-3}zt_{-2}t_2 + v^{-1}z^3t_{-1}^2t_1^2 - v^{-2}z^2t_{-1}^2t_2 - vzt_{-1}t_1 - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(4_{16}) = vzt_1t_2 + v^2t_3$
$\mathcal{S}_3(\overline{4_{16}}) = v^{-2}z^2t_1^3 + v^{-4}z^2t_1^3 - v^{-3}zt_1t_2 - 2v^{-5}zt_1t_2 + v^{-6}t_3$
$\mathcal{S}_3(4_{17}) = v^{-1}z^3t_{-1}t_1^3 - 2v^{-2}z^2t_{-1}t_1t_2 + v^{-3}zt_{-1}t_3 - vzt_1^2 + t_2$
$\mathcal{S}_3(\overline{4_{17}}) = -v^{-1}zt_{-1}t_3 + v^{-2}t_2$
$\mathcal{S}_3(4_{18}) = -v^{-1}zt_2t_3 + v^{-2}t_5$
$\mathcal{S}_3(\overline{4_{18}}) = -v^{-3}z^3t_1^3t_2 + 2v^{-4}z^2t_1^2t_3 + v^{-4}z^2t_1t_2^2 - 2v^{-5}zt_1t_4 - v^{-5}zt_2t_3 + v^{-6}t_5$
$\mathcal{S}_3(4_{19}) = v^{-2}z^2t_{-2}t_1^2 - v^{-3}zt_{-2}t_2 - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{4_{19}}) = -z^2t_{-2}t_1^2 + v^{-1}zt_{-2}t_2 - v^3z^{-1} + vz^{-1}$
$\mathcal{S}_3(4_{20}) = vzt_{-2}t_2 + v^3zt_{-1}t_1 - v^5z^{-1} + v^3z^{-1}$
$\mathcal{S}_3(\overline{4_{20}}) = v^{-4}z^2t_{-2}t_1^2 - v^{-5}zt_{-2}t_2 - v^{-3}z^3t_{-1}^2t_1^2 + v^{-4}z^2t_{-1}^2t_2 - v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{21}) = v^{-2}z^2t_1^2t_3 - v^{-3}zt_1t_4 - v^{-3}zt_2t_3 + v^{-4}t_5$
$\mathcal{S}_3(4_{22}) = v^3zt_1t_2 + v^2z^2t_3 + v^2t_3$
$\mathcal{S}_3(\overline{4_{22}}) = v^{-4}z^4t_1^3 + 2v^{-4}z^2t_1^3 - 2v^{-5}z^3t_1t_2 - 3v^{-5}zt_1t_2 + v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(4_{23}) = -v^{-3}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(\overline{4_{23}}) = -v^2z^2t_{-1}t_1^2 + vzt_{-1}t_2 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(4_{24}) = t_5$
$\mathcal{S}_3(\overline{4_{24}}) = v^{-4}z^4t_1^5 - 4v^{-5}z^3t_1^3t_2 + 3v^{-6}z^2t_1^2t_3 + 3v^{-6}z^2t_1t_2^2 - 2v^{-7}zt_1t_4 - 2v^{-7}zt_2t_3 + v^{-8}t_5$

$\mathcal{S}_3(4_{25}) = v^{-1}zt_{-1}t_1 - v^3z^{-1} + vz + vz^{-1} - v^{-1}z$
$\mathcal{S}_3(\overline{4_{25}}) = -vzt_{-1}t_1 + vz - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(4_{26}) = -vzt_1^2 - z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(\overline{4_{26}}) = -vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2 + t_2 - v^{-2}z^2t_2$
$\mathcal{S}_3(4_{27}) = z^2t_1^3 + v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2 - v^{-2}z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(5_1) = v^4z^2t_{-1}t_1^2 + v^3z^3t_{-1}t_2 + v^3zt_{-1}t_2 - v^6t_1 + v^4z^2t_1 + 2v^4t_1$
$\mathcal{S}_3(\overline{5_1}) = v^{-4}z^4t_{-1}t_1^2 + 2v^{-4}z^2t_{-1}t_1^2 - v^{-5}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 + v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-6}t_1$
$\mathcal{S}_3(5_2) = -v^2z^2t_{-1}t_1t_2 + vzt_{-1}t_3 + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{5_2}) = v^{-4}z^2t_{-1}t_1t_2 - v^{-5}zt_{-1}t_3 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(5_3) = -v^{-2}z^2t_{-1}t_1^2 + t_1 + v^{-2}z^2t_1$
$\mathcal{S}_3(\overline{5_3}) = -v^2z^2t_{-1}t_1^2 + v^2z^2t_1 + t_1$
$\mathcal{S}_3(5_4) = v^{-4}z^4t_1^2t_2 + 2v^{-4}z^2t_1^2t_2 - v^{-5}z^3t_1t_3 - 2v^{-5}zt_1t_3 - v^{-5}z^3t_2^2 - v^{-5}zt_2^2 + v^{-6}z^2t_4 + v^{-6}t_4$
$\mathcal{S}_3(\overline{5_4}) = -v^2z^2t_1^2t_2 - vz^3t_1t_3 + z^2t_4 + t_4$
$\mathcal{S}_3(5_5) = vzt_{-1}t_4 + v^2t_3$
$\mathcal{S}_3(\overline{5_5}) = v^{-4}z^4t_{-1}t_1^4 - 3v^{-5}z^3t_{-1}t_1^2t_2 + 2v^{-6}z^2t_{-1}t_1t_3 + v^{-6}z^2t_{-1}t_2^2 - v^{-7}zt_{-1}t_4 + v^{-4}z^2t_1^3 - 2v^{-5}zt_1t_2 + v^{-6}t_3$
$\mathcal{S}_3(5_6) = v^{-4}z^4t_1^4t_2 - v^{-5}z^3t_1^3t_3 - 3v^{-5}z^3t_1^2t_2^2 + v^{-6}z^2t_1^2t_4 + 4v^{-6}z^2t_1t_2t_3 - v^{-7}zt_1t_5 + v^{-6}z^2t_2^3 - 2v^{-7}zt_2t_4 - v^{-7}zt_3^2 + v^{-8}t_6$
$\mathcal{S}_3(\overline{5_6}) = -v^{-1}zt_1t_5 + v^{-2}t_6$
$\mathcal{S}_3(5_7) = v^2z^2t_{-1}^2t_2 + 2v^3zt_{-1}t_1 - v^5z^{-1} + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_7}) = v^{-4}z^2t_{-2}t_1^2 - v^{-3}z^3t_{-1}^2t_1^2 - 2v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(5_8) = -z^2t_{-1}t_1t_3 + v^{-1}zt_{-1}t_4 - vzt_1t_2 + t_3$
$\mathcal{S}_3(\overline{5_8}) = -v^{-3}z^3t_{-1}t_1^2t_2 + v^{-4}z^2t_{-1}t_1t_3 + v^{-4}z^2t_{-1}t_2^2 - v^{-5}zt_{-1}t_4 - v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(5_9) = -v^{-3}z^3t_1^2t_2^2 + 2v^{-4}z^2t_1t_2t_3 + v^{-4}z^2t_2^3 - 2v^{-5}zt_2t_4 - v^{-5}zt_3^2 + v^{-6}t_6$
$\mathcal{S}_3(\overline{5_9}) = v^{-2}z^2t_1^2t_4 - 2v^{-3}zt_1t_5 + v^{-4}t_6$
$\mathcal{S}_3(5_{10}) = v^2z^2t_{-1}t_1^2 + v^3zt_{-1}t_2 + v^2t_1$
$\mathcal{S}_3(\overline{5_{10}}) = v^{-2}z^2t_{-1}t_1^2 + v^{-4}z^2t_{-1}t_1^2 - v^{-5}zt_{-1}t_2 + v^{-2}t_1$
$\mathcal{S}_3(5_{11}) = -vz^3t_{-1}t_1^3 + z^2t_{-1}t_1t_2 + v^3zt_1^2 - vzt_1^2 + t_2$
$\mathcal{S}_3(\overline{5_{11}}) = v^{-2}z^2t_{-1}t_1t_2 - v^{-3}zt_1^2 + v^{-2}t_2$
$\mathcal{S}_3(5_{12}) = -z^2t_{-1}t_2^2 + v^{-1}zt_{-1}t_4 - vzt_1t_2 + t_3$
$\mathcal{S}_3(\overline{5_{12}}) = -v^{-3}z^3t_{-1}t_1^2t_2 + 2v^{-4}z^2t_{-1}t_1t_3 - v^{-5}zt_{-1}t_4 - v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(5_{13}) = -z^2t_{-1}^2t_2 - vzt_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_{13}}) = -v^{-2}z^2t_{-2}t_1^2 + v^{-1}z^3t_{-1}^2t_1^2 - vzt_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(5_{14}) = v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 + v^{-3}zt_{-1}t_4 - vzt_1t_2 + t_3$
$\mathcal{S}_3(\overline{5_{14}}) = v^{-2}z^2t_{-1}t_1t_3 - v^{-3}zt_{-1}t_4 - v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(5_{15}) = -v^{-3}z^3t_1^2t_2^2 + 3v^{-4}z^2t_1t_2t_3 - v^{-5}zt_1t_5 - v^{-5}zt_2t_4 - v^{-5}zt_3^2 + v^{-6}t_6$
$\mathcal{S}_3(\overline{5_{15}}) = v^{-2}z^2t_1t_2t_3 - v^{-3}zt_1t_5 - v^{-3}zt_2t_4 + v^{-4}t_6$
$\mathcal{S}_3(5_{16}) = v^{-2}z^2t_{-2}t_1t_2 - v^{-3}zt_{-2}t_3 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{16}}) = -v^{-2}z^2t_{-2}t_1t_2 + v^{-3}zt_{-2}t_3 + v^{-1}z^3t_{-1}^2t_1t_2 - v^{-2}z^2t_{-1}^2t_3 + v^2t_1$
$\mathcal{S}_3(5_{17}) = v^{-2}z^2t_1^2t_2 + v^{-4}z^2t_1^2t_2 - v^{-3}zt_1t_3 - v^{-5}zt_1t_3 - v^{-5}zt_2^2 + v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{17}}) = -z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_1t_3 + t_4$
$\mathcal{S}_3(5_{18}) = v^2z^2t_{-1}t_1t_2 + v^3zt_1^2 + v^2t_2$
$\mathcal{S}_3(\overline{5_{18}}) = -v^{-3}z^3t_{-1}t_1^3 + v^{-4}z^2t_{-1}t_1t_2 - 2v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(5_{19}) = v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 + v^{-3}zt_{-1}t_4 + z^2t_1^3 - 2v^{-1}zt_1t_2 + v^{-2}t_3$
$\mathcal{S}_3(\overline{5_{19}}) = v^{-2}z^2t_{-1}t_1t_3 - v^{-3}zt_{-1}t_4 + v^{-2}t_3$
$\mathcal{S}_3(5_{20}) = v^{-2}z^2t_1t_2t_3 - v^{-3}zt_2t_4 - v^{-3}zt_3^2 + v^{-4}t_6$

$\mathcal{S}_3(\overline{5_{20}}) = -v^{-3}z^3t_1^3t_3 + 2v^{-4}z^2t_1^2t_4 + v^{-4}z^2t_1t_2t_3 - 2v^{-5}zt_1t_5 - v^{-5}zt_2t_4 + v^{-6}t_6$
$\mathcal{S}_3(5_{21}) = v^{-4}z^2t_{-2}t_1t_2 - v^{-5}zt_{-2}t_3 - v^{-3}z^3t_{-1}^2t_1t_2 + v^{-4}z^2t_{-1}^2t_3 - v^{-3}zt_{-1}t_2 + v^{-4}t_1$
$\mathcal{S}_3(\overline{5_{21}}) = -z^2t_{-2}t_1t_2 + v^{-1}zt_{-2}t_3 - v^2z^2t_{-1}t_1^2 + vzt_{-1}t_2 + v^4t_1$
$\mathcal{S}_3(5_{22}) = -v^{-1}zt_{-1}t_3 + v^{-1}zt_1^2 + t_2$
$\mathcal{S}_3(\overline{5_{22}}) = v^{-1}z^3t_{-1}t_1^3 - 2v^{-2}z^2t_{-1}t_1t_2 + v^{-3}zt_{-1}t_3 - vzt_1^2 - v^{-1}zt_1^2 + v^{-2}t_2$
$\mathcal{S}_3(5_{23}) = -v^{-2}z^2t_{-1}t_1^2 + v^{-3}zt_{-1}t_2 + 2t_1 - v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{23}}) = -vzt_{-1}t_2 - v^2t_1 + 2t_1$
$\mathcal{S}_3(5_{24}) = vzt_1t_3 - vzt_2^2 + t_4$
$\mathcal{S}_3(\overline{5_{24}}) = -v^{-3}z^3t_1^4 + 3v^{-4}z^2t_1^2t_2 - 3v^{-5}zt_1t_3 + v^{-6}t_4$
$\mathcal{S}_3(5_{25}) = v^3zt_{-1}t_3 + v^3z^3t_1^2 + v^3zt_1^2 + v^4z^2t_2 + v^4t_2$
$\mathcal{S}_3(\overline{5_{25}}) = -v^{-5}z^3t_{-1}t_1^3 + 2v^{-6}z^2t_{-1}t_1t_2 - v^{-7}zt_{-1}t_3 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{26}) = -v^2z^4t_{-1}t_1^2 - v^2z^2t_{-1}t_1^2 + vz^3t_{-1}t_2 + vzt_{-1}t_2 + v^4z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{5_{26}}) = -v^{-3}z^3t_{-1}t_2 - v^{-3}zt_{-1}t_2 + v^{-2}t_1 + v^{-4}z^2t_1$
$\mathcal{S}_3(5_{27}) = -v^2z^4t_{-1}t_1^2 - v^2z^2t_{-1}t_1^2 + vz^3t_{-1}t_2 + vzt_{-1}t_2 + v^4z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{5_{27}}) = -v^{-3}z^3t_{-1}t_2 - v^{-3}zt_{-1}t_2 + v^{-2}t_1 + v^{-4}z^2t_1$
$\mathcal{S}_3(5_{28}) = -v^{-3}zt_1t_3 - v^{-3}z^3t_2^2 - v^{-3}zt_2^2 + v^{-4}z^2t_4 + v^{-4}t_4$
$\mathcal{S}_3(\overline{5_{28}}) = vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - v^{-1}zt_1t_3 + v^{-2}z^2t_4 + v^{-2}t_4$
$\mathcal{S}_3(5_{29}) = -vz^3t_{-1}t_1 - vzt_{-1}t_1 + v^{-1}zt_{-1}t_1 + v^3z - vz - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_{29}}) = -vzt_{-1}t_1 + v^{-1}z^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz^{-1} + v^{-1}z + v^{-1}z^{-1} - v^{-3}z$
$\mathcal{S}_3(5_{30}) = -vzt_{-1}t_2 - z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{30}}) = -v^{-2}z^2t_{-1}t_1^2 + v^{-3}zt_{-1}t_2 + v^2t_1 - z^2t_1$
$\mathcal{S}_3(5_{31}) = z^4t_{-1}t_1^2 + z^2t_{-1}t_1^2 - v^{-1}z^3t_{-1}t_2 - v^2z^2t_1 + t_1$
$\mathcal{S}_3(\overline{5_{31}}) = z^2t_{-1}t_1^2 + v^{-1}z^3t_{-1}t_2 + t_1 - v^{-2}z^2t_1$
$\mathcal{S}_3(5_{32}) = -vzt_1t_2 + v^{-1}z^3t_1t_2 + v^{-1}zt_1t_2 + t_3 - v^{-2}z^2t_3$
$\mathcal{S}_3(\overline{5_{32}}) = z^2t_1^3 + v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2 - v^{-3}zt_1t_2 - v^{-2}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(5_{33}) = v^{-1}zt_1^2 + v^2t_2 - z^2t_2$
$\mathcal{S}_3(\overline{5_{33}}) = -vzt_1^2 + v^{-1}z^3t_1^2 - v^{-3}zt_1^2 - v^{-2}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(5_{34}) = z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-1}z^3t_2^2 - v^{-2}z^2t_4 + v^{-2}t_4$
$\mathcal{S}_3(\overline{5_{34}}) = -v^{-1}z^3t_1^4 - v^{-2}z^4t_1^2t_2 + 2v^{-2}z^2t_1^2t_2 + 2v^{-3}z^3t_1t_3 - v^{-3}zt_1t_3 - v^{-3}zt_2^2 - v^{-4}z^2t_4 + v^{-4}t_4$
$\mathcal{S}_3(5_{35}) = v^3zt_{-2}t_2 + v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 - v^5z - v^5z^{-1} + v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_{35}}) = v^{-6}z^2t_{-2}t_1^2 - v^{-7}zt_{-2}t_2 - v^{-5}z^3t_{-1}^2t_1^2 + v^{-6}z^2t_{-1}^2t_2 - v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(5_{36}) = v^3zt_1t_2 - vz^3t_1t_2 - vzt_1t_2 + z^2t_3 + t_3$
$\mathcal{S}_3(\overline{5_{36}}) = v^{-4}z^2t_1^3 - v^{-3}z^3t_1t_2 - v^{-3}zt_1t_2 - v^{-5}zt_1t_2 + v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(5_{37}) = v^{-1}z^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - v^{-3}zt_{-1}t_1 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_{37}}) = v^3zt_{-1}t_1 - vz^3t_{-1}t_1 - vzt_{-1}t_1 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(5_{38}) = -v^2z^2t_{-2}t_1^2 + vzt_{-2}t_2 - v^3z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{5_{38}}) = v^{-4}z^2t_{-2}t_1^2 - v^{-5}zt_{-2}t_2 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(5_{39}) = -v^{-3}zt_1t_2 + v^{-2}z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(\overline{5_{39}}) = -v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 + vzt_1t_2 - 2v^{-1}z^3t_1t_2 - 2v^{-1}zt_1t_2 + v^{-2}z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(5_{40}) = v^3z^3t_{-1}t_2 + 2v^3zt_{-1}t_2 + v^4z^2t_1 + v^4t_1$
$\mathcal{S}_3(\overline{5_{40}}) = v^{-4}z^4t_{-1}t_1^2 + 2v^{-4}z^2t_{-1}t_1^2 - v^{-5}z^3t_{-1}t_2 - 2v^{-5}zt_{-1}t_2 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(5_{41}) = v^{-4}z^4t_1^2t_2 + 2v^{-4}z^2t_1^2t_2 - v^{-5}z^3t_1t_3 - v^{-5}zt_1t_3 - v^{-5}z^3t_2^2 - 2v^{-5}zt_2^2 + v^{-6}z^2t_4 + v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{41}}) = -v^2z^2t_1^2t_2 - vz^3t_1t_3 - vzt_1t_3 + vzt_2^2 + z^2t_4 + t_4$
$\mathcal{S}_3(5_{42}) = v^3zt_{-1}t_2 + vzt_{-1}t_2 + v^4t_1$

$\mathcal{S}_3(\overline{5_{42}}) = v^{-2}z^2t_{-1}t_1^2 + v^{-4}z^2t_{-1}t_1^2 - v^{-3}zt_{-1}t_2 - v^{-5}zt_{-1}t_2 + v^{-4}t_1$
$\mathcal{S}_3(5_{43}) = v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-3}zt_1t_3 - v^{-1}zt_2^2 + v^{-2}t_4$
$\mathcal{S}_3(\overline{5_{43}}) = -v^{-1}zt_1t_3 - v^{-3}zt_1t_3 + v^{-4}t_4$
$\mathcal{S}_3(5_{44}) = -z^2t_{-1}t_1t_3 + v^{-1}zt_{-1}t_4 + v^2t_3$
$\mathcal{S}_3(\overline{5_{44}}) = -v^{-3}z^3t_{-1}t_1^2t_2 + v^{-4}z^2t_{-1}t_1t_3 + v^{-4}z^2t_{-1}t_2^2 - v^{-5}zt_{-1}t_4 + v^{-4}z^2t_1^3 - 2v^{-5}zt_1t_2 + v^{-6}t_3$
$\mathcal{S}_3(5_{45}) = v^{-1}z^3t_{-1}t_1^3 - v^{-2}z^2t_{-1}t_1t_2 - vzt_1^2 + v^{-1}zt_1^2 + t_2$
$\mathcal{S}_3(\overline{5_{45}}) = -z^2t_{-1}t_1t_2 - vzt_1^2 + v^{-2}t_2$
$\mathcal{S}_3(5_{46}) = -z^2t_{-2}t_1t_2 + v^{-1}zt_{-2}t_3 + v^2t_1$
$\mathcal{S}_3(\overline{5_{46}}) = v^{-4}z^2t_{-2}t_1t_2 - v^{-5}zt_{-2}t_3 - v^{-3}z^3t_{-1}^2t_1t_2 + v^{-4}z^2t_{-1}^2t_3 + v^{-2}t_1$
$\mathcal{S}_3(5_{47}) = -v^{-3}z^3t_1^2t_2^2 + v^{-4}z^2t_1^2t_4 + v^{-4}z^2t_1t_2t_3 - v^{-5}zt_1t_5 + v^{-4}z^2t_2^3 - 2v^{-5}zt_2t_4 + v^{-6}t_6$
$\mathcal{S}_3(\overline{5_{47}}) = v^{-2}z^2t_1t_2t_3 - v^{-3}zt_1t_5 - v^{-3}zt_3^2 + v^{-4}t_6$
$\mathcal{S}_3(5_{48}) = -v^{-1}zt_{-2}t_3 + v^{-1}zt_{-1}t_2 + t_1$
$\mathcal{S}_3(\overline{5_{48}}) = v^{-3}z^3t_{-2}t_1^3 - 2v^{-4}z^2t_{-2}t_1t_2 + v^{-5}zt_{-2}t_3 - v^{-2}z^4t_{-1}^2t_1^3 + 2v^{-3}z^3t_{-1}^2t_1t_2 - v^{-4}z^2t_{-1}^2t_3 + z^2t_{-1}t_1^2 - v^{-1}zt_{-1}t_2 + t_1$
$\mathcal{S}_3(5_{49}) = v^{-2}z^2t_1^2t_2 + v^{-4}z^2t_1^2t_2 - v^{-5}zt_1t_3 - v^{-3}zt_2^2 - v^{-5}zt_2^2 + v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{49}}) = -z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_2^2 + t_4$
$\mathcal{S}_3(5_{50}) = v^3z^3t_1^2 + 2v^3zt_1^2 + v^4z^2t_2 + v^4t_2$
$\mathcal{S}_3(\overline{5_{50}}) = -v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{51}) = v^4z^2t_{-1}t_1^2 - v^6t_1 + v^4t_1 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{5_{51}}) = v^{-4}z^2t_{-1}t_1^2 + v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}t_1 - v^{-6}t_1$
$\mathcal{S}_3(5_{52}) = -v^{-2}z^4t_{-1}t_1^2 - 2v^{-2}z^2t_{-1}t_1^2 + v^{-3}z^3t_{-1}t_2 + v^{-3}zt_{-1}t_2 + z^2t_1 + 2t_1 - v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{52}}) = -v^2z^2t_{-1}t_1^2 - vz^3t_{-1}t_2 - vzt_{-1}t_2 - v^2t_1 + z^2t_1 + 2t_1$
$\mathcal{S}_3(5_{53}) = -v^2z^2t_1^2t_2 + vzt_1t_3 - vz^3t_2^2 - vzt_2^2 + z^2t_4 + t_4$
$\mathcal{S}_3(\overline{5_{53}}) = v^{-4}z^4t_1^2t_2 + 2v^{-4}z^2t_1^2t_2 - 2v^{-5}z^3t_1t_3 - 3v^{-5}zt_1t_3 + v^{-6}z^2t_4 + v^{-6}t_4$
$\mathcal{S}_3(5_{54}) = -v^{-3}z^3t_{-1}t_2 - 2v^{-3}zt_{-1}t_2 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(\overline{5_{54}}) = -v^2z^4t_{-1}t_1^2 - 2v^2z^2t_{-1}t_1^2 + vz^3t_{-1}t_2 + 2vzt_{-1}t_2 + v^4z^2t_1 + v^4t_1$
$\mathcal{S}_3(5_{55}) = v^{-4}z^2t_{-1}t_1t_2 - v^{-5}zt_{-1}t_3 + v^{-2}z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(\overline{5_{55}}) = -v^2z^2t_{-1}t_1t_2 + vzt_{-1}t_3 - vz^3t_1^2 - vzt_1^2 + z^2t_2 + t_2$
$\mathcal{S}_3(5_{56}) = -v^{-3}z^3t_2^2 - 2v^{-3}zt_2^2 + v^{-4}z^2t_4 + v^{-4}t_4$
$\mathcal{S}_3(\overline{5_{56}}) = vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - 2v^{-1}zt_1t_3 + v^{-1}zt_2^2 + v^{-2}z^2t_4 + v^{-2}t_4$
$\mathcal{S}_3(5_{57}) = v^3zt_1^2 + vzt_1^2 + v^4t_2$
$\mathcal{S}_3(\overline{5_{57}}) = -v^{-1}zt_1^2 - v^{-3}zt_1^2 - v^{-5}zt_1^2 + v^{-6}t_2$
$\mathcal{S}_3(5_{58}) = -v^2z^2t_{-1}t_1^2 - z^2t_{-1}t_1^2 + vzt_{-1}t_2 + v^{-1}zt_{-1}t_2 + v^4t_1$
$\mathcal{S}_3(\overline{5_{58}}) = -v^{-1}zt_{-1}t_2 - v^{-3}zt_{-1}t_2 + v^{-4}t_1$
$\mathcal{S}_3(5_{59}) = -z^2t_1^2t_2 + v^{-1}zt_1t_3 - vzt_2^2 + t_4$
$\mathcal{S}_3(\overline{5_{59}}) = v^{-2}z^2t_1^2t_2 + v^{-4}z^2t_1^2t_2 - v^{-3}zt_1t_3 - 2v^{-5}zt_1t_3 + v^{-6}t_4$
$\mathcal{S}_3(5_{60}) = -z^2t_{-1}t_1^2 - vzt_{-1}t_2 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{60}}) = -z^2t_{-1}t_1^2 - v^{-2}z^2t_{-1}t_1^2 + v^{-3}zt_{-1}t_2 + v^2t_1$
$\mathcal{S}_3(5_{61}) = -vz^3t_{-1}^2t_1^2 + z^2t_{-1}^2t_2 + 2v^3zt_{-1}t_1 - v^5z^{-1} + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_{61}}) = v^{-2}z^2t_{-2}t_1^2 - 2v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(5_{62}) = v^{-1}z^3t_{-1}t_1^2t_2 - 2v^{-2}z^2t_{-1}t_1t_3 + v^{-3}zt_{-1}t_4 - vzt_1t_2 + t_3$
$\mathcal{S}_3(\overline{5_{62}}) = v^{-2}z^2t_{-1}t_2^2 - v^{-3}zt_{-1}t_4 - v^{-3}zt_1t_2 + v^{-4}t_3$
$\mathcal{S}_3(5_{63}) = -z^2t_{-1}t_1t_2 + v^{-1}zt_1^2 + v^2t_2$
$\mathcal{S}_3(\overline{5_{63}}) = v^{-1}z^3t_{-1}t_1^3 - v^{-2}z^2t_{-1}t_1t_2 - vzt_1^2 - v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(5_{64}) = -v^{-2}z^2t_{-2}t_1t_2 + v^{-3}zt_{-2}t_3 + v^{-1}z^3t_{-1}^2t_1t_2 - v^{-2}z^2t_{-1}^2t_3 - vzt_{-1}t_2 + t_1$

$\mathcal{S}_3(\overline{5_{64}}) = v^{-2}z^2t_{-2}t_1t_2 - v^{-3}zt_{-2}t_3 - v^{-2}z^2t_{-1}t_1^2 + v^{-3}zt_{-1}t_2 + t_1$
$\mathcal{S}_3(5_{65}) = v^{-2}z^2t_2^3 - 2v^{-3}zt_2t_4 + v^{-4}t_6$
$\mathcal{S}_3(\overline{5_{65}}) = -v^{-3}z^3t_1^2t_2^2 + v^{-4}z^2t_1^2t_4 + 2v^{-4}z^2t_1t_2t_3 - 2v^{-5}zt_1t_5 - v^{-5}zt_3^2 + v^{-6}t_6$
$\mathcal{S}_3(5_{66}) = -v^{-1}zt_2^2 - v^{-3}zt_2^2 + v^{-4}t_4$
$\mathcal{S}_3(\overline{5_{66}}) = v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - 2v^{-1}zt_1t_3 + v^{-3}zt_2^2 + v^{-2}t_4$
$\mathcal{S}_3(5_{67}) = vzt_1t_3 + v^2t_4$
$\mathcal{S}_3(\overline{5_{67}}) = -v^{-3}z^3t_1^4 - v^{-5}z^3t_1^4 + 2v^{-4}z^2t_1^2t_2 + 3v^{-6}z^2t_1^2t_2 - v^{-5}zt_1t_3 - 2v^{-7}zt_1t_3 - v^{-7}zt_2^2 + v^{-8}t_4$
$\mathcal{S}_3(5_{68}) = -v^{-2}z^4t_{-1}t_1^4 + 3v^{-3}z^3t_{-1}t_1^2t_2 - 2v^{-4}z^2t_{-1}t_1t_3 - v^{-4}z^2t_{-1}t_2^2 + v^{-5}zt_{-1}t_4 + z^2t_1^3 - 2v^{-1}zt_1t_2 + v^{-2}t_3$
$\mathcal{S}_3(\overline{5_{68}}) = -v^{-1}zt_{-1}t_4 + v^{-2}t_3$
$\mathcal{S}_3(5_{69}) = -v^{-1}zt_2t_4 + v^{-2}t_6$
$\mathcal{S}_3(\overline{5_{69}}) = v^{-4}z^4t_1^4t_2 - 2v^{-5}z^3t_1^3t_3 - 2v^{-5}z^3t_1^2t_2^2 + 2v^{-6}z^2t_1^2t_4 + 4v^{-6}z^2t_1t_2t_3 - 2v^{-7}zt_1t_5 - v^{-7}zt_2t_4 - v^{-7}zt_3^2 + v^{-8}t_6$
$\mathcal{S}_3(5_{70}) = -v^{-3}z^3t_{-2}t_1^3 + 2v^{-4}z^2t_{-2}t_1t_2 - v^{-5}zt_{-2}t_3 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{70}}) = v^{-1}zt_{-2}t_3 - z^2t_{-1}^2t_3 + v^2t_1$
$\mathcal{S}_3(5_{71}) = -v^{-3}zt_{-2}t_2 + v^{-1}z^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_{71}}) = -z^2t_{-2}t_1^2 + v^{-1}zt_{-2}t_2 + vz^3t_{-1}^2t_1^2 - z^2t_{-1}^2t_2 - vz^3t_{-1}t_1 - vzt_{-1}t_1 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(5_{72}) = vzt_{-2}t_3 + v^3zt_{-1}t_2 + v^4t_1$
$\mathcal{S}_3(\overline{5_{72}}) = -v^{-5}z^3t_{-2}t_1^3 + 2v^{-6}z^2t_{-2}t_1t_2 - v^{-7}zt_{-2}t_3 + v^{-4}z^4t_{-1}^2t_1^3 - 2v^{-5}z^3t_{-1}^2t_1t_2 + v^{-6}z^2t_{-1}^2t_3 + v^{-4}z^2t_{-1}t_1^2 - v^{-5}zt_{-1}t_2 + v^{-4}t_1$
$\mathcal{S}_3(5_{73}) = -v^{-3}z^3t_1^3t_3 + v^{-4}z^2t_1^2t_4 + 2v^{-4}z^2t_1t_2t_3 - v^{-5}zt_1t_5 - v^{-5}zt_2t_4 - v^{-5}zt_3^2 + v^{-6}t_6$
$\mathcal{S}_3(\overline{5_{73}}) = v^{-2}z^2t_1^2t_4 - v^{-3}zt_1t_5 - v^{-3}zt_2t_4 + v^{-4}t_6$
$\mathcal{S}_3(5_{74}) = v^3zt_1t_3 + v^2z^2t_4 + v^2t_4$
$\mathcal{S}_3(\overline{5_{74}}) = -v^{-5}z^5t_1^4 - 2v^{-5}z^3t_1^4 + 3v^{-6}z^4t_1^2t_2 + 5v^{-6}z^2t_1^2t_2 - 2v^{-7}z^3t_1t_3 - 3v^{-7}zt_1t_3 - v^{-7}z^3t_2^2 - v^{-7}zt_2^2 + v^{-8}z^2t_4 + v^{-8}t_4$
$\mathcal{S}_3(5_{75}) = -v^{-3}zt_{-1}t_3 + v^{-2}z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(\overline{5_{75}}) = vz^3t_{-1}t_1^3 - 2z^2t_{-1}t_1t_2 + v^{-1}zt_{-1}t_3 - vz^3t_1^2 - vzt_1^2 + z^2t_2 + t_2$
$\mathcal{S}_3(5_{76}) = -vzt_1t_2 - z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(\overline{5_{76}}) = z^2t_1^3 - v^{-2}z^4t_1^3 - v^{-2}z^2t_1^3 - 2v^{-1}zt_1t_2 + 2v^{-3}z^3t_1t_2 + v^{-3}zt_1t_2 + v^{-2}t_3 - v^{-4}z^2t_3$
$\mathcal{S}_3(5_{77}) = z^2t_1^2t_2 + v^{-1}z^3t_1t_3 - v^{-1}zt_2^2 - v^{-2}z^2t_4 + v^{-2}t_4$
$\mathcal{S}_3(\overline{5_{77}}) = -v^{-1}z^3t_1^4 - v^{-2}z^4t_1^2t_2 + 2v^{-2}z^2t_1^2t_2 + v^{-3}z^3t_1t_3 - 2v^{-3}zt_1t_3 + v^{-3}z^3t_2^2 - v^{-4}z^2t_4 + v^{-4}t_4$
$\mathcal{S}_3(5_{78}) = v^{-1}zt_{-1}t_2 + v^2t_1 - z^2t_1$
$\mathcal{S}_3(\overline{5_{78}}) = z^2t_{-1}t_1^2 - v^{-1}zt_{-1}t_2 - z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(5_{79}) = v^3zt_{-1}t_3 + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{5_{79}}) = -v^{-5}z^3t_{-1}t_1^3 + 2v^{-6}z^2t_{-1}t_1t_2 - v^{-7}zt_{-1}t_3 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(5_{80}) = -v^{-3}z^3t_1t_3 - 2v^{-3}zt_1t_3 + v^{-4}z^2t_4 + v^{-4}t_4$
$\mathcal{S}_3(\overline{5_{80}}) = vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - v^{-1}z^3t_1t_3 - v^{-1}z^3t_2^2 - v^{-1}zt_2^2 + v^{-2}z^2t_4 + v^{-2}t_4$
$\mathcal{S}_3(5_{81}) = v^3zt_2^2 + v^2z^2t_4 + v^2t_4$
$\mathcal{S}_3(\overline{5_{81}}) = -v^{-5}z^5t_1^4 - 2v^{-5}z^3t_1^4 + 3v^{-6}z^4t_1^2t_2 + 5v^{-6}z^2t_1^2t_2 - 2v^{-7}z^3t_1t_3 - 2v^{-7}zt_1t_3 - v^{-7}z^3t_2^2 - 2v^{-7}zt_2^2 + v^{-8}z^2t_4 + v^{-8}t_4$
$\mathcal{S}_3(5_{82}) = vzt_2^2 + v^2t_4$
$\mathcal{S}_3(\overline{5_{82}}) = -v^{-3}z^3t_1^4 - v^{-5}z^3t_1^4 + 2v^{-4}z^2t_1^2t_2 + 3v^{-6}z^2t_1^2t_2 - 2v^{-7}zt_1t_3 - v^{-5}zt_2^2 - v^{-7}zt_2^2 + v^{-8}t_4$
$\mathcal{S}_3(5_{83}) = v^{-1}z^3t_{-2}t_1^3 - 2v^{-2}z^2t_{-2}t_1t_2 + v^{-3}zt_{-2}t_3 + v^2t_1$
$\mathcal{S}_3(\overline{5_{83}}) = -v^{-3}zt_{-2}t_3 + v^{-2}z^2t_{-1}^2t_3 + v^{-2}t_1$
$\mathcal{S}_3(5_{84}) = -v^{-1}zt_3^2 + v^{-2}t_6$
$\mathcal{S}_3(\overline{5_{84}}) = v^{-4}z^4t_1^4t_2 - 2v^{-5}z^3t_1^3t_3 - 2v^{-5}z^3t_1^2t_2^2 + 3v^{-6}z^2t_1^2t_4 + 2v^{-6}z^2t_1t_2t_3 - 2v^{-7}zt_1t_5 + v^{-6}z^2t_2^3 - 2v^{-7}zt_2t_4 +$

$v^{-8}t_6$

$\mathcal{S}_3(5_{85}) = t_6$

$\mathcal{S}_3(\overline{5_{85}}) = -v^{-5}z^5t_1^6 + 5v^{-6}z^4t_1^4t_2 - 4v^{-7}z^3t_1^3t_3 - 6v^{-7}z^3t_1^2t_2^2 + 3v^{-8}z^2t_1^2t_4 + 6v^{-8}z^2t_1t_2t_3 - 2v^{-9}zt_1t_5 + v^{-8}z^2t_2^3 - 2v^{-9}zt_2t_4 - v^{-9}zt_3^2 + v^{-10}t_6$

$\mathcal{S}_3(5_{86}) = v^{-1}z^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz - 2vz^{-1} + 2v^{-1}z + 3v^{-1}z^{-1} - v^{-3}z - v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{86}}) = -vz^3t_{-1}t_1 - vzt_{-1}t_1 + v^3z + v^3z^{-1} - 2vz - 3vz^{-1} + v^{-1}z + 2v^{-1}z^{-1}$

$\mathcal{S}_3(5_{87}) = v^5zt_1^2 + v^3zt_1^2 + v^4z^2t_2 + v^4t_2 + v^2z^2t_2$

$\mathcal{S}_3(\overline{5_{87}}) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 + v^{-4}z^2t_2 + v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(5_{88}) = v^5zt_1^2 + v^4z^2t_2 + v^2z^2t_2 + v^2t_2$

$\mathcal{S}_3(\overline{5_{88}}) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-4}z^2t_2 + v^{-4}t_2 + v^{-6}z^2t_2$

$\mathcal{S}_3(5_{89}) = v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 + v^7z^{-1} - 2v^5z - 3v^5z^{-1} + 2v^3z + 2v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{89}}) = -v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 - 2v^{-3}z - 2v^{-3}z^{-1} + 2v^{-5}z + 3v^{-5}z^{-1} - v^{-7}z^{-1}$

$\mathcal{S}_3(5_{90}) = v^5zt_{-1}t_1 + v^3zt_{-1}t_1 - v^5z - v^5z^{-1} + v^3z^{-1} + vz$

$\mathcal{S}_3(\overline{5_{90}}) = -v^{-3}zt_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-1}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{91}) = v^5zt_{-1}t_1 - v^5z - v^3z^{-1} + vz + vz^{-1}$

$\mathcal{S}_3(\overline{5_{91}}) = -v^{-5}zt_{-1}t_1 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1} + v^{-5}z$

$\mathcal{S}_3(5_{92}) = v^3zt_{-1}t_2 - v^6t_1 + v^4z^2t_1 + 2v^4t_1 + v^2z^2t_1$

$\mathcal{S}_3(\overline{5_{92}}) = v^{-4}z^2t_{-1}t_1^2 - v^{-5}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{93}) = v^5z^3t_1^2 + 2v^5zt_1^2 + v^4z^4t_2 + 3v^4z^2t_2 + v^4t_2$

$\mathcal{S}_3(\overline{5_{93}}) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 3v^{-5}zt_1^2 + v^{-6}z^4t_2 + 3v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(5_{94}) = v^5z^3t_{-1}t_1 + 2v^5zt_{-1}t_1 - v^5z^3 - 3v^5z - v^5z^{-1} + v^3z^3 + 3v^3z + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{94}}) = -v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{95}) = v^5z^3t_{-1}t_1 + v^5zt_{-1}t_1 + v^7z^{-1} - v^5z^3 - 3v^5z - 3v^5z^{-1} + v^3z^3 + 3v^3z + 2v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{95}}) = -v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - 2v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + 3v^{-5}z^{-1} - v^{-7}z^{-1}$

## Knots in $L(2,1)$

$\mathcal{S}_3(\overline{1_1}) = -v^{-1}zt_1^2 - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(2_1) = v^3zt_1^2 - v^3z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{2_1}) = -vzt_1^2 + vz - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(2_2) = v^{-2}z^2t_1 + 2v^{-2}t_1 - v^{-4}t_1$
$\mathcal{S}_3(2_3) = t_1$
$\mathcal{S}_3(\overline{2_3}) = v^{-2}z^2t_1^3 + 2v^{-4}z^2t_1 + 3v^{-4}t_1 - 2v^{-6}t_1$
$\mathcal{S}_3(3_1) = -v^4t_1 + v^2z^2t_1 + 2v^2t_1$
$\mathcal{S}_3(\overline{3_1}) = v^2z^2t_1^3 - z^4t_1 - 2z^2t_1 + t_1 + v^{-2}z^2t_1$
$\mathcal{S}_3(3_2) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 3v^{-5}zt_1^2 + v^{-7}zt_1^2 - v^{-5}z^3 - 3v^{-5}z - 2v^{-5}z^{-1} + v^{-7}z^3 + 3v^{-7}z + 3v^{-7}z^{-1} - v^{-9}z^{-1}$
$\mathcal{S}_3(\overline{3_2}) = -v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(3_3) = vzt_1^2 - v^3z^{-1} + vz^{-1}$
$\mathcal{S}_3(\overline{3_3}) = -v^{-1}zt_1^2 - v^{-3}zt_1^2 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(3_4) = -v^4z^2t_1^3 + v^2z^4t_1 + 2v^2z^2t_1 - z^2t_1 + t_1$
$\mathcal{S}_3(\overline{3_4}) = v^2t_1 - z^2t_1 - t_1 + v^{-2}t_1$
$\mathcal{S}_3(3_5) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - 2v^{-3}z - 2v^{-3}z^{-1} + 2v^{-5}z + 3v^{-5}z^{-1} - v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{3_5}) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 4v^{-5}zt_1^2 + v^{-7}zt_1^2 - v^{-5}z^3 - 2v^{-5}z - v^{-5}z^{-1} + v^{-7}z^3 + 2v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{3_6}) = -v^{-3}z^3t_1^4 - 3v^{-7}z^5t_1^2 - 9v^{-7}z^3t_1^2 - 6v^{-7}zt_1^2 + 3v^{-9}zt_1^2 - 3v^{-7}z^3 - 4v^{-7}z - 2v^{-7}z^{-1} + 3v^{-9}z^3 + 4v^{-9}z + 3v^{-9}z^{-1} - v^{-11}z^{-1}$
$\mathcal{S}_3(3_8) = v^5zt_1^2 - v^5z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{4_1}) = -v^3z^3t_1^4 + v^{-1}z^7t_1^2 + 3v^{-1}z^5t_1^2 + v^{-1}z^3t_1^2 - 3v^{-1}zt_1^2 - v^{-3}z^3t_1^2 + v^{-3}zt_1^2 + v^{-1}z^5 + 2v^{-1}z^3 + 3v^{-1}z - v^{-3}z^5 - 2v^{-3}z^3 - 4v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(4_2) = v^{-8}z^8t_1^3 + 6v^{-8}z^6t_1^3 + 10v^{-8}z^4t_1^3 + 4v^{-8}z^2t_1^3 - v^{-10}z^2t_1^3 + 2v^{-10}z^8t_1 + 14v^{-10}z^6t_1 + 32v^{-10}z^4t_1 + 28v^{-10}z^2t_1 + 8v^{-10}t_1 - 2v^{-12}z^6t_1 - 11v^{-12}z^4t_1 - 18v^{-12}z^2t_1 - 9v^{-12}t_1 + 2v^{-14}t_1$
$\mathcal{S}_3(\overline{4_2}) = v^{-4}z^4t_1^3 + 2v^{-4}z^2t_1^3 + 2v^{-6}z^4t_1 + 6v^{-6}z^2t_1 + 4v^{-6}t_1 - 2v^{-8}z^2t_1 - 3v^{-8}t_1$
$\mathcal{S}_3(4_3) = v^6z^2t_1^3 - v^4z^4t_1 - 2v^4z^2t_1 - v^4t_1 + 2v^2z^2t_1 + 2v^2t_1$
$\mathcal{S}_3(\overline{4_3}) = v^2z^2t_1^3 - z^4t_1 - 2z^2t_1 + 2v^{-2}z^2t_1 + 2v^{-2}t_1 - v^{-4}t_1$
$\mathcal{S}_3(4_4) = v^3z^3t_1^2 + 2v^3zt_1^2 - vz^3t_1^2 - 2vzt_1^2 - v^3z^3 - 2v^3z + vz^3 + 2vz - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_4}) = -vz^3t_1^2 - 2vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2 + vz^3 + 2vz - v^{-1}z^3 - 2v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(4_5) = v^{-6}z^6t_1^3 + 3v^{-6}z^4t_1^3 + v^{-6}z^2t_1^3 + 3v^{-8}z^6t_1 + 14v^{-8}z^4t_1 + 18v^{-8}z^2t_1 + 6v^{-8}t_1 - 3v^{-10}z^4t_1 - 10v^{-10}z^2t_1 - 6v^{-10}t_1 + v^{-12}t_1$
$\mathcal{S}_3(\overline{4_5}) = v^{-6}z^6t_1^3 + 4v^{-6}z^4t_1^3 + 4v^{-6}z^2t_1^3 + 2v^{-8}z^6t_1 + 9v^{-8}z^4t_1 + 12v^{-8}z^2t_1 + 5v^{-8}t_1 - 2v^{-10}z^4t_1 - 6v^{-10}z^2t_1 - 4v^{-10}t_1$
$\mathcal{S}_3(4_6) = v^5z^3t_1^2 + 2v^5zt_1^2 - v^5z^3 - 3v^5z - v^5z^{-1} + v^3z^3 + 3v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{4_6}) = -v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2 + v^{-1}z^3 + 2v^{-1}z - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(4_7) = -v^6t_1 + v^4z^2t_1 + v^4t_1 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{4_7}) = z^2t_1^3 - v^{-2}z^4t_1 - v^{-2}z^2t_1 + 2v^{-2}t_1 + v^{-4}z^2t_1 - v^{-4}t_1$
$\mathcal{S}_3(4_8) = v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 3v^{-4}t_1 - v^{-6}z^2t_1 - 2v^{-6}t_1$
$\mathcal{S}_3(\overline{4_8}) = -v^2z^2t_1^3 + z^4t_1 + 2z^2t_1 + t_1 - v^{-2}z^2t_1$
$\mathcal{S}_3(4_9) = -v^{-3}zt_1^2 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{4_9}) = v^3zt_1^2 - vz^3t_1^2 - vzt_1^2 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(4_{10}) = v^5zt_1^2 + v^3zt_1^2 - v^5z - v^5z^{-1} + v^3z^{-1} + vz$
$\mathcal{S}_3(\overline{4_{10}}) = -vzt_1^2 - v^{-1}zt_1^2 + vz - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{11}) = -z^2t_1^3 + z^2t_1 + 2t_1 - v^{-2}z^2t_1 - 2v^{-2}t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{4_{11}}) = v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}z^2t_1 + v^{-4}t_1 - v^{-6}t_1$
$\mathcal{S}_3(4_{12}) = v^3z^3t_1^2 + 2v^3zt_1^2 - vz^3t_1^2 - vzt_1^2 - v^3z^3 - 2v^3z - v^3z^{-1} + vz^3 + 2vz + vz^{-1}$
$\mathcal{S}_3(\overline{4_{12}}) = -vz^3t_1^2 - 2vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2 - v^{-3}zt_1^2 + vz^3 + 2vz - v^{-1}z^3 - 2v^{-1}z - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{13}) = -v^4z^2t_1^3 + v^2z^4t_1 + 2v^2z^2t_1 + v^2t_1 - 2z^2t_1 - t_1 + v^{-2}t_1$
$\mathcal{S}_3(4_{14}) = v^{-6}z^6t_1^3 + 3v^{-6}z^4t_1^3 + 2v^{-6}z^2t_1^3 + 3v^{-8}z^6t_1 + 13v^{-8}z^4t_1 + 16v^{-8}z^2t_1 + 6v^{-8}t_1 - 3v^{-10}z^4t_1 - 9v^{-10}z^2t_1 - 6v^{-10}t_1 + v^{-12}t_1$
$\mathcal{S}_3(4_{15}) = -v^3z^3t_1^2 + vzt_1^2 - v^5z^{-1} + v^3z^3 + 2v^3z + 2v^3z^{-1} - vz^3 - 3vz - 2vz^{-1} + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_{15}}) = v^7z^3t_1^4 - v^3z^7t_1^2 - 4v^3z^5t_1^2 - 4v^3z^3t_1^2 + v^3zt_1^2 + vz^3t_1^2 - 2vzt_1^2 + 2v^3z^3 + v^3z + v^3z^{-1} - 3vz^3 - 3vz - 3vz^{-1} + v^{-1}z^3 + 2v^{-1}z + 2v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_{16}}) = v^{-2}z^2t_1^3 + v^{-4}z^2t_1^3 + v^{-4}z^2t_1 + v^{-4}t_1 + 2v^{-6}z^2t_1 + 2v^{-6}t_1 - 2v^{-8}t_1$
$\mathcal{S}_3(4_{17}) = v^5z^3t_1^4 - vz^7t_1^2 - 3vz^5t_1^2 - vz^3t_1^2 + 2vzt_1^2 + v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2 - vz^5 - 2vz^3 - 3vz - vz^{-1} + v^{-1}z^5 + 2v^{-1}z^3 + 4v^{-1}z + v^{-1}z^{-1} - v^{-3}z$
$\mathcal{S}_3(\overline{4_{17}}) = -v^3zt_1^2 + v^3z - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(4_{18}) = v^{-4}z^4t_1^3 + v^{-4}z^2t_1^3 + 3v^{-6}z^4t_1 + 8v^{-6}z^2t_1 + 4v^{-6}t_1 - 3v^{-8}z^2t_1 - 3v^{-8}t_1$
$\mathcal{S}_3(\overline{4_{18}}) = v^{-8}z^8t_1^3 + 6v^{-8}z^6t_1^3 + 11v^{-8}z^4t_1^3 + 6v^{-8}z^2t_1^3 - v^{-10}z^2t_1^3 + 2v^{-10}z^8t_1 + 13v^{-10}z^6t_1 + 28v^{-10}z^4t_1 + 24v^{-10}z^2t_1 + 7v^{-10}t_1 - 2v^{-12}z^6t_1 - 10v^{-12}z^4t_1 - 15v^{-12}z^2t_1 - 7v^{-12}t_1 + v^{-14}t_1$
$\mathcal{S}_3(4_{19}) = -v^3zt_1^2 + vz^3t_1^2 + vzt_1^2 - v^3z^{-1} + 2vz + 2vz^{-1} - 2v^{-1}z - 2v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{4_{19}}) = v^5zt_1^2 - v^3z^3t_1^2 - v^3zt_1^2 + v^5z^{-1} - 2v^3z - 3v^3z^{-1} + 2vz + 2vz^{-1}$
$\mathcal{S}_3(4_{20}) = v^5z^3t_1^2 + v^5zt_1^2 + v^7z^{-1} - v^5z^3 - 3v^5z - 3v^5z^{-1} + v^3z^3 + 3v^3z + 2v^3z^{-1}$
$\mathcal{S}_3(\overline{4_{20}}) = -v^5z^3t_1^4 + vz^7t_1^2 + 4vz^5t_1^2 + 4vz^3t_1^2 - 2vzt_1^2 - v^{-1}z^3t_1^2 + v^{-1}zt_1^2 - 2vz^3 - vz^{-1} + 3v^{-1}z^3 + 3v^{-1}z + 2v^{-1}z^{-1} - v^{-3}z^3 - 3v^{-3}z - 2v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{21}) = v^{-6}z^6t_1^3 + 4v^{-6}z^4t_1^3 + 3v^{-6}z^2t_1^3 + 2v^{-8}z^6t_1 + 10v^{-8}z^4t_1 + 14v^{-8}z^2t_1 + 5v^{-8}t_1 - 2v^{-10}z^4t_1 - 7v^{-10}z^2t_1 - 4v^{-10}t_1$
$\mathcal{S}_3(\overline{4_{23}}) = -v^6z^2t_1^3 + v^4z^4t_1 + 2v^4z^2t_1 - v^4t_1 + 2v^2t_1$
$\mathcal{S}_3(\overline{4_{24}}) = v^{-4}z^4t_1^5 + 4v^{-10}z^8t_1^3 + 18v^{-10}z^6t_1^3 + 25v^{-10}z^4t_1^3 + 10v^{-10}z^2t_1^3 - 4v^{-12}z^2t_1^3 + 8v^{-12}z^8t_1 + 39v^{-12}z^6t_1 + 61v^{-12}z^4t_1 + 40v^{-12}z^2t_1 + 10v^{-12}t_1 - 8v^{-14}z^6t_1 - 27v^{-14}z^4t_1 - 28v^{-14}z^2t_1 - 12v^{-14}t_1 + 3v^{-16}t_1$
$\mathcal{S}_3(\overline{4_{26}}) = -vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2 - vz^{-1} + v^{-1}z + v^{-1}z^{-1} - v^{-3}z$
$\mathcal{S}_3(5_1) = v^8z^2t_1^3 - v^6z^4t_1 - 3v^6z^2t_1 - 2v^6t_1 + v^4z^4t_1 + 5v^4z^2t_1 + 3v^4t_1$
$\mathcal{S}_3(\overline{5_1}) = z^4t_1^3 + 2z^2t_1^3 - v^{-2}z^6t_1 - 4v^{-2}z^4t_1 - 3v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}z^4t_1 + 3v^{-4}z^2t_1 + v^{-4}t_1 - v^{-6}t_1$
$\mathcal{S}_3(5_2) = v^5z^3t_1^2 + 2v^5zt_1^2 - v^3z^3t_1^2 - v^3zt_1^2 - v^5z^3 - 2v^5z + v^3z^3 + v^3z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{5_3}) = -v^6z^2t_1^3 + v^4z^4t_1 + 2v^4z^2t_1 - v^2z^2t_1 + t_1$
$\mathcal{S}_3(5_4) = -v^{-7}z^7t_1^2 - 6v^{-7}z^5t_1^2 - 10v^{-7}z^3t_1^2 - 5v^{-7}zt_1^2 + v^{-9}z^3t_1^2 + 2v^{-9}zt_1^2 - v^{-7}z^5 - 5v^{-7}z^3 - 6v^{-7}z - 2v^{-7}z^{-1} + v^{-9}z^5 + 5v^{-9}z^3 + 7v^{-9}z + 3v^{-9}z^{-1} - v^{-11}z - v^{-11}z^{-1}$
$\mathcal{S}_3(\overline{5_5}) = v^4z^4t_1^5 - v^{-2}z^{10}t_1^3 - 4v^{-2}z^8t_1^3 - 3v^{-2}z^6t_1^3 + 5v^{-2}z^4t_1^3 + 6v^{-2}z^2t_1^3 + v^{-4}z^4t_1^3 - 2v^{-4}z^2t_1^3 - 2v^{-4}z^{10}t_1 - 10v^{-4}z^8t_1 - 19v^{-4}z^6t_1 - 21v^{-4}z^4t_1 - 8v^{-4}z^2t_1 + 2v^{-4}t_1 + 2v^{-6}z^8t_1 + 7v^{-6}z^6t_1 + 11v^{-6}z^4t_1 + 10v^{-6}z^2t_1 - 2v^{-8}z^2t_1 - v^{-8}t_1$
$\mathcal{S}_3(5_6) = -v^{-11}z^{11}t_1^4 - 8v^{-11}z^9t_1^4 - 21v^{-11}z^7t_1^4 - 20v^{-11}z^5t_1^4 - 5v^{-11}z^3t_1^4 + v^{-13}z^3t_1^4 - 3v^{-15}z^{13}t_1^2 - 32v^{-15}z^{11}t_1^2 - 131v^{-15}z^9t_1^2 - 260v^{-15}z^7t_1^2 - 259v^{-15}z^5t_1^2 - 120v^{-15}z^3t_1^2 - 20v^{-15}zt_1^2 + 3v^{-17}z^9t_1^2 + 23v^{-17}z^7t_1^2 + 60v^{-17}z^5t_1^2 + 59v^{-17}z^3t_1^2 + 18v^{-17}zt_1^2 - 3v^{-19}zt_1^2 - 3v^{-15}z^{11} - 27v^{-15}z^9 - 88v^{-15}z^7 - 131v^{-15}z^5 - 96v^{-15}z^3 - 35v^{-15}z - 5v^{-15}z^{-1} + 3v^{-17}z^{11} + 27v^{-17}z^9 + 89v^{-17}z^7 + 138v^{-17}z^5 + 115v^{-17}z^3 + 52v^{-17}z + 10v^{-17}z^{-1} - v^{-19}z^7 - 7v^{-19}z^5 - 19v^{-19}z^3 - 17v^{-19}z - 6v^{-19}z^{-1} + v^{-21}z^{-1}$
$\mathcal{S}_3(\overline{5_6}) = -v^{-5}z^5t_1^4 - 2v^{-5}z^3t_1^4 - 3v^{-9}z^7t_1^2 - 14v^{-9}z^5t_1^2 - 20v^{-9}z^3t_1^2 - 9v^{-9}zt_1^2 + 3v^{-11}z^3t_1^2 + 5v^{-11}zt_1^2 - 3v^{-9}z^5 - 9v^{-9}z^3 - 7v^{-9}z - 2v^{-9}z^{-1} + 3v^{-11}z^5 + 9v^{-11}z^3 + 8v^{-11}z + 3v^{-11}z^{-1} - v^{-13}z - v^{-13}z^{-1}$
$\mathcal{S}_3(5_7) = -v^7zt_1^2 + 2v^5z^3t_1^2 + 3v^5zt_1^2 - v^5z^3 - 2v^5z - v^5z^{-1} + v^3z^3 + 2v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_7}) = -v^5z^3t_1^4 + vz^7t_1^2 + 4vz^5t_1^2 + 4vz^3t_1^2 - vzt_1^2 - 2v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - 2vz^3 + 3v^{-1}z^3 + 2v^{-1}z - v^{-3}z^3 -$

$2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_8) = -v^4z^4t_1^3 - 2v^4z^2t_1^3 + v^2z^6t_1 + 4v^2z^4t_1 + 3v^2z^2t_1 - v^2t_1 + z^2t_1 + 3t_1 - v^{-2}z^2t_1 - v^{-2}t_1$

$\mathcal{S}_3(\overline{5_8}) = z^6t_1^3 + 4z^4t_1^3 + 3z^2t_1^3 - v^{-2}z^4t_1^3 - v^{-2}z^2t_1^3 - v^{-2}z^8t_1 - 6v^{-2}z^6t_1 - 10v^{-2}z^4t_1 - 3v^{-2}z^2t_1 + 2v^{-2}t_1 + v^{-4}z^6t_1 + 3v^{-4}z^4t_1 + v^{-4}z^2t_1 - v^{-4}t_1 + v^{-6}z^2t_1$

$\mathcal{S}_3(5_9) = -v^{-9}z^9t_1^4 - 5v^{-9}z^7t_1^4 - 6v^{-9}z^5t_1^4 - v^{-9}z^3t_1^4 - 4v^{-13}z^{11}t_1^2 - 33v^{-13}z^9t_1^2 - 98v^{-13}z^7t_1^2 - 128v^{-13}z^5t_1^2 - 70v^{-13}z^3t_1^2 - 12v^{-13}zt_1^2 + 4v^{-15}z^7t_1^2 + 21v^{-15}z^5t_1^2 + 29v^{-15}z^3t_1^2 + 9v^{-15}zt_1^2 - v^{-17}zt_1^2 - 4v^{-13}z^9 - 27v^{-13}z^7 - 62v^{-13}z^5 - 63v^{-13}z^3 - 30v^{-13}z - 5v^{-13}z^{-1} + 4v^{-15}z^9 + 27v^{-15}z^7 + 64v^{-15}z^5 + 73v^{-15}z^3 + 44v^{-15}z + 10v^{-15}z^{-1} - 2v^{-17}z^5 - 10v^{-17}z^3 - 14v^{-17}z - 6v^{-17}z^{-1} + v^{-19}z^{-1}$

$\mathcal{S}_3(\overline{5_9}) = -v^{-7}z^7t_1^4 - 4v^{-7}z^5t_1^4 - 4v^{-7}z^3t_1^4 - 3v^{-11}z^9t_1^2 - 19v^{-11}z^7t_1^2 - 42v^{-11}z^5t_1^2 - 39v^{-11}z^3t_1^2 - 13v^{-11}zt_1^2 + 3v^{-13}z^5t_1^2 + 10v^{-13}z^3t_1^2 + 8v^{-13}zt_1^2 - 3v^{-11}z^7 - 14v^{-11}z^5 - 20v^{-11}z^3 - 10v^{-11}z - 2v^{-11}z^{-1} + 3v^{-13}z^7 + 14v^{-13}z^5 + 21v^{-13}z^3 + 12v^{-13}z + 3v^{-13}z^{-1} - v^{-15}z^3 - 2v^{-15}z - v^{-15}z^{-1}$

$\mathcal{S}_3(5_{10}) = v^6z^2t_1^3 - v^6t_1 - v^4z^4t_1 - v^4z^2t_1 + v^4t_1 + 2v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(\overline{5_{10}}) = v^2z^2t_1^3 + z^2t_1^3 - z^4t_1 - 2z^2t_1 - v^{-2}z^4t_1 + 2v^{-2}t_1 + v^{-4}z^2t_1 - v^{-4}t_1$

$\mathcal{S}_3(5_{11}) = -v^7z^3t_1^4 + v^3z^7t_1^2 + 3v^3z^5t_1^2 + 2v^3z^3t_1^2 - 2vz^3t_1^2 + v^3z^5 + v^3z^3 + v^3z - vz^5 - vz^3 - 2vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{11}}) = -vz^3t_1^2 - vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2 - v^{-3}zt_1^2 + vz^3 + vz - v^{-1}z^3 - 2v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(5_{12}) = -v^4z^4t_1^3 - v^4z^2t_1^3 + v^2z^6t_1 + 3v^2z^4t_1 - 2v^2t_1 + z^4t_1 + 5z^2t_1 + 5t_1 - 2v^{-2}z^2t_1 - 2v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{12}}) = z^6t_1^3 + 4z^4t_1^3 + 4z^2t_1^3 - v^{-2}z^4t_1^3 - v^{-2}z^2t_1^3 - v^{-2}z^8t_1 - 6v^{-2}z^6t_1 - 11v^{-2}z^4t_1 - 6v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}z^6t_1 + 4v^{-4}z^4t_1 + 5v^{-4}z^2t_1 + v^{-4}t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{13}) = v^5zt_1^2 - 2v^3z^3t_1^2 - 2v^3zt_1^2 + vzt_1^2 + v^3z^3 + v^3z - vz^3 - 2vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{13}}) = v^7z^3t_1^4 - v^3z^7t_1^2 - 4v^3z^5t_1^2 - 4v^3z^3t_1^2 + 2vz^3t_1^2 + 2v^3z^3 + v^3z - 3vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{14}) = -v^2z^6t_1^3 - 4v^2z^4t_1^3 - 3v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 + z^8t_1 + 6z^6t_1 + 10z^4t_1 + 4z^2t_1 - v^{-2}z^6t_1 - 3v^{-2}z^4t_1 + 2v^{-2}t_1 - v^{-4}z^2t_1 - v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{14}}) = v^2z^4t_1^3 + 2v^2z^2t_1^3 - z^6t_1 - 4z^4t_1 - 3z^2t_1 + t_1 - v^{-2}t_1 + 2v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{15}) = -v^{-9}z^9t_1^4 - 5v^{-9}z^7t_1^4 - 7v^{-9}z^5t_1^4 - 2v^{-9}z^3t_1^4 - 4v^{-13}z^{11}t_1^2 - 32v^{-13}z^9t_1^2 - 94v^{-13}z^7t_1^2 - 125v^{-13}z^5t_1^2 - 74v^{-13}z^3t_1^2 - 15v^{-13}zt_1^2 + 4v^{-15}z^7t_1^2 + 20v^{-15}z^5t_1^2 + 29v^{-15}z^3t_1^2 + 11v^{-15}zt_1^2 - v^{-17}zt_1^2 - 4v^{-13}z^9 - 26v^{-13}z^7 - 58v^{-13}z^5 - 54v^{-13}z^3 - 22v^{-13}z - 3v^{-13}z^{-1} + 4v^{-15}z^9 + 26v^{-15}z^7 + 60v^{-15}z^5 + 62v^{-15}z^3 + 30v^{-15}z + 5v^{-15}z^{-1} - 2v^{-17}z^5 - 8v^{-17}z^3 - 8v^{-17}z - 2v^{-17}z^{-1}$

$\mathcal{S}_3(\overline{5_{15}}) = -v^{-7}z^7t_1^4 - 3v^{-7}z^5t_1^4 - 2v^{-7}z^3t_1^4 - 4v^{-11}z^9t_1^2 - 24v^{-11}z^7t_1^2 - 49v^{-11}z^5t_1^2 - 40v^{-11}z^3t_1^2 - 11v^{-11}zt_1^2 + 4v^{-13}z^5t_1^2 + 11v^{-13}z^3t_1^2 + 7v^{-13}zt_1^2 - 4v^{-11}z^7 - 18v^{-11}z^5 - 26v^{-11}z^3 - 14v^{-11}z - 3v^{-11}z^{-1} + 4v^{-13}z^7 + 18v^{-13}z^5 + 28v^{-13}z^3 + 18v^{-13}z + 5v^{-13}z^{-1} - 2v^{-15}z^3 - 4v^{-15}z - 2v^{-15}z^{-1}$

$\mathcal{S}_3(5_{16}) = v^4z^4t_1^3 + v^4z^2t_1 + 2v^4t_1 - v^2z^6t_1 - 3v^2z^4t_1 - 4v^2z^2t_1 - 3v^2t_1 + z^4t_1 + t_1 + v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{16}}) = -v^6z^4t_1^3 - 2v^6z^2t_1^3 + v^4z^4t_1^3 + v^4z^2t_1^3 + v^4z^6t_1 + 4v^4z^4t_1 + 3v^4z^2t_1 - 2v^4t_1 - v^2z^6t_1 - 3v^2z^4t_1 + 4v^2t_1 + z^4t_1 - t_1$

$\mathcal{S}_3(5_{17}) = -v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - v^{-7}z^5t_1^2 - 4v^{-7}z^3t_1^2 - 2v^{-7}zt_1^2 + v^{-9}zt_1^2 - v^{-5}z^3 - v^{-5}z - 2v^{-7}z - 2v^{-7}z^{-1} + v^{-9}z^3 + 3v^{-9}z + 3v^{-9}z^{-1} - v^{-11}z^{-1}$

$\mathcal{S}_3(\overline{5_{17}}) = -v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2 + v^{-3}z^5t_1^2 + 3v^{-3}z^3t_1^2 + 2v^{-3}zt_1^2 - v^{-5}zt_1^2 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^3 + 2v^{-3}z + v^{-3}z^{-1} - v^{-5}z^3 - v^{-5}z$

$\mathcal{S}_3(5_{18}) = -v^5z^3t_1^2 - v^5zt_1^2 + v^3z^3t_1^2 + 2v^3zt_1^2 + v^5z^3 + v^5z - v^3z^3 - 2v^3z - v^3z^{-1} + vz + vz^{-1}$

$\mathcal{S}_3(\overline{5_{18}}) = -v^3z^3t_1^4 + v^{-1}z^7t_1^2 + 3v^{-1}z^5t_1^2 + 2v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - 2v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-1}z^5 + v^{-1}z^3 + v^{-1}z - v^{-3}z^5 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{19}) = -v^2z^6t_1^3 - 4v^2z^4t_1^3 - 3v^2z^2t_1^3 + z^4t_1^3 + 2z^2t_1^3 + z^8t_1 + 6z^6t_1 + 10z^4t_1 + 3z^2t_1 - 2t_1 - v^{-2}z^6t_1 - 3v^{-2}z^4t_1 + 2v^{-2}z^2t_1 + 6v^{-2}t_1 - v^{-4}z^2t_1 - 3v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{19}}) = v^2z^4t_1^3 + 2v^2z^2t_1^3 - z^6t_1 - 4z^4t_1 - 3z^2t_1 + t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(5_{20}) = -v^{-7}z^7t_1^4 - 3v^{-7}z^5t_1^4 - v^{-7}z^3t_1^4 - 4v^{-11}z^9t_1^2 - 25v^{-11}z^7t_1^2 - 52v^{-11}z^5t_1^2 - 41v^{-11}z^3t_1^2 - 9v^{-11}zt_1^2 + 4v^{-13}z^5t_1^2 + 12v^{-13}z^3t_1^2 + 5v^{-13}zt_1^2 - 4v^{-11}z^7 - 19v^{-11}z^5 - 28v^{-11}z^3 - 17v^{-11}z - 3v^{-11}z^{-1} + 4v^{-13}z^7 + 19v^{-13}z^5 +$

$30v^{-13}z^3 + 22v^{-13}z + 5v^{-13}z^{-1} - 2v^{-15}z^3 - 5v^{-15}z - 2v^{-15}z^{-1}$

$\mathcal{S}_3(\overline{5_{20}}) = -v^{-9}z^9t_1^4 - 6v^{-9}z^7t_1^4 - 11v^{-9}z^5t_1^4 - 6v^{-9}z^3t_1^4 - 3v^{-13}z^{11}t_1^2 - 25v^{-13}z^9t_1^2 - 77v^{-13}z^7t_1^2 - 109v^{-13}z^5t_1^2 - 70v^{-13}z^3t_1^2 - 16v^{-13}zt_1^2 + 3v^{-15}z^7t_1^2 + 16v^{-15}z^5t_1^2 + 25v^{-15}z^3t_1^2 + 11v^{-15}zt_1^2 - 3v^{-13}z^9 - 20v^{-13}z^7 - 45v^{-13}z^5 - 42v^{-13}z^3 - 17v^{-13}z - 3v^{-13}z^{-1} + 3v^{-15}z^9 + 20v^{-15}z^7 + 46v^{-15}z^5 + 46v^{-15}z^3 + 22v^{-15}z + 5v^{-15}z^{-1} - v^{-17}z^5 - 4v^{-17}z^3 - 5v^{-17}z - 2v^{-17}z^{-1}$

$\mathcal{S}_3(5_{21}) = v^4z^4t_1^3 + 2v^4z^2t_1^3 - v^2z^4t_1^3 - v^2z^2t_1^3 - v^2z^6t_1 - 4v^2z^4t_1 - 3v^2z^2t_1 + 2v^2t_1 + z^6t_1 + 3z^4t_1 - 2t_1 - v^{-2}z^4t_1 - v^{-2}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{21}}) = -v^6z^4t_1^3 - v^6z^2t_1^3 - v^6z^2t_1 - 2v^6t_1 + v^4z^6t_1 + 4v^4z^4t_1 + 6v^4z^2t_1 + 3v^4t_1 - v^2z^4t_1 - v^2z^2t_1$

$\mathcal{S}_3(5_{22}) = -v^3zt_1^2 + v^{-1}zt_1^2 + v^3z - vz^{-1} - v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{22}}) = v^5z^3t_1^4 - vz^7t_1^2 - 3vz^5t_1^2 - vz^3t_1^2 + 2vzt_1^2 + v^{-1}z^3t_1^2 - 3v^{-1}zt_1^2 - vz^5 - 2vz^3 - 3vz + v^{-1}z^5 + 2v^{-1}z^3 + 4v^{-1}z - v^{-1}z^{-1} - v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{23}}) = v^4t_1 - v^2z^2t_1 - 2v^2t_1 + 2t_1$

$\mathcal{S}_3(\overline{5_{24}}) = -v^{-3}z^3t_1^4 - 3v^{-7}z^5t_1^2 - 9v^{-7}z^3t_1^2 - 7v^{-7}zt_1^2 + 3v^{-9}zt_1^2 - 3v^{-7}z^3 - 3v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^3 + 3v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(5_{25}) = v^7zt_1^2 + v^3z^3t_1^2 + v^3zt_1^2 - v^7z - v^5z - v^5z^{-1} + 2v^3z + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{25}}) = -vz^3t_1^4 + v^{-3}z^7t_1^2 + 3v^{-3}z^5t_1^2 - 4v^{-3}zt_1^2 - 2v^{-5}z^3t_1^2 + v^{-5}zt_1^2 + v^{-3}z^5 + 2v^{-3}z^3 + 3v^{-3}z - v^{-5}z^5 - 2v^{-5}z^3 - 5v^{-5}z - v^{-5}z^{-1} + 2v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{26}) = -v^6z^4t_1^3 - v^6z^2t_1^3 + v^4z^6t_1 + 3v^4z^4t_1 + 2v^4z^2t_1 - v^4t_1 - v^2z^4t_1 + 2v^2t_1$

$\mathcal{S}_3(\overline{5_{26}}) = z^2t_1 + t_1 - v^{-2}z^4t_1 - 2v^{-2}z^2t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(5_{27}) = -v^6z^4t_1^3 - v^6z^2t_1^3 + v^4z^6t_1 + 3v^4z^4t_1 + 2v^4z^2t_1 - v^4t_1 - v^2z^4t_1 + 2v^2t_1$

$\mathcal{S}_3(\overline{5_{27}}) = z^2t_1 + t_1 - v^{-2}z^4t_1 - 2v^{-2}z^2t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(5_{28}) = -v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - 2v^{-5}z^3 - 5v^{-5}z - 2v^{-5}z^{-1} + 2v^{-7}z^3 + 6v^{-7}z + 3v^{-7}z^{-1} - v^{-9}z - v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{28}}) = vz^3t_1^4 - v^{-3}z^7t_1^2 - 3v^{-3}z^5t_1^2 - 3v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-3}z^5 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^5 + v^{-5}z^3 + 2v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{29}) = -v^3z^3t_1^2 - v^3zt_1^2 + vzt_1^2 + v^3z^3 + 2v^3z - vz^3 - 3vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{29}}) = -v^3zt_1^2 + vz^3t_1^2 + vzt_1^2 + v^3z - vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1} - v^{-3}z$

$\mathcal{S}_3(5_{30}) = v^4t_1 - v^2z^2t_1 - v^2t_1 - z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{30}}) = -v^2z^2t_1^3 + v^2t_1 + z^4t_1 + z^2t_1 - t_1 - v^{-2}z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{31}) = v^4z^4t_1^3 + v^4z^2t_1^3 - v^2z^6t_1 - 3v^2z^4t_1 - 2v^2z^2t_1 + z^4t_1 + z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{31}}) = v^4z^2t_1^3 - v^2z^4t_1 - 3v^2z^2t_1 + z^4t_1 + 3z^2t_1 + t_1 - v^{-2}z^2t_1$

$\mathcal{S}_3(5_{32}) = z^2t_1 + 2t_1 - v^{-2}z^4t_1 - 3v^{-2}z^2t_1 - 2v^{-2}t_1 + v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{32}}) = z^2t_1^3 - v^{-2}z^4t_1 - v^{-2}z^2t_1 + v^{-2}t_1 + 2v^{-4}z^2t_1 + v^{-4}t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{33}) = v^{-1}zt_1^2 - v^3z^{-1} + vz + vz^{-1} - v^{-1}z$

$\mathcal{S}_3(\overline{5_{33}}) = -vzt_1^2 + v^{-1}z^3t_1^2 - v^{-3}zt_1^2 + v^{-1}z - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{34}) = -2v^{-3}z^3t_1^2 - 3v^{-3}zt_1^2 + v^{-5}zt_1^2 + v^{-3}z^3 + v^{-3}z - v^{-3}z^{-1} - v^{-5}z^3 - 2v^{-5}z + v^{-5}z^{-1} + v^{-7}z$

$\mathcal{S}_3(\overline{5_{34}}) = -v^{-1}z^3t_1^4 + v^{-5}z^7t_1^2 + 2v^{-5}z^5t_1^2 - 2v^{-5}z^3t_1^2 - 4v^{-5}zt_1^2 - v^{-7}z^3t_1^2 + 2v^{-7}zt_1^2 + v^{-5}z^5 - 2v^{-5}z - 2v^{-5}z^{-1} - v^{-7}z^5 + 2v^{-7}z + 3v^{-7}z^{-1} - v^{-9}z^{-1}$

$\mathcal{S}_3(5_{35}) = v^7z^3t_1^2 + v^5z^3t_1^2 + v^5zt_1^2 + v^9z^{-1} - v^7z^3 - 2v^7z - 2v^7z^{-1} + v^3z^3 + 2v^3z + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{35}}) = -v^3z^3t_1^4 + v^{-1}z^7t_1^2 + 4v^{-1}z^5t_1^2 + 3v^{-1}z^3t_1^2 - 3v^{-1}zt_1^2 - v^{-3}z^3t_1^2 + 2v^{-3}zt_1^2 - v^{-1}z^3 + v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z^3 + v^{-3}z^{-1} - v^{-5}z^3 - v^{-5}z$

$\mathcal{S}_3(5_{36}) = -v^2z^2t_1 - v^2t_1 + z^4t_1 + 3z^2t_1 + 3t_1 - v^{-2}z^2t_1 - v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{36}}) = v^{-4}z^2t_1^3 + v^{-4}z^4t_1 + 3v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-8}t_1$

$\mathcal{S}_3(5_{37}) = vz^3t_1^2 + vzt_1^2 - v^{-1}zt_1^2 - vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + 3v^{-1}z + v^{-1}z^{-1} - v^{-3}z$

$\mathcal{S}_3(\overline{5_{37}}) = v^5zt_1^2 - v^3z^3t_1^2 - v^3zt_1^2 - v^5z + v^3z^3 + 2v^3z - vz^3 - 2vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{38}) = v^7zt_1^2 - v^5z^3t_1^2 - v^5zt_1^2 + v^7z^{-1} - 2v^5z - 2v^5z^{-1} + v^3z + vz + vz^{-1}$

$\mathcal{S}_3(\overline{5_{39}}) = -v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 - z^2t_1 - t_1 + 2v^{-2}z^4t_1 + 5v^{-2}z^2t_1 + 4v^{-2}t_1 - 2v^{-4}z^2t_1 - 2v^{-4}t_1$

$\mathcal{S}_3(5_{40}) = -v^6z^2t_1 - 2v^6t_1 + v^4z^4t_1 + 4v^4z^2t_1 + 3v^4t_1$

$\mathcal{S}_3(\overline{5_{40}}) = z^4t_1^3 + 2z^2t_1^3 - v^{-2}z^6t_1 - 4v^{-2}z^4t_1 - 3v^{-2}z^2t_1 + 2v^{-2}t_1 + v^{-4}z^4t_1 + 2v^{-4}z^2t_1 - v^{-4}t_1$

$\mathcal{S}_3(5_{41}) = -v^{-7}z^7t_1^2 - 6v^{-7}z^5t_1^2 - 10v^{-7}z^3t_1^2 - 4v^{-7}zt_1^2 + v^{-9}z^3t_1^2 + 2v^{-9}zt_1^2 - v^{-7}z^5 - 5v^{-7}z^3 - 7v^{-7}z - 3v^{-7}z^{-1} + v^{-9}z^5 + 5v^{-9}z^3 + 8v^{-9}z + 5v^{-9}z^{-1} - v^{-11}z - 2v^{-11}z^{-1}$

$\mathcal{S}_3(\overline{5_{42}}) = v^2z^2t_1^3 + z^2t_1^3 - z^4t_1 - 2z^2t_1 + t_1 - v^{-2}z^4t_1 - v^{-2}z^2t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(5_{43}) = v^{-1}z^3t_1^4 - v^{-3}z^5t_1^2 - 4v^{-3}z^3t_1^2 - 3v^{-3}zt_1^2 + 2v^{-5}z^5t_1^2 + 5v^{-5}z^3t_1^2 + 4v^{-5}zt_1^2 - 2v^{-7}zt_1^2 - v^{-3}z^3 - 3v^{-3}z - 2v^{-3}z^{-1} + 3v^{-5}z^3 + 4v^{-5}z + 3v^{-5}z^{-1} - 2v^{-7}z^3 - v^{-7}z - v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{43}}) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - v^{-3}z - v^{-5}z^{-1} + v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{44}) = -v^4z^4t_1^3 - 2v^4z^2t_1^3 + v^2z^6t_1 + 4v^2z^4t_1 + 3v^2z^2t_1 + t_1 - v^{-2}z^2t_1$

$\mathcal{S}_3(\overline{5_{44}}) = z^6t_1^3 + 4z^4t_1^3 + 3z^2t_1^3 - v^{-2}z^4t_1^3 - v^{-2}z^2t_1^3 + v^{-4}z^2t_1^3 - v^{-2}z^8t_1 - 6v^{-2}z^6t_1 - 10v^{-2}z^4t_1 - 3v^{-2}z^2t_1 + 2v^{-2}t_1 + v^{-4}z^6t_1 + 3v^{-4}z^4t_1 - 3v^{-4}t_1 + 3v^{-6}z^2t_1 + 4v^{-6}t_1 - 2v^{-8}t_1$

$\mathcal{S}_3(5_{45}) = v^5z^3t_1^4 - vz^7t_1^2 - 3vz^5t_1^2 - 2vz^3t_1^2 + 2v^{-1}z^3t_1^2 - vz^5 - vz^3 - vz - vz^{-1} + v^{-1}z^5 + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1} - v^{-3}z$

$\mathcal{S}_3(\overline{5_{45}}) = v^3z^3t_1^2 + v^3zt_1^2 - vz^3t_1^2 - 2vzt_1^2 - v^3z^3 - v^3z + vz^3 + 2vz - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$

$\mathcal{S}_3(5_{46}) = -v^6z^4t_1^3 - v^6z^2t_1 - 2v^6t_1 + v^4z^6t_1 + 3v^4z^4t_1 + 4v^4z^2t_1 + 3v^4t_1 - v^2z^4t_1$

$\mathcal{S}_3(\overline{5_{46}}) = v^4z^4t_1^3 + 2v^4z^2t_1^3 - v^2z^4t_1^3 - v^2z^2t_1^3 - v^2z^6t_1 - 4v^2z^4t_1 - 3v^2z^2t_1 + 2v^2t_1 + z^6t_1 + 3z^4t_1 - 3t_1 - v^{-2}z^4t_1 + 2v^{-2}t_1$

$\mathcal{S}_3(5_{47}) = -v^{-9}z^9t_1^4 - 5v^{-9}z^7t_1^4 - 7v^{-9}z^5t_1^4 - 3v^{-9}z^3t_1^4 - 4v^{-13}z^{11}t_1^2 - 32v^{-13}z^9t_1^2 - 93v^{-13}z^7t_1^2 - 121v^{-13}z^5t_1^2 - 69v^{-13}z^3t_1^2 - 14v^{-13}zt_1^2 + 4v^{-15}z^7t_1^2 + 20v^{-15}z^5t_1^2 + 28v^{-15}z^3t_1^2 + 11v^{-15}zt_1^2 - v^{-17}zt_1^2 - 4v^{-13}z^9 - 26v^{-13}z^7 - 58v^{-13}z^5 - 57v^{-13}z^3 - 26v^{-13}z - 5v^{-13}z^{-1} + 4v^{-15}z^9 + 26v^{-15}z^7 + 60v^{-15}z^5 + 66v^{-15}z^3 + 38v^{-15}z + 10v^{-15}z^{-1} - 2v^{-17}z^5 - 9v^{-17}z^3 - 12v^{-17}z - 6v^{-17}z^{-1} + v^{-19}z^{-1}$

$\mathcal{S}_3(\overline{5_{47}}) = -v^{-7}z^7t_1^4 - 3v^{-7}z^5t_1^4 - 2v^{-7}z^3t_1^4 - 4v^{-11}z^9t_1^2 - 24v^{-11}z^7t_1^2 - 49v^{-11}z^5t_1^2 - 40v^{-11}z^3t_1^2 - 11v^{-11}zt_1^2 + 4v^{-13}z^5t_1^2 + 11v^{-13}z^3t_1^2 + 6v^{-13}zt_1^2 - 4v^{-11}z^7 - 18v^{-11}z^5 - 26v^{-11}z^3 - 14v^{-11}z - 2v^{-11}z^{-1} + 4v^{-13}z^7 + 18v^{-13}z^5 + 28v^{-13}z^3 + 18v^{-13}z + 3v^{-13}z^{-1} - 2v^{-15}z^3 - 4v^{-15}z - v^{-15}z^{-1}$

$\mathcal{S}_3(5_{48}) = v^6t_1 - v^4t_1 - v^2z^4t_1 - 3v^2z^2t_1 - v^2t_1 + z^2t_1 + 2t_1$

$\mathcal{S}_3(\overline{5_{48}}) = -v^{10}z^4t_1^5 + v^4z^{10}t_1^3 + 5v^4z^8t_1^3 + 9v^4z^6t_1^3 + 4v^4z^4t_1^3 - 3v^4z^2t_1^3 - v^2z^4t_1^3 + 3v^2z^2t_1^3 + v^2z^{10}t_1 + 2v^2z^8t_1 - 2v^2z^6t_1 + 2v^2z^2t_1 - 2v^2t_1 - z^8t_1 + 3z^4t_1 + z^2t_1 + 5t_1 - v^{-2}z^4t_1 - v^{-2}z^2t_1 - 2v^{-2}t_1$

$\mathcal{S}_3(5_{49}) = -v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-7}z^5t_1^2 - 4v^{-7}z^3t_1^2 - 2v^{-7}zt_1^2 + v^{-9}zt_1^2 - v^{-5}z^3 - 2v^{-5}z - v^{-5}z^{-1} - v^{-7}z + v^{-9}z^3 + 3v^{-9}z + 2v^{-9}z^{-1} - v^{-11}z^{-1}$

$\mathcal{S}_3(\overline{5_{49}}) = -v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2 + v^{-3}z^5t_1^2 + 3v^{-3}z^3t_1^2 + v^{-3}zt_1^2 - v^{-5}zt_1^2 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^3 + 3v^{-3}z + 2v^{-3}z^{-1} - v^{-5}z^3 - 2v^{-5}z - 2v^{-5}z^{-1} + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{50}) = v^3z^3t_1^2 + 2v^3zt_1^2 - v^5z - v^5z^{-1} + v^3z + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{50}}) = -v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-5}z - v^{-5}z^{-1} + v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{51}) = v^8z^2t_1^3 - v^6z^4t_1 - 2v^6z^2t_1 - v^6t_1 + 2v^4z^2t_1 + v^4t_1 + v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(5_{52}) = -v^2z^4t_1^3 - 2v^2z^2t_1^3 + z^6t_1 + 4z^4t_1 + 4z^2t_1 + t_1 - v^{-2}z^4t_1 - 2v^{-2}z^2t_1$

$\mathcal{S}_3(\overline{5_{52}}) = -v^6z^2t_1^3 + v^4z^4t_1 + 3v^4z^2t_1 + v^4t_1 - v^2z^4t_1 - 4v^2z^2t_1 - 2v^2t_1 + z^2t_1 + 2t_1$

$\mathcal{S}_3(5_{53}) = v^{-1}z^3t_1^2 + v^{-1}zt_1^2 - v^{-3}zt_1^2 - v^{-1}z^3 - 3v^{-1}z - 2v^{-1}z^{-1} + v^{-3}z^3 + 4v^{-3}z + 3v^{-3}z^{-1} - v^{-5}z - v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{53}}) = -v^{-7}z^7t_1^2 - 6v^{-7}z^5t_1^2 - 11v^{-7}z^3t_1^2 - 6v^{-7}zt_1^2 + v^{-9}z^3t_1^2 + 2v^{-9}zt_1^2 - v^{-7}z^5 - 4v^{-7}z^3 - 4v^{-7}z - v^{-7}z^{-1} + v^{-9}z^5 + 4v^{-9}z^3 + 4v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{54}}) = -v^6z^4t_1^3 - 2v^6z^2t_1^3 + v^4z^6t_1 + 4v^4z^4t_1 + 4v^4z^2t_1 - v^4t_1 - v^2z^4t_1 - v^2z^2t_1 + 2v^2t_1$

$\mathcal{S}_3(5_{55}) = -v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2 + v^{-3}z^3t_1^2 + v^{-3}zt_1^2 + v^{-1}z^3 + v^{-1}z - v^{-1}z^{-1} - v^{-3}z^3 - v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{55}}) = v^5z^3t_1^2 + 2v^5zt_1^2 - v^3z^3t_1^2 - v^3zt_1^2 - vz^3t_1^2 - vzt_1^2 - v^5z^3 - 2v^5z + v^3z^3 + 2v^3z - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{56}) = -v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - 2v^{-5}z^3 - 6v^{-5}z - 3v^{-5}z^{-1} + 2v^{-7}z^3 + 7v^{-7}z + 5v^{-7}z^{-1} - v^{-9}z - 2v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{56}}) = vz^3t_1^4 - v^{-3}z^7t_1^2 - 3v^{-3}z^5t_1^2 - 3v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2 + v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-3}z^5 - v^{-3}z^3 - v^{-3}z + v^{-5}z^5 + v^{-5}z^3 + v^{-5}z - v^{-5}z^{-1} + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{57}) = v^3zt_1^2 + vzt_1^2 - v^5z^{-1} + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{57}}) = -v^{-1}zt_1^2 - v^{-3}zt_1^2 - v^{-5}zt_1^2 - v^{-5}z^{-1} + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{58}) = -v^6z^2t_1^3 - v^4z^2t_1^3 + v^4z^4t_1 + 2v^4z^2t_1 + v^2z^4t_1 + v^2z^2t_1 - z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{58}}) = v^2t_1 - z^2t_1 - v^{-2}z^2t_1 - v^{-2}t_1 + v^{-4}t_1$

$\mathcal{S}_3(5_{59}) = -v^{-1}z^3t_1^2 - v^{-1}zt_1^2 + v^{-3}z^5t_1^2 + 3v^{-3}z^3t_1^2 + 2v^{-3}zt_1^2 - v^{-5}zt_1^2 - 2v^{-1}z - 2v^{-1}z^{-1} + v^{-3}z^3 + 3v^{-3}z + 3v^{-3}z^{-1} - v^{-5}z^3 - v^{-5}z - v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{59}}) = -v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - v^{-7}z^5t_1^2 - 4v^{-7}z^3t_1^2 - 3v^{-7}zt_1^2 + v^{-9}zt_1^2 - v^{-5}z^3 - v^{-5}z - v^{-7}z - v^{-7}z^{-1} + v^{-9}z^3 + 2v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(5_{60}) = -v^4z^2t_1^3 + v^4t_1 + v^2z^4t_1 + v^2z^2t_1 - v^2t_1 - 2z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{60}}) = -v^4z^2t_1^3 - v^2z^2t_1^3 + v^2z^4t_1 + 2v^2z^2t_1 + v^2t_1 + z^4t_1 - t_1 - v^{-2}z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{61}) = -v^9z^3t_1^4 + v^5z^7t_1^2 + 4v^5z^5t_1^2 + 4v^5z^3t_1^2 + v^5zt_1^2 - 2v^3z^3t_1^2 + v^3zt_1^2 - 2v^5z^3 - 2v^5z - v^5z^{-1} + 3v^3z^3 + 2v^3z + v^3z^{-1} - vz^3$

$\mathcal{S}_3(\overline{5_{61}}) = -v^3zt_1^2 + 2vz^3t_1^2 + vzt_1^2 - 2v^{-1}zt_1^2 - vz^3 + v^{-1}z^3 + 2v^{-1}z - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{62}) = -v^2z^6t_1^3 - 4v^2z^4t_1^3 - 4v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 + z^8t_1 + 6z^6t_1 + 11z^4t_1 + 7z^2t_1 + t_1 - v^{-2}z^6t_1 - 4v^{-2}z^4t_1 - 4v^{-2}z^2t_1$

$\mathcal{S}_3(\overline{5_{62}}) = v^2z^4t_1^3 + v^2z^2t_1^3 - z^6t_1 - 3z^4t_1 + 2t_1 - v^{-2}z^4t_1 - 4v^{-2}z^2t_1 - 3v^{-2}t_1 + 3v^{-4}z^2t_1 + 3v^{-4}t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{63}) = v^3z^3t_1^2 + v^3zt_1^2 - vz^3t_1^2 - vzt_1^2 + v^{-1}zt_1^2 - v^3z^3 - v^3z - v^3z^{-1} + vz^3 + 2vz + vz^{-1} - v^{-1}z$

$\mathcal{S}_3(\overline{5_{63}}) = v^5z^3t_1^4 - vz^7t_1^2 - 3vz^5t_1^2 - 2vz^3t_1^2 + 2v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - v^{-3}zt_1^2 - vz^5 - vz^3 - vz + v^{-1}z^5 + v^{-1}z^3 + 2v^{-1}z - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{64}) = -v^6z^4t_1^3 - 2v^6z^2t_1^3 + v^4z^4t_1^3 + v^4z^2t_1^3 + v^4z^6t_1 + 4v^4z^4t_1 + 3v^4z^2t_1 - v^4t_1 - v^2z^6t_1 - 3v^2z^4t_1 - v^2z^2t_1 + 2v^2t_1 + z^4t_1$

$\mathcal{S}_3(\overline{5_{64}}) = v^4z^4t_1^3 - v^2z^2t_1^3 + v^4z^2t_1 + 2v^4t_1 - v^2z^6t_1 - 3v^2z^4t_1 - 4v^2z^2t_1 - 3v^2t_1 + 2z^4t_1 + 2z^2t_1 + t_1 - v^{-2}z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{65}) = -v^{-7}z^7t_1^4 - 2v^{-7}z^5t_1^4 - v^{-7}z^3t_1^4 - 5v^{-11}z^9t_1^2 - 28v^{-11}z^7t_1^2 - 52v^{-11}z^5t_1^2 - 36v^{-11}z^3t_1^2 - 8v^{-11}zt_1^2 + 5v^{-13}z^5t_1^2 + 11v^{-13}z^3t_1^2 + 5v^{-13}zt_1^2 - 5v^{-11}z^7 - 22v^{-11}z^5 - 35v^{-11}z^3 - 22v^{-11}z - 5v^{-11}z^{-1} + 5v^{-13}z^7 + 22v^{-13}z^5 + 39v^{-13}z^3 + 32v^{-13}z + 10v^{-13}z^{-1} - 4v^{-15}z^3 - 10v^{-15}z - 6v^{-15}z^{-1} + v^{-17}z^{-1}$

$\mathcal{S}_3(\overline{5_{65}}) = -v^{-9}z^9t_1^4 - 5v^{-9}z^7t_1^4 - 8v^{-9}z^5t_1^4 - 4v^{-9}z^3t_1^4 - 4v^{-13}z^{11}t_1^2 - 31v^{-13}z^9t_1^2 - 89v^{-13}z^7t_1^2 - 118v^{-13}z^5t_1^2 - 73v^{-13}z^3t_1^2 - 17v^{-13}zt_1^2 + 4v^{-15}z^7t_1^2 + 19v^{-15}z^5t_1^2 + 28v^{-15}z^3t_1^2 + 12v^{-15}zt_1^2 - v^{-17}zt_1^2 - 4v^{-13}z^9 - 25v^{-13}z^7 - 54v^{-13}z^5 - 48v^{-13}z^3 - 18v^{-13}z - 2v^{-13}z^{-1} + 4v^{-15}z^9 + 25v^{-15}z^7 + 56v^{-15}z^5 + 55v^{-15}z^3 + 24v^{-15}z + 3v^{-15}z^{-1} - 2v^{-17}z^5 - 7v^{-17}z^3 - 6v^{-17}z - v^{-17}z^{-1}$

$\mathcal{S}_3(5_{66}) = -v^{-3}z^3t_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - 2v^{-3}z - v^{-3}z^{-1} + 2v^{-7}z + 2v^{-7}z^{-1} - v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{66}}) = v^{-1}z^3t_1^4 - v^{-3}z^5t_1^2 - 4v^{-3}z^3t_1^2 - 4v^{-3}zt_1^2 + 2v^{-5}z^5t_1^2 + 5v^{-5}z^3t_1^2 + 3v^{-5}zt_1^2 - 2v^{-7}zt_1^2 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + 3v^{-5}z^3 + 4v^{-5}z + 2v^{-5}z^{-1} - 2v^{-7}z^3 - 2v^{-7}z - 2v^{-7}z^{-1} + v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{67}}) = -v^{-3}z^3t_1^4 - v^{-5}z^3t_1^4 - 2v^{-7}z^5t_1^2 - 5v^{-7}z^3t_1^2 - 3v^{-7}zt_1^2 - 3v^{-9}z^5t_1^2 - 9v^{-9}z^3t_1^2 - 4v^{-9}zt_1^2 + 3v^{-11}zt_1^2 - 2v^{-7}z^3 - v^{-7}z - v^{-9}z^3 - 3v^{-9}z - 2v^{-9}z^{-1} + 3v^{-11}z^3 + 4v^{-11}z + 3v^{-11}z^{-1} - v^{-13}z^{-1}$

$\mathcal{S}_3(5_{68}) = -v^6z^4t_1^5 + z^{10}t_1^3 + 4z^8t_1^3 + 3z^6t_1^3 - 5z^4t_1^3 - 5z^2t_1^3 - v^{-2}z^4t_1^3 + 3v^{-2}z^2t_1^3 + 2v^{-2}z^{10}t_1 + 10v^{-2}z^8t_1 + 19v^{-2}z^6t_1 + 21v^{-2}z^4t_1 + 10v^{-2}z^2t_1 + v^{-2}t_1 - 2v^{-4}z^8t_1 - 7v^{-4}z^6t_1 - 11v^{-4}z^4t_1 - 8v^{-4}z^2t_1 + v^{-4}t_1 + 2v^{-6}z^2t_1 - v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{68}}) = v^4z^2t_1^3 - v^2z^4t_1 - 2v^2z^2t_1 + v^2t_1 - t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{69}) = -v^{-5}z^5t_1^4 - v^{-5}z^3t_1^4 - 4v^{-9}z^7t_1^2 - 17v^{-9}z^5t_1^2 - 21v^{-9}z^3t_1^2 - 7v^{-9}zt_1^2 + 4v^{-11}z^3t_1^2 + 4v^{-11}zt_1^2 - 4v^{-9}z^5 - 11v^{-9}z^3 - 10v^{-9}z - 3v^{-9}z^{-1} + 4v^{-11}z^5 + 11v^{-11}z^3 + 12v^{-11}z + 5v^{-11}z^{-1} - 2v^{-13}z - 2v^{-13}z^{-1}$

$\mathcal{S}_3(\overline{5_{69}}) = -v^{-11}z^{11}t_1^4 - 8v^{-11}z^9t_1^4 - 22v^{-11}z^7t_1^4 - 24v^{-11}z^5t_1^4 - 8v^{-11}z^3t_1^4 + v^{-13}z^3t_1^4 - 3v^{-15}z^{13}t_1^2 - 31v^{-15}z^{11}t_1^2 - 124v^{-15}z^9t_1^2 - 244v^{-15}z^7t_1^2 - 247v^{-15}z^5t_1^2 - 121v^{-15}z^3t_1^2 - 22v^{-15}zt_1^2 + 3v^{-17}z^9t_1^2 + 22v^{-17}z^7t_1^2 + 56v^{-17}z^5t_1^2 + 56v^{-17}z^3t_1^2 + 18v^{-17}zt_1^2 - 2v^{-19}zt_1^2 - 3v^{-15}z^{11} - 26v^{-15}z^9 - 82v^{-15}z^7 - 118v^{-15}z^5 - 81v^{-15}z^3 - 26v^{-15}z - 3v^{-15}z^{-1} + 3v^{-17}z^{11} + 26v^{-17}z^9 + 83v^{-17}z^7 + 124v^{-17}z^5 + 95v^{-17}z^3 + 36v^{-17}z + 5v^{-17}z^{-1} - v^{-19}z^7 - 6v^{-19}z^5 - 14v^{-19}z^3 - 10v^{-19}z - 2v^{-19}z^{-1}$

$\mathcal{S}_3(5_{70}) = v^4z^2t_1^3 - v^2z^4t_1^3 - v^2z^2t_1^3 + 2v^2z^2t_1 + 3v^2t_1 - 3z^4t_1 - 8z^2t_1 - 5t_1 + 3v^{-2}z^2t_1 + 3v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{70}}) = -v^8z^2t_1^3 + v^6z^4t_1 + 2v^6z^2t_1 - v^6t_1 + v^4t_1 + v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(\overline{5_{71}}) = v^9z^3t_1^4 - v^5z^7t_1^2 - 4v^5z^5t_1^2 - 4v^5z^3t_1^2 + 2v^5zt_1^2 - 3v^3zt_1^2 + 2v^5z^3 + v^5z^{-1} - 2v^3z^3 - v^3z - 2v^3z^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{72}) = -v^8t_1 + v^4z^4t_1 + 4v^4z^2t_1 + 2v^4t_1$

$\mathcal{S}_3(\overline{5_{72}}) = v^8z^4t_1^5 - v^2z^{10}t_1^3 - 5v^2z^8t_1^3 - 9v^2z^6t_1^3 - 4v^2z^4t_1^3 + 4v^2z^2t_1^3 + z^4t_1^3 - 2z^2t_1^3 - z^{10}t_1 - 2z^8t_1 + 2z^6t_1 - z^4t_1 - 4z^2t_1 + 3t_1 + v^{-2}z^8t_1 - 4v^{-2}z^4t_1 - 2v^{-2}z^2t_1 - 4v^{-2}t_1 + v^{-4}z^4t_1 + 2v^{-4}z^2t_1 + 2v^{-4}t_1$

$\mathcal{S}_3(5_{73}) = -v^{-9}z^9t_1^4 - 6v^{-9}z^7t_1^4 - 10v^{-9}z^5t_1^4 - 4v^{-9}z^3t_1^4 - 3v^{-13}z^{11}t_1^2 - 26v^{-13}z^9t_1^2 - 82v^{-13}z^7t_1^2 - 116v^{-13}z^5t_1^2 - 71v^{-13}z^3t_1^2 - 14v^{-13}zt_1^2 + 3v^{-15}z^7t_1^2 + 17v^{-15}z^5t_1^2 + 26v^{-15}z^3t_1^2 + 9v^{-15}zt_1^2 - 3v^{-13}z^9 - 21v^{-13}z^7 - 49v^{-13}z^5 - 48v^{-13}z^3 - 21v^{-13}z - 3v^{-13}z^{-1} + 3v^{-15}z^9 + 21v^{-15}z^7 + 50v^{-15}z^5 + 53v^{-15}z^3 + 28v^{-15}z + 5v^{-15}z^{-1} - v^{-17}z^5 - 5v^{-17}z^3 - 7v^{-17}z - 2v^{-17}z^{-1}$

$\mathcal{S}_3(\overline{5_{73}}) = -v^{-7}z^7t_1^4 - 4v^{-7}z^5t_1^4 - 3v^{-7}z^3t_1^4 - 3v^{-11}z^9t_1^2 - 20v^{-11}z^7t_1^2 - 45v^{-11}z^5t_1^2 - 40v^{-11}z^3t_1^2 - 11v^{-11}zt_1^2 + 3v^{-13}z^5t_1^2 + 11v^{-13}z^3t_1^2 + 7v^{-13}zt_1^2 - 3v^{-11}z^7 - 15v^{-11}z^5 - 22v^{-11}z^3 - 13v^{-11}z - 3v^{-11}z^{-1} + 3v^{-13}z^7 + 15v^{-13}z^5 + 23v^{-13}z^3 + 16v^{-13}z + 5v^{-13}z^{-1} - v^{-15}z^3 - 3v^{-15}z - 2v^{-15}z^{-1}$

$\mathcal{S}_3(\overline{5_{75}}) = v^7z^3t_1^4 - v^3z^7t_1^2 - 3v^3z^5t_1^2 - v^3z^3t_1^2 + 3v^3zt_1^2 - 3vzt_1^2 - v^3z^5 - 2v^3z^3 - 3v^3z + vz^5 + 2vz^3 + 3vz - vz^{-1} + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{76}}) = z^2t_1^3 - v^{-2}z^4t_1^3 - v^{-2}z^2t_1^3 + 2v^{-2}z^2t_1 + 3v^{-2}t_1 - 2v^{-4}z^4t_1 - 4v^{-4}z^2t_1 - 3v^{-4}t_1 + 2v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{77}}) = -v^{-1}z^3t_1^4 + v^{-5}z^7t_1^2 + 2v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 5v^{-5}zt_1^2 - v^{-7}z^3t_1^2 + 2v^{-7}zt_1^2 + v^{-5}z^5 + v^{-5}z^3 - v^{-5}z^{-1} - v^{-7}z^5 - v^{-7}z^3 - v^{-7}z + v^{-7}z^{-1} + v^{-9}z$

$\mathcal{S}_3(5_{79}) = v^7zt_1^2 - v^7z - v^3z^{-1} + vz + vz^{-1}$

$\mathcal{S}_3(\overline{5_{79}}) = -vz^3t_1^4 + v^{-3}z^7t_1^2 + 3v^{-3}z^5t_1^2 - 4v^{-3}zt_1^2 - v^{-5}z^3t_1^2 + 2v^{-5}zt_1^2 + v^{-3}z^5 + 2v^{-3}z^3 + 2v^{-3}z - v^{-3}z^{-1} - v^{-5}z^5 - 2v^{-5}z^3 - 3v^{-5}z + v^{-5}z^{-1} + v^{-7}z$

$\mathcal{S}_3(5_{80}) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 3v^{-5}zt_1^2 - v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + v^{-7}z^3 + 3v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{80}}) = vz^3t_1^4 - v^{-3}z^7t_1^2 - 3v^{-3}z^5t_1^2 - 2v^{-3}z^3t_1^2 + v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-3}z^5 - 2v^{-3}z^3 - 4v^{-3}z - 2v^{-3}z^{-1} + v^{-5}z^5 + 2v^{-5}z^3 + 5v^{-5}z + 3v^{-5}z^{-1} - v^{-7}z - v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{81}}) = -v^{-5}z^5t_1^4 - 2v^{-5}z^3t_1^4 - 3v^{-9}z^7t_1^2 - 14v^{-9}z^5t_1^2 - 20v^{-9}z^3t_1^2 - 8v^{-9}zt_1^2 + 3v^{-11}z^3t_1^2 + 5v^{-11}zt_1^2 - 3v^{-9}z^5 - 9v^{-9}z^3 - 8v^{-9}z - 3v^{-9}z^{-1} + 3v^{-11}z^5 + 9v^{-11}z^3 + 9v^{-11}z + 5v^{-11}z^{-1} - v^{-13}z - 2v^{-13}z^{-1}$

$\mathcal{S}_3(5_{82}) = -vzt_1^2 + v^{-1}z^3t_1^2 - vz^{-1} + 2v^{-1}z + 2v^{-1}z^{-1} - 2v^{-3}z - 2v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{82}}) = -v^{-3}z^3t_1^4 - v^{-5}z^3t_1^4 - 2v^{-7}z^5t_1^2 - 5v^{-7}z^3t_1^2 - 2v^{-7}zt_1^2 - 3v^{-9}z^5t_1^2 - 9v^{-9}z^3t_1^2 - 4v^{-9}zt_1^2 + 3v^{-11}zt_1^2 - 2v^{-7}z^3 - 2v^{-7}z - v^{-7}z^{-1} - v^{-9}z^3 - 2v^{-9}z + 3v^{-11}z^3 + 4v^{-11}z + 2v^{-11}z^{-1} - v^{-13}z^{-1}$

$\mathcal{S}_3(5_{83}) = -v^6z^2t_1^3 + v^4z^4t_1^3 + v^4z^2t_1^3 - 2v^4z^2t_1 - 3v^4t_1 + 3v^2z^4t_1 + 8v^2z^2t_1 + 6v^2t_1 - 3z^2t_1 - 2t_1$

$\mathcal{S}_3(\overline{5_{83}}) = v^6z^2t_1^3 - v^4z^4t_1 - 2v^4z^2t_1 + v^4t_1 - v^2t_1 - z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{84}) = -v^{-5}z^5t_1^4 - v^{-5}z^3t_1^4 - 4v^{-9}z^7t_1^2 - 17v^{-9}z^5t_1^2 - 21v^{-9}z^3t_1^2 - 7v^{-9}zt_1^2 + 4v^{-11}z^3t_1^2 + 3v^{-11}zt_1^2 - 4v^{-9}z^5 - 11v^{-9}z^3 - 10v^{-9}z - 2v^{-9}z^{-1} + 4v^{-11}z^5 + 11v^{-11}z^3 + 12v^{-11}z + 3v^{-11}z^{-1} - 2v^{-13}z - v^{-13}z^{-1}$

$\mathcal{S}_3(\overline{5_{84}}) = -v^{-11}z^{11}t_1^4 - 8v^{-11}z^9t_1^4 - 22v^{-11}z^7t_1^4 - 24v^{-11}z^5t_1^4 - 9v^{-11}z^3t_1^4 + v^{-13}z^3t_1^4 - 3v^{-15}z^{13}t_1^2 - 31v^{-15}z^{11}t_1^2 - 124v^{-15}z^9t_1^2 - 243v^{-15}z^7t_1^2 - 243v^{-15}z^5t_1^2 - 116v^{-15}z^3t_1^2 - 21v^{-15}zt_1^2 + 3v^{-17}z^9t_1^2 + 22v^{-17}z^7t_1^2 + 56v^{-17}z^5t_1^2 + 55v^{-17}z^3t_1^2 + 18v^{-17}zt_1^2 - 2v^{-19}zt_1^2 - 3v^{-15}z^{11} - 26v^{-15}z^9 - 82v^{-15}z^7 - 118v^{-15}z^5 - 84v^{-15}z^3 - 30v^{-15}z - 5v^{-15}z^{-1} + 3v^{-17}z^{11} + 26v^{-17}z^9 + 83v^{-17}z^7 + 124v^{-17}z^5 + 99v^{-17}z^3 + 44v^{-17}z + 10v^{-17}z^{-1} - v^{-19}z^7 - 6v^{-19}z^5 - 15v^{-19}z^3 - 14v^{-19}z - 6v^{-19}z^{-1} + v^{-21}z^{-1}$

$\mathcal{S}_3(\overline{5_{85}}) = -v^{-5}z^5t_1^6 - 5v^{-13}z^{11}t_1^4 - 30v^{-13}z^9t_1^4 - 64v^{-13}z^7t_1^4 - 55v^{-13}z^5t_1^4 - 15v^{-13}z^3t_1^4 + 5v^{-15}z^3t_1^4 - 15v^{-17}z^{13}t_1^2 - 121v^{-17}z^{11}t_1^2 - 378v^{-17}z^9t_1^2 - 587v^{-17}z^7t_1^2 - 475v^{-17}z^5t_1^2 - 190v^{-17}z^3t_1^2 - 30v^{-17}zt_1^2 + 15v^{-19}z^9t_1^2 + 76v^{-19}z^7t_1^2 + 140v^{-19}z^5t_1^2 + 105v^{-19}z^3t_1^2 + 30v^{-19}zt_1^2 - 6v^{-21}zt_1^2 - 15v^{-17}z^{11} - 96v^{-17}z^9 - 222v^{-17}z^7 - 238v^{-17}z^5 - 136v^{-17}z^3 - 40v^{-17}z - 5v^{-17}z^{-1} + 15v^{-19}z^{11} + 96v^{-19}z^9 + 227v^{-19}z^7 + 254v^{-19}z^5 + 167v^{-19}z^3 + 60v^{-19}z + 10v^{-19}z^{-1} - 5v^{-21}z^7 - 16v^{-21}z^5 - 31v^{-21}z^3 - 20v^{-21}z - 6v^{-21}z^{-1} + v^{-23}z^{-1}$

$\mathcal{S}_3(5_{86}) = vz^3t_1^2 + vzt_1^2 - vz^3 - 2vz - 2vz^{-1} + v^{-1}z^3 + 3v^{-1}z + 3v^{-1}z^{-1} - v^{-3}z - v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{86}}) = -v^3z^3t_1^2 - v^3zt_1^2 + v^3z^3 + 2v^3z + v^3z^{-1} - vz^3 - 3vz - 3vz^{-1} + v^{-1}z + 2v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{88}}) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1} + v^{-7}z$
$\mathcal{S}_3(\overline{5_{89}}) = -v^{-1}z^3t_1^2 - v^{-1}zt_1^2 + v^{-1}z^3 + v^{-1}z - v^{-3}z^3 - 3v^{-3}z - 2v^{-3}z^{-1} + 2v^{-5}z + 3v^{-5}z^{-1} - v^{-7}z^{-1}$
$\mathcal{S}_3(5_{90}) = v^7zt_1^2 + v^5zt_1^2 - v^7z - v^5z - v^5z^{-1} + v^3z + v^3z^{-1} + vz$
$\mathcal{S}_3(5_{92}) = -2v^6t_1 + 2v^4z^2t_1 + 3v^4t_1 + v^2z^2t_1$
$\mathcal{S}_3(5_{94}) = v^7z^3t_1^2 + 2v^7zt_1^2 - v^7z^3 - 2v^7z - v^5z - v^5z^{-1} + v^3z^3 + 3v^3z + v^3z^{-1}$
$\mathcal{S}_3(5_{95}) = v^7z^3t_1^2 + v^7zt_1^2 - v^7z^3 - v^7z + v^7z^{-1} - 2v^5z - 3v^5z^{-1} + v^3z^3 + 3v^3z + 2v^3z^{-1}$

## Knots in $L(3,1)$

$\mathcal{S}_3(1_1) = t_{-1}$
$\mathcal{S}_3(\overline{1_1}) = -v^{-1}zt_{-1}^2 + v^{-2}t_1$
$\mathcal{S}_3(2_2) = -v^{-3}zt_{-1}t_1 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{2_3}) = v^{-2}z^2t_{-1}^3 - 2v^{-5}zt_{-1}t_1 - 2v^{-3}z - v^{-3}z^{-1} + 2v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(3_1) = v^3zt_{-1}^2 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{3_1}) = v^{-2}z^2t_{-1}^2t_1 - v^{-2}z^2t_{-1} + v^{-2}t_{-1} - v^{-1}zt_1^2$
$\mathcal{S}_3(3_2) = v^{-6}z^2t_{-1}^2t_1 + v^{-4}z^4t_{-1} + 4v^{-4}z^2t_{-1} + 2v^{-4}t_{-1} - 2v^{-6}z^2t_{-1} - v^{-6}t_{-1} - v^{-5}zt_1^2$
$\mathcal{S}_3(\overline{3_2}) = v^{-2}z^2t_1 + 2v^{-2}t_1 - v^{-4}t_1$
$\mathcal{S}_3(3_3) = v^2t_{-1} + vzt_1^2$
$\mathcal{S}_3(\overline{3_3}) = -v^{-1}zt_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-4}t_1$
$\mathcal{S}_3(3_4) = vzt_{-1}^2 - z^2t_{-1}t_1^2 + v^2t_1 + z^2t_1$
$\mathcal{S}_3(\overline{3_4}) = -z^2t_{-1} + v^{-2}t_{-1} - vzt_1^2$
$\mathcal{S}_3(3_5) = -v^{-3}zt_{-1}^2 + v^{-2}z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(\overline{3_5}) = v^{-6}z^2t_{-1}^2t_1 + v^{-4}z^4t_{-1} + 4v^{-4}z^2t_{-1} + 3v^{-4}t_{-1} - 2v^{-6}z^2t_{-1} - 2v^{-6}t_{-1}$
$\mathcal{S}_3(\overline{3_6}) = -v^{-3}z^3t_{-1}^4 + 3v^{-8}z^2t_{-1}^2t_1 + 3v^{-6}z^4t_{-1} + 9v^{-6}z^2t_{-1} + 3v^{-6}t_{-1} - 6v^{-8}z^2t_{-1} - 2v^{-8}t_{-1} - v^{-7}zt_1^2$
$\mathcal{S}_3(\overline{3_7}) = -v^{-3}z^3t_{-1}^2 - 2v^{-3}zt_{-1}^2 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(4_1) = -v^4t_{-1} + v^2z^2t_{-1} + 2v^2t_{-1}$
$\mathcal{S}_3(\overline{4_1}) = -v^{-3}z^3t_{-1}^3t_1 + 2v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + 2v^{-4}z^2t_{-1}t_1^2 + 2v^{-2}z^2t_1 + v^{-2}t_1 - 3v^{-4}z^2t_1$
$\mathcal{S}_3(4_2) = -v^{-9}z^3t_{-1}^3t_1 - v^{-7}z^7t_{-1}^2 - 6v^{-7}z^5t_{-1}^2 - 10v^{-7}z^3t_{-1}^2 - 3v^{-7}zt_{-1}^2 + 3v^{-9}z^3t_{-1}^2 + v^{-9}zt_{-1}^2 + 2v^{-10}z^2t_{-1}t_1^2 + v^{-8}z^6t_1 + 5v^{-8}z^4t_1 + 9v^{-8}z^2t_1 + 3v^{-8}t_1 - 4v^{-10}z^2t_1 - 2v^{-10}t_1$
$\mathcal{S}_3(4_4) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} - v^2t_{-1} + 2z^2t_{-1} + 2t_{-1} - vz^3t_1^2 - vzt_1^2$
$\mathcal{S}_3(\overline{4_4}) = v^{-1}z^3t_{-1}^2 + v^{-2}z^2t_{-1}t_1^2 + z^2t_1 + t_1 - 2v^{-2}z^2t_1$
$\mathcal{S}_3(4_5) = v^{-8}z^2t_{-1}^2t_1 + 2v^{-6}z^4t_{-1} + 6v^{-6}z^2t_{-1} + 3v^{-6}t_{-1} - 2v^{-8}z^2t_{-1} - 2v^{-8}t_{-1} - v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - v^{-5}zt_1^2$
$\mathcal{S}_3(\overline{4_5}) = v^{-6}z^4t_{-1} + 2v^{-6}z^2t_{-1} + v^{-6}t_{-1} - v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 4v^{-5}zt_1^2 + v^{-7}zt_1^2$
$\mathcal{S}_3(4_7) = v^5zt_{-1}^2 + v^4z^2t_1 + v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{4_7}) = v^{-4}z^2t_{-1}^2t_1 + v^{-2}z^2t_{-1} + v^{-2}t_{-1} - v^{-4}z^2t_{-1} - v^{-3}zt_1^2$
$\mathcal{S}_3(4_8) = -v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{4_8}) = -v^{-1}z^3t_{-1}t_1 - v^2z^2t_1^3 - vz^3 - vz - vz^{-1} + v^{-1}z^3 + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_9}) = z^2t_{-1} + t_{-1} + v^3zt_1^2 - vz^3t_1^2 - vzt_1^2$
$\mathcal{S}_3(4_{11}) = -v^{-1}zt_{-1}t_1 + v^{-3}zt_{-1}t_1 - z^2t_1^3 - vz - vz^{-1} + 2v^{-1}z + v^{-1}z^{-1} - v^{-3}z$
$\mathcal{S}_3(\overline{4_{11}}) = -v^{-3}zt_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-1}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{12}) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} + 2z^2t_{-1} + t_{-1} - vz^3t_1^2$
$\mathcal{S}_3(\overline{4_{12}}) = v^{-1}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-2}z^2t_{-1}t_1^2 + z^2t_1 + t_1 - 2v^{-2}z^2t_1 - v^{-2}t_1 + v^{-4}t_1$
$\mathcal{S}_3(4_{14}) = v^{-8}z^2t_{-1}^2t_1 + 2v^{-6}z^4t_{-1} + 5v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - 2v^{-8}z^2t_{-1} - v^{-8}t_{-1} - v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2$

$\mathcal{S}_3(4_{15}) = -vz^3t_{-1}t_1 - vzt_{-1}t_1 + v^{-1}zt_{-1}t_1 + v^3z - vz - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_{15}}) = -v^2z^2t_{-1}^3 + v^{-1}z^3t_{-1}^2t_1^2 - vzt_{-1}t_1 - 3v^{-1}z^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - v^2z^2t_1^3 - 2vz^3 - vz - vz^{-1} + 2v^{-1}z^3 + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(4_{16}) = v^{-1}zt_{-1}t_1 - v^3z^{-1} + vz + vz^{-1} - v^{-1}z$
$\mathcal{S}_3(\overline{4_{16}}) = v^{-2}z^2t_{-1}^3 + v^{-4}z^2t_{-1}^3 - v^{-5}zt_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-3}z - v^{-5}z - v^{-5}z^{-1} + 2v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(4_{17}) = -2v^{-2}z^2t_{-1}^2t_1 + v^{-1}z^3t_{-1}t_1^3 - 2z^2t_{-1} + 3v^{-2}z^2t_{-1} + v^{-2}t_{-1} - vzt_1^2 - 2v^{-1}z^3t_1^2$
$\mathcal{S}_3(\overline{4_{17}}) = v^2t_{-1} - z^2t_{-1} - t_{-1} + v^{-2}t_{-1}$
$\mathcal{S}_3(4_{18}) = 2v^{-4}z^2t_{-1} + 2v^{-4}t_{-1} - v^{-6}t_{-1} - v^{-3}z^3t_1^2 - v^{-3}zt_1^2$
$\mathcal{S}_3(\overline{4_{18}}) = -v^{-9}z^3t_{-1}^3t_1 - v^{-7}z^7t_{-1}^2 - 6v^{-7}z^5t_{-1}^2 - 11v^{-7}z^3t_{-1}^2 - 5v^{-7}zt_{-1}^2 + 3v^{-9}z^3t_{-1}^2 + 2v^{-9}zt_{-1}^2 + v^{-10}z^2t_{-1}t_1^2 + v^{-8}z^6t_1 + 4v^{-8}z^4t_1 + 6v^{-8}z^2t_1 + 2v^{-8}t_1 - 2v^{-10}z^2t_1 - v^{-10}t_1$
$\mathcal{S}_3(4_{19}) = v^{-1}z^3t_{-1}t_1 - v^{-1}zt_{-1}t_1 + v^2z^2t_1^3 + vz^3 + vz - v^{-1}z^3 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{4_{19}}) = -vz^3t_{-1}t_1 + vzt_{-1}t_1 - v^4z^2t_1^3 - v^3z^3 - v^3z - v^3z^{-1} + vz^3 + vz + vz^{-1}$
$\mathcal{S}_3(\overline{4_{20}}) = z^2t_{-1}^3 - v^{-3}z^3t_{-1}^2t_1^2 + 3v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + z^2t_1^3 + 2v^{-1}z^3 + v^{-1}z - 2v^{-3}z^3 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{21}) = v^{-6}z^4t_{-1} + 3v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - v^{-8}t_{-1} - v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 3v^{-5}zt_1^2 + v^{-7}zt_1^2$
$\mathcal{S}_3(\overline{4_{22}}) = v^{-4}z^4t_{-1}^3 + 2v^{-4}z^2t_{-1}^3 - 2v^{-7}z^3t_{-1}t_1 - 3v^{-7}zt_{-1}t_1 - 2v^{-5}z^3 - 4v^{-5}z - v^{-5}z^{-1} + 2v^{-7}z^3 + 4v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{4_{23}}) = v^3zt_{-1}^2 - v^2z^2t_{-1}t_1^2 + 2v^2z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{4_{24}}) = v^{-4}z^4t_{-1}^5 - 4v^{-11}z^3t_{-1}^3t_1 - 4v^{-9}z^7t_{-1}^2 - 18v^{-9}z^5t_{-1}^2 - 25v^{-9}z^3t_{-1}^2 - 6v^{-9}zt_{-1}^2 + 12v^{-11}z^3t_{-1}^2 + 3v^{-11}zt_{-1}^2 + 3v^{-12}z^2t_{-1}t_1^2 + 4v^{-10}z^6t_1 + 9v^{-10}z^4t_1 + 12v^{-10}z^2t_1 + 3v^{-10}t_1 - 6v^{-12}z^2t_1 - 2v^{-12}t_1$
$\mathcal{S}_3(\overline{4_{26}}) = t_{-1} - v^{-2}z^2t_{-1} - vzt_1^2 + v^{-1}z^3t_1^2 + v^{-1}zt_1^2$
$\mathcal{S}_3(4_{27}) = v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 + z^2t_1^3 + v^{-1}z^3 - v^{-1}z^{-1} - v^{-3}z^3 + v^{-3}z^{-1}$
$\mathcal{S}_3(5_1) = v^5z^3t_{-1}^2 + v^5zt_{-1}^2 + v^4z^2t_{-1}t_1^2 - v^6t_1 + v^4z^4t_1 + 2v^4z^2t_1 + 2v^4t_1$
$\mathcal{S}_3(\overline{5_1}) = v^{-4}z^4t_{-1}^2t_1 + 2v^{-4}z^2t_{-1}^2t_1 - v^{-4}z^4t_{-1} + 2v^{-4}t_{-1} - v^{-6}t_{-1} - v^{-3}z^3t_1^2 - v^{-3}zt_1^2$
$\mathcal{S}_3(5_2) = -v^2z^2t_{-1}^2t_1 - v^4z^2t_{-1} - v^4t_{-1} + 3v^2z^2t_{-1} + 2v^2t_{-1} - v^3z^3t_1^2$
$\mathcal{S}_3(5_4) = v^{-8}z^4t_{-1}^2t_1 + 2v^{-8}z^2t_{-1}^2t_1 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 9v^{-6}z^2t_{-1} + 3v^{-6}t_{-1} - 2v^{-8}z^4t_{-1} - 5v^{-8}z^2t_{-1} - 2v^{-8}t_{-1} - v^{-7}z^3t_1^2 - v^{-7}zt_1^2$
$\mathcal{S}_3(\overline{5_5}) = v^{-4}z^4t_{-1}^4t_1 - 3v^{-4}z^4t_{-1}^3 + v^{-4}z^2t_{-1}^3 - 3v^{-7}z^3t_{-1}^2t_1^2 - 3v^{-5}z^5t_{-1}t_1 - 9v^{-5}z^3t_{-1}t_1 - 3v^{-5}zt_{-1}t_1 + 10v^{-7}z^3t_{-1}t_1 + v^{-4}z^2t_1^3 + 4v^{-5}z^3 - v^{-5}z - v^{-5}z^{-1} - 4v^{-7}z^3 + v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(5_6) = v^{-12}z^4t_{-1}^4t_1 + v^{-10}z^{10}t_{-1}^3 + 8v^{-10}z^8t_{-1}^3 + 21v^{-10}z^6t_{-1}^3 + 20v^{-10}z^4t_{-1}^3 + 4v^{-10}z^2t_{-1}^3 - 4v^{-12}z^4t_{-1}^3 - v^{-12}z^2t_{-1}^3 - 3v^{-15}z^3t_{-1}^2t_1^2 - 2v^{-13}z^9t_{-1}t_1 - 15v^{-13}z^7t_{-1}t_1 - 39v^{-13}z^5t_{-1}t_1 - 40v^{-13}z^3t_{-1}t_1 - 9v^{-13}zt_{-1}t_1 + 12v^{-15}z^3t_{-1}t_1 + 4v^{-15}zt_{-1}t_1 + v^{-12}z^2t_1^3 - 2v^{-11}z^9 - 16v^{-11}z^7 - 43v^{-11}z^5 - 46v^{-11}z^3 - 17v^{-11}z - 2v^{-11}z^{-1} + 2v^{-13}z^9 + 16v^{-13}z^7 + 43v^{-13}z^5 + 52v^{-13}z^3 + 21v^{-13}z + 3v^{-13}z^{-1} - 6v^{-15}z^3 - 4v^{-15}z - v^{-15}z^{-1}$
$\mathcal{S}_3(5_7) = v^6z^2t_{-1}^3 + 2v^3z^3t_{-1}t_1 + 2v^3zt_{-1}t_1 + v^5z^3 - v^5z^{-1} - v^3z^3 + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_7}) = -v^{-3}z^3t_{-1}^2t_1^2 + 2v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + z^2t_1^3 + v^{-1}z^3 - v^{-3}z^3 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(5_8) = vz^3t_{-1}t_1 + 2vzt_{-1}t_1 - v^{-1}z^3t_{-1}t_1 - 2v^{-1}zt_{-1}t_1 - vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_8}) = -v^{-2}z^4t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1^2 - v^{-3}z^5t_{-1}t_1 - 4v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + 5v^{-5}z^3t_{-1}t_1 + v^{-2}z^2t_1^3 + 3v^{-3}z^3 - v^{-3}z^{-1} - 3v^{-5}z^3 + v^{-5}z^{-1}$
$\mathcal{S}_3(5_9) = v^{-8}z^8t_{-1}^3 + 5v^{-8}z^6t_{-1}^3 + 6v^{-8}z^4t_{-1}^3 + v^{-8}z^2t_{-1}^3 - v^{-13}z^3t_{-1}^2t_1^2 - 3v^{-11}z^7t_{-1}t_1 - 16v^{-11}z^5t_{-1}t_1 - 22v^{-11}z^3t_{-1}t_1 - 6v^{-11}zt_{-1}t_1 + 4v^{-13}z^3t_{-1}t_1 + 2v^{-13}zt_{-1}t_1 + v^{-10}z^2t_1^3 - 3v^{-9}z^7 - 17v^{-9}z^5 - 27v^{-9}z^3 - 13v^{-9}z - 2v^{-9}z^{-1} + 3v^{-11}z^7 + 17v^{-11}z^5 + 29v^{-11}z^3 + 15v^{-11}z + 3v^{-11}z^{-1} - 2v^{-13}z^3 - 2v^{-13}z - v^{-13}z^{-1}$
$\mathcal{S}_3(\overline{5_9}) = -2v^{-9}z^5t_{-1}t_1 - 6v^{-9}z^3t_{-1}t_1 - 4v^{-9}zt_{-1}t_1 + v^{-6}z^6t_1^3 + 4v^{-6}z^4t_1^3 + 4v^{-6}z^2t_1^3 - 2v^{-7}z^5 - 7v^{-7}z^3 - 6v^{-7}z - v^{-7}z^{-1} + 2v^{-9}z^5 + 7v^{-9}z^3 + 6v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(5_{10}) = v^5zt_{-1}^2 + v^2z^2t_{-1}t_1^2 + v^4z^2t_1 + v^2t_1$
$\mathcal{S}_3(\overline{5_{10}}) = v^{-2}z^2t_{-1}^2t_1 + v^{-4}z^2t_{-1}^2t_1 + v^{-2}t_{-1} - v^{-4}z^2t_{-1} - v^{-3}zt_1^2$
$\mathcal{S}_3(5_{11}) = z^2t_{-1}^2t_1 - vz^3t_{-1}t_1^3 + v^2z^2t_{-1} - z^2t_{-1} + t_{-1} + v^3zt_1^2 + vz^3t_1^2 - vzt_1^2$
$\mathcal{S}_3(\overline{5_{11}}) = v^{-1}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-2}z^2t_{-1}t_1^2 + z^2t_1 - v^{-2}z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(5_{12}) = -v^2z^2t_{-1}^3+vz^3t_{-1}t_1+vzt_{-1}t_1-2v^{-1}z^3t_{-1}t_1-v^{-1}zt_{-1}t_1-2vz^3-2vz-vz^{-1}+2v^{-1}z^3+2v^{-1}z+v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{12}}) = -v^{-2}z^4t_{-1}^3-v^{-5}z^3t_{-1}^2t_1^2-v^{-3}z^5t_{-1}t_1-4v^{-3}z^3t_{-1}t_1-3v^{-3}zt_{-1}t_1+4v^{-5}z^3t_{-1}t_1+v^{-5}zt_{-1}t_1+2v^{-3}z^3-v^{-3}z^{-1}-2v^{-5}z^3+v^{-5}z^{-1}$

$\mathcal{S}_3(5_{13}) = -v^4z^2t_{-1}^3-2vz^3t_{-1}t_1-vzt_{-1}t_1+v^{-1}zt_{-1}t_1-v^3z^3+vz^3-vz^{-1}+v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{13}}) = v^{-1}z^3t_{-1}^2t_1^2-vzt_{-1}t_1-2v^{-1}z^3t_{-1}t_1+v^{-1}zt_{-1}t_1-v^2z^2t_1^3-vz^3-vz^{-1}+v^{-1}z^3+v^{-1}z^{-1}$

$\mathcal{S}_3(5_{14}) = -z^2t_{-1}^3+v^{-3}z^3t_{-1}^2t_1^2+v^{-1}z^5t_{-1}t_1+4v^{-1}z^3t_{-1}t_1+v^{-1}zt_{-1}t_1-5v^{-3}z^3t_{-1}t_1-v^{-3}zt_{-1}t_1+z^4t_1^3-vz-vz^{-1}-3v^{-1}z^3+v^{-1}z^{-1}+3v^{-3}z^3+v^{-3}z$

$\mathcal{S}_3(\overline{5_{14}}) = -v^{-1}z^3t_{-1}t_1-2v^{-1}zt_{-1}t_1+v^{-3}z^3t_{-1}t_1+v^{-3}zt_{-1}t_1-v^{-5}zt_{-1}t_1+v^{-1}z^3+v^{-1}z-v^{-3}z^3-2v^{-3}z-v^{-3}z^{-1}+v^{-5}z+v^{-5}z^{-1}$

$\mathcal{S}_3(5_{15}) = v^{-8}z^8t_{-1}^3+5v^{-8}z^6t_{-1}^3+7v^{-8}z^4t_{-1}^3+2v^{-8}z^2t_{-1}^3-v^{-13}z^3t_{-1}^2t_1^2-3v^{-11}z^7t_{-1}t_1-15v^{-11}z^5t_{-1}t_1-21v^{-11}z^3t_{-1}t_1-7v^{-11}zt_{-1}t_1+4v^{-13}z^3t_{-1}t_1+3v^{-13}zt_{-1}t_1-3v^{-9}z^7-16v^{-9}z^5-25v^{-9}z^3-13v^{-9}z-2v^{-9}z^{-1}+3v^{-11}z^7+16v^{-11}z^5+27v^{-11}z^3+16v^{-11}z+3v^{-11}z^{-1}-2v^{-13}z^3-3v^{-13}z-v^{-13}z^{-1}$

$\mathcal{S}_3(\overline{5_{15}}) = -3v^{-9}z^5t_{-1}t_1-8v^{-9}z^3t_{-1}t_1-5v^{-9}zt_{-1}t_1+v^{-11}zt_{-1}t_1+v^{-6}z^6t_1^3+3v^{-6}z^4t_1^3+2v^{-6}z^2t_1^3-3v^{-7}z^5-10v^{-7}z^3-8v^{-7}z-v^{-7}z^{-1}+3v^{-9}z^5+10v^{-9}z^3+9v^{-9}z+v^{-9}z^{-1}-v^{-11}z$

$\mathcal{S}_3(5_{16}) = z^4t_{-1}^2t_1+z^2t_{-1}^2t_1+v^2t_{-1}-z^4t_{-1}-2z^2t_{-1}-t_{-1}+v^{-2}t_{-1}$

$\mathcal{S}_3(\overline{5_{16}}) = vz^3t_{-1}^3t_1+v^3z^3t_{-1}^2+v^3zt_{-1}^2-3vz^3t_{-1}^2-vzt_{-1}^2+z^4t_{-1}t_1^2-z^2t_{-1}t_1^2+v^2z^4t_1-z^4t_1+2z^2t_1+t_1$

$\mathcal{S}_3(5_{17}) = v^{-6}z^2t_{-1}^2t_1+v^{-8}z^2t_{-1}^2t_1+v^{-4}z^4t_{-1}+3v^{-4}z^2t_{-1}+v^{-4}t_{-1}+v^{-6}z^4t_{-1}+2v^{-6}z^2t_{-1}+v^{-6}t_{-1}-2v^{-8}z^2t_{-1}-v^{-8}t_{-1}-v^{-7}zt_1^2$

$\mathcal{S}_3(\overline{5_{17}}) = -v^{-4}z^2t_{-1}t_1^2+z^2t_1+2t_1-v^{-2}z^4t_1-3v^{-2}z^2t_1-2v^{-2}t_1+2v^{-4}z^2t_1+v^{-4}t_1$

$\mathcal{S}_3(5_{18}) = v^2z^2t_{-1}^2t_1+v^4z^2t_{-1}-v^2z^2t_{-1}+v^2t_{-1}+v^3z^3t_1^2+v^3zt_1^2$

$\mathcal{S}_3(\overline{5_{18}}) = -v^{-3}z^3t_{-1}^3t_1+v^{-3}z^3t_{-1}^2-2v^{-3}zt_{-1}^2+v^{-4}z^2t_{-1}t_1^2+v^{-2}z^2t_1-v^{-4}z^2t_1+v^{-4}t_1$

$\mathcal{S}_3(5_{19}) = -z^2t_{-1}^3+v^{-3}z^3t_{-1}^2t_1^2+v^{-1}z^5t_{-1}t_1+4v^{-1}z^3t_{-1}t_1+2v^{-1}zt_{-1}t_1-5v^{-3}z^3t_{-1}t_1-3v^{-3}zt_{-1}t_1+z^4t_1^3+z^2t_1^3-3v^{-1}z^3-3v^{-1}z-v^{-1}z^{-1}+3v^{-3}z^3+3v^{-3}z+v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{19}}) = -v^{-1}z^3t_{-1}t_1-2v^{-1}zt_{-1}t_1+v^{-3}z^3t_{-1}t_1+v^{-3}zt_{-1}t_1+v^{-1}z^3+v^{-1}z-v^{-1}z^{-1}-v^{-3}z^3-v^{-3}z+v^{-3}z^{-1}$

$\mathcal{S}_3(5_{20}) = -3v^{-9}z^5t_{-1}t_1-9v^{-9}z^3t_{-1}t_1-4v^{-9}zt_{-1}t_1+v^{-11}zt_{-1}t_1+v^{-6}z^6t_1^3+3v^{-6}z^4t_1^3+v^{-6}z^2t_1^3-3v^{-7}z^5-11v^{-7}z^3-9v^{-7}z-2v^{-7}z^{-1}+3v^{-9}z^5+11v^{-9}z^3+10v^{-9}z+3v^{-9}z^{-1}-v^{-11}z-v^{-11}z^{-1}$

$\mathcal{S}_3(\overline{5_{20}}) = v^{-8}z^8t_{-1}^3+6v^{-8}z^6t_{-1}^3+11v^{-8}z^4t_{-1}^3+6v^{-8}z^2t_{-1}^3-v^{-10}z^2t_{-1}^3-2v^{-11}z^7t_{-1}t_1-10v^{-11}z^5t_{-1}t_1-14v^{-11}z^3t_{-1}t_1-6v^{-11}zt_{-1}t_1+v^{-13}zt_{-1}t_1-2v^{-9}z^7-11v^{-9}z^5-18v^{-9}z^3-10v^{-9}z-v^{-9}z^{-1}+2v^{-11}z^7+11v^{-11}z^5+18v^{-11}z^3+11v^{-11}z+v^{-11}z^{-1}-v^{-13}z$

$\mathcal{S}_3(5_{21}) = -v^{-2}z^4t_{-1}^2t_1+v^{-2}z^2t_{-1}^2t_1-v^{-1}z^3t_{-1}t_1^3-z^4t_{-1}+t_{-1}+v^{-2}z^4t_{-1}-3v^{-2}z^2t_{-1}-v^{-2}t_{-1}+v^{-4}t_{-1}-vz^3t_1^2-vzt_1^2+3v^{-1}z^3t_1^2$

$\mathcal{S}_3(\overline{5_{21}}) = v^3zt_{-1}^2-v^2z^4t_{-1}t_1^2-2v^2z^2t_{-1}t_1^2+v^2z^4t_1+3v^2z^2t_1+v^2t_1$

$\mathcal{S}_3(5_{22}) = v^2t_{-1}-z^2t_{-1}+v^{-1}zt_1^2$

$\mathcal{S}_3(\overline{5_{22}}) = v^{-1}z^3t_{-1}^3t_1-vzt_{-1}^2-2v^{-1}z^3t_{-1}^2-v^{-1}zt_{-1}^2-2v^{-2}z^2t_{-1}t_1^2-2z^2t_1-t_1+3v^{-2}z^2t_1+2v^{-2}t_1$

$\mathcal{S}_3(5_{23}) = -v^{-2}z^2t_{-1}^2t_1+2t_{-1}+v^{-2}z^2t_{-1}-v^{-2}t_{-1}+v^{-1}zt_1^2$

$\mathcal{S}_3(\overline{5_{23}}) = -v^3zt_{-1}^2-v^2z^2t_1-v^2t_1+2t_1$

$\mathcal{S}_3(\overline{5_{24}}) = -v^{-3}z^3t_{-1}^4+3v^{-8}z^2t_{-1}^2t_1+3v^{-6}z^4t_{-1}+9v^{-6}z^2t_{-1}+4v^{-6}t_{-1}-6v^{-8}z^2t_{-1}-3v^{-8}t_{-1}$

$\mathcal{S}_3(5_{25}) = -v^6t_{-1}+2v^4z^2t_{-1}+2v^4t_{-1}+v^3z^3t_1^2+v^3zt_1^2$

$\mathcal{S}_3(\overline{5_{25}}) = -v^{-5}z^3t_{-1}^3t_1-v^{-3}z^3t_{-1}^2-v^{-3}zt_{-1}^2+v^{-5}z^3t_{-1}^2-v^{-5}zt_{-1}^2+2v^{-6}z^2t_{-1}t_1^2+2v^{-4}z^2t_1+v^{-4}t_1-2v^{-6}z^2t_1$

$\mathcal{S}_3(5_{26}) = v^3z^3t_{-1}^2+v^3zt_{-1}^2-v^2z^4t_{-1}t_1^2-v^2z^2t_{-1}t_1^2+v^4z^2t_1+v^2z^4t_1+v^2z^2t_1+v^2t_1$

$\mathcal{S}_3(\overline{5_{26}}) = -v^{-2}z^4t_{-1}-v^{-2}z^2t_{-1}+v^{-2}t_{-1}+v^{-4}z^2t_{-1}-v^{-1}z^3t_1^2-v^{-1}zt_1^2$

$\mathcal{S}_3(5_{27}) = v^3z^3t_{-1}^2+v^3zt_{-1}^2-v^2z^4t_{-1}t_1^2-v^2z^2t_{-1}t_1^2+v^4z^2t_1+v^2z^4t_1+v^2z^2t_1+v^2t_1$

$\mathcal{S}_3(\overline{5_{27}}) = -v^{-2}z^4t_{-1}-v^{-2}z^2t_{-1}+v^{-2}t_{-1}+v^{-4}z^2t_{-1}-v^{-1}z^3t_1^2-v^{-1}zt_1^2$

$\mathcal{S}_3(5_{28}) = v^{-4}z^4t_{-1}+3v^{-4}z^2t_{-1}+2v^{-4}t_{-1}-v^{-6}t_{-1}-v^{-5}z^3t_1^2-v^{-5}zt_1^2$

$\mathcal{S}_3(\overline{5_{28}}) = v^{-4}z^4t_{-1}t_1^2-v^{-4}z^2t_{-1}t_1^2+vz^3t_1^4+v^{-2}z^6t_1+3v^{-2}z^4t_1+2v^{-2}z^2t_1+2v^{-2}t_1-2v^{-4}z^4t_1-v^{-4}t_1$

$\mathcal{S}_3(5_{30}) = -v^3zt_{-1}^2 - v^2z^2t_1 - z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{30}}) = -v^{-2}z^2t_{-1}^2t_1 + v^2t_{-1} - z^2t_{-1} + v^{-2}z^2t_{-1} + v^{-1}zt_1^2$
$\mathcal{S}_3(5_{31}) = -vz^3t_{-1}^2 + z^4t_{-1}t_1^2 + z^2t_{-1}t_1^2 - v^2z^2t_1 - z^4t_1 + t_1$
$\mathcal{S}_3(\overline{5_{31}}) = z^2t_{-1}^2t_1 + z^4t_{-1} + t_{-1} - v^{-2}z^2t_{-1} + vz^3t_1^2$
$\mathcal{S}_3(5_{32}) = -v^{-1}zt_{-1}t_1 + v^{-3}z^3t_{-1}t_1 + v^{-3}zt_{-1}t_1 - vz - vz^{-1} + v^{-1}z^3 + 3v^{-1}z + v^{-1}z^{-1} - v^{-3}z^3 - 2v^{-3}z$
$\mathcal{S}_3(\overline{5_{32}}) = z^2t_{-1}^3 + v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 - v^{-5}zt_{-1}t_1 + v^{-1}z^3 - v^{-3}z^3 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{5_{33}}) = -vzt_{-1}^2 + v^{-1}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-2}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(5_{34}) = v^{-3}z^3t_{-1}^2 + v^{-4}z^2t_{-1}t_1^2 + 2v^{-2}z^2t_1 + 2v^{-2}t_1 - 2v^{-4}z^2t_1 - v^{-4}t_1$
$\mathcal{S}_3(\overline{5_{34}}) = -v^{-1}z^3t_{-1}^4 - v^{-6}z^4t_{-1}^2t_1 + 2v^{-6}z^2t_{-1}^2t_1 - v^{-4}z^6t_{-1} - 2v^{-4}z^4t_{-1} + 3v^{-4}z^2t_{-1} + 2v^{-4}t_{-1} + 2v^{-6}z^4t_{-1} - 2v^{-6}z^2t_{-1} - v^{-6}t_{-1} - v^{-5}zt_1^2$
$\mathcal{S}_3(5_{35}) = v^5z^3t_{-1}t_1 + v^5zt_{-1}t_1 + v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 - v^7z - v^5z^{-1} + v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{5_{35}}) = v^{-2}z^2t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1^2 - v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 + 3v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 + v^{-2}z^2t_1^3 + 2v^{-3}z^3 - v^{-3}z^{-1} - 2v^{-5}z^3 + v^{-5}z^{-1}$
$\mathcal{S}_3(5_{36}) = vzt_{-1}t_1 - v^{-1}z^3t_{-1}t_1 - v^{-1}zt_{-1}t_1 + v^3z - vz^3 - 3vz - vz^{-1} + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{5_{36}}) = v^{-4}z^2t_{-1}^3 - v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + v^{-5}z + v^{-5}z^{-1} + v^{-7}z$
$\mathcal{S}_3(5_{38}) = -v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 - v^6z^2t_1^3 - v^5z^3 - v^5z + v^3z^3 - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(5_{39}) = -v^{-5}zt_{-1}t_1 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1} + v^{-5}z$
$\mathcal{S}_3(\overline{5_{39}}) = v^{-1}zt_{-1}t_1 - 2v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 - v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 + vz - 2v^{-1}z^3 - 4v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z^3 + 3v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(5_{40}) = v^5z^3t_{-1}^2 + 2v^5zt_{-1}^2 + v^4z^4t_1 + 3v^4z^2t_1 + v^4t_1$
$\mathcal{S}_3(\overline{5_{40}}) = v^{-4}z^4t_{-1}^2t_1 + 2v^{-4}z^2t_{-1}^2t_1 - v^{-4}z^4t_{-1} - v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-3}z^3t_1^2 - 2v^{-3}zt_1^2$
$\mathcal{S}_3(5_{41}) = v^{-8}z^4t_{-1}^2t_1 + 2v^{-8}z^2t_{-1}^2t_1 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 9v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - 2v^{-8}z^4t_{-1} - 5v^{-8}z^2t_{-1} - v^{-8}t_{-1} - v^{-7}z^3t_1^2 - 2v^{-7}zt_1^2$
$\mathcal{S}_3(5_{42}) = v^5zt_{-1}^2 + v^3zt_{-1}^2 + v^4z^2t_1 + v^4t_1 + v^2z^2t_1$
$\mathcal{S}_3(\overline{5_{42}}) = v^{-2}z^2t_{-1}^2t_1 + v^{-4}z^2t_{-1}^2t_1 - v^{-2}z^2t_{-1} - v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-1}zt_1^2 - v^{-3}zt_1^2$
$\mathcal{S}_3(5_{43}) = -v^{-3}zt_{-1}^2 + v^{-4}z^2t_{-1}t_1^2 - 2v^{-6}z^2t_{-1}t_1^2 + v^{-1}z^3t_1^4 + v^{-2}z^4t_1 + 4v^{-2}z^2t_1 + 2v^{-2}t_1 - 2v^{-4}z^4t_1 - 7v^{-4}z^2t_1 - 2v^{-4}t_1 + 4v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_{43}}) = v^{-2}z^2t_{-1} + v^{-2}t_{-1} + v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-6}t_{-1}$
$\mathcal{S}_3(5_{44}) = vz^3t_{-1}t_1 + 2vzt_{-1}t_1 - v^{-1}z^3t_{-1}t_1 - v^{-1}zt_{-1}t_1 - v^3z^{-1} - vz^3 - vz + vz^{-1} + v^{-1}z^3 + v^{-1}z$
$\mathcal{S}_3(\overline{5_{44}}) = -v^{-2}z^4t_{-1}^3 + v^{-4}z^2t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1^2 - v^{-3}z^5t_{-1}t_1 - 4v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + 5v^{-5}z^3t_{-1}t_1 + v^{-5}zt_{-1}t_1 - 2v^{-7}zt_{-1}t_1 + v^{-2}z^2t_1^3 + 3v^{-3}z^3 + v^{-3}z - 3v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + 2v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(5_{45}) = v^{-1}z^3t_{-1}^3t_1 - vzt_{-1}^2 - v^{-1}z^3t_{-1}^2 + v^{-1}zt_{-1}^2 - v^{-2}z^2t_{-1}t_1^2 - z^2t_1 + t_1 + v^{-2}z^2t_1$
$\mathcal{S}_3(\overline{5_{45}}) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} + z^2t_{-1} + v^{-2}t_{-1} - vz^3t_1^2 - vzt_1^2$
$\mathcal{S}_3(5_{46}) = -v^2z^4t_{-1}t_1^2 - v^2z^2t_{-1}t_1^2 - v^4t_1 + v^2z^4t_1 + 2v^2z^2t_1 + 2v^2t_1$
$\mathcal{S}_3(\overline{5_{46}}) = -v^{-2}z^4t_{-1}^2t_1 + v^{-2}z^2t_{-1}^2t_1 - v^{-1}z^3t_{-1}t_1^3 - z^4t_{-1} + t_{-1} + v^{-2}z^4t_{-1} - 2v^{-2}z^2t_{-1} - vz^3t_1^2 - vzt_1^2 + 3v^{-1}z^3t_1^2 + v^{-1}zt_1^2$
$\mathcal{S}_3(5_{47}) = v^{-8}z^8t_{-1}^3 + 5v^{-8}z^6t_{-1}^3 + 7v^{-8}z^4t_{-1}^3 + 3v^{-8}z^2t_{-1}^3 - v^{-13}z^3t_{-1}^2t_1^2 - 3v^{-11}z^7t_{-1}t_1 - 15v^{-11}z^5t_{-1}t_1 - 20v^{-11}z^3t_{-1}t_1 - 6v^{-11}zt_{-1}t_1 + 4v^{-13}z^3t_{-1}t_1 + v^{-13}zt_{-1}t_1 + v^{-10}z^2t_1^3 - 3v^{-9}z^7 - 16v^{-9}z^5 - 24v^{-9}z^3 - 11v^{-9}z - v^{-9}z^{-1} + 3v^{-11}z^7 + 16v^{-11}z^5 + 26v^{-11}z^3 + 12v^{-11}z + v^{-11}z^{-1} - 2v^{-13}z^3 - v^{-13}z$
$\mathcal{S}_3(\overline{5_{47}}) = -3v^{-9}z^5t_{-1}t_1 - 8v^{-9}z^3t_{-1}t_1 - 4v^{-9}zt_{-1}t_1 + v^{-11}zt_{-1}t_1 + v^{-6}z^6t_1^3 + 3v^{-6}z^4t_1^3 + 2v^{-6}z^2t_1^3 - 3v^{-7}z^5 - 10v^{-7}z^3 - 8v^{-7}z - 2v^{-7}z^{-1} + 3v^{-9}z^5 + 10v^{-9}z^3 + 9v^{-9}z + 3v^{-9}z^{-1} - v^{-11}z - v^{-11}z^{-1}$
$\mathcal{S}_3(5_{48}) = -v^3z^3t_{-1}^2 + vzt_{-1}^2 + v^4t_1 - v^2z^4t_1 - 2v^2z^2t_1 - v^2t_1 + z^2t_1 + t_1$
$\mathcal{S}_3(\overline{5_{48}}) = v^3z^3t_{-1}^4 - v^{-2}z^4t_{-1}^3t_1^2 + z^2t_{-1}^2t_1 + 5v^{-2}z^4t_{-1}^2t_1 - 2v^{-2}z^2t_{-1}^2t_1 + 2v^{-1}z^3t_{-1}t_1^3 + z^6t_{-1} + 5z^4t_{-1} - z^2t_{-1} - 5v^{-2}z^4t_{-1} + 2v^{-2}z^2t_{-1} + v^{-2}t_{-1} + 2vz^3t_1^2 - 5v^{-1}z^3t_1^2 - v^{-1}zt_1^2$
$\mathcal{S}_3(5_{49}) = v^{-6}z^2t_{-1}^2t_1 + v^{-8}z^2t_{-1}^2t_1 + v^{-4}z^4t_{-1} + 3v^{-4}z^2t_{-1} + v^{-6}z^4t_{-1} + 2v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - 2v^{-8}z^2t_{-1} -$

$v^{-8}t_{-1} - v^{-5}zt_1^2 - v^{-7}zt_1^2$

$\mathcal{S}_3(\overline{5_{49}}) = v^{-3}zt_{-1}^2 - v^{-4}z^2t_{-1}t_1^2 + z^2t_1 + 2t_1 - v^{-2}z^4t_1 - 3v^{-2}z^2t_1 - v^{-2}t_1 + 2v^{-4}z^2t_1$

$\mathcal{S}_3(5_{50}) = v^4z^2t_{-1} + v^4t_{-1} + v^3z^3t_1^2 + 2v^3zt_1^2$

$\mathcal{S}_3(\overline{5_{50}}) = -v^{-3}z^3t_{-1}^2 - 2v^{-3}zt_{-1}^2 - v^{-5}z^3t_{-1}^2 - v^{-5}zt_{-1}^2 + v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(5_{52}) = -v^{-2}z^4t_{-1}^2t_1 - 2v^{-2}z^2t_{-1}^2t_1 + z^2t_{-1} + 2t_{-1} + v^{-2}z^4t_{-1} + v^{-2}z^2t_{-1} - v^{-2}t_{-1} + v^{-1}z^3t_1^2 + v^{-1}zt_1^2$

$\mathcal{S}_3(\overline{5_{52}}) = -v^3z^3t_{-1}^2 - v^3zt_{-1}^2 - v^2z^2t_{-1}t_1^2 - v^2z^4t_1 - v^2z^2t_1 - v^2t_1 + z^2t_1 + 2t_1$

$\mathcal{S}_3(5_{53}) = -v^{-1}z^3t_{-1}^2 - v^{-1}zt_{-1}^2 - v^{-2}z^2t_{-1}t_1^2 - z^2t_1 + 2v^{-2}z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{53}}) = v^{-8}z^4t_{-1}^2t_1 + 2v^{-8}z^2t_{-1}^2t_1 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 10v^{-6}z^2t_{-1} + 4v^{-6}t_{-1} - 2v^{-8}z^4t_{-1} - 6v^{-8}z^2t_{-1} - 3v^{-8}t_{-1}$

$\mathcal{S}_3(5_{54}) = -v^{-2}z^4t_{-1} - 2v^{-2}z^2t_{-1} + v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-1}z^3t_1^2 - 2v^{-1}zt_1^2$

$\mathcal{S}_3(\overline{5_{54}}) = v^3z^3t_{-1}^2 + 2v^3zt_{-1}^2 - v^2z^4t_{-1}t_1^2 - 2v^2z^2t_{-1}t_1^2 + v^4z^2t_1 + v^4t_1 + v^2z^4t_1 + 2v^2z^2t_1$

$\mathcal{S}_3(\overline{5_{55}}) = -v^2z^2t_{-1}^2t_1 - v^4z^2t_{-1} - v^4t_{-1} + 2v^2z^2t_{-1} + v^2t_{-1} + z^2t_{-1} + t_{-1} - v^3z^3t_1^2 - vz^3t_1^2 - vzt_1^2$

$\mathcal{S}_3(5_{56}) = v^{-4}z^4t_{-1} + 3v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2$

$\mathcal{S}_3(\overline{5_{56}}) = v^{-3}zt_{-1}^2 + v^{-4}z^4t_{-1}t_1^2 - v^{-4}z^2t_{-1}t_1^2 + vz^3t_1^4 + v^{-2}z^6t_1 + 3v^{-2}z^4t_1 + 2v^{-2}z^2t_1 + 3v^{-2}t_1 - 2v^{-4}z^4t_1 - 2v^{-4}t_1$

$\mathcal{S}_3(5_{57}) = v^4t_{-1} + v^3zt_1^2 + vzt_1^2$

$\mathcal{S}_3(\overline{5_{57}}) = -v^{-1}zt_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-5}zt_{-1}^2 + v^{-6}t_1$

$\mathcal{S}_3(5_{58}) = v^3zt_{-1}^2 + vzt_{-1}^2 - v^2z^2t_{-1}t_1^2 - z^2t_{-1}t_1^2 + v^4t_1 + v^2z^2t_1 + z^2t_1$

$\mathcal{S}_3(\overline{5_{58}}) = -z^2t_{-1} - v^{-2}z^2t_{-1} + v^{-4}t_{-1} - vzt_1^2 - v^{-1}zt_1^2$

$\mathcal{S}_3(5_{59}) = -v^{-1}zt_{-1}^2 - v^{-4}z^2t_{-1}t_1^2 + z^2t_1 + t_1 - v^{-2}z^4t_1 - 3v^{-2}z^2t_1 - v^{-2}t_1 + 2v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{59}}) = v^{-6}z^2t_{-1}^2t_1 + v^{-8}z^2t_{-1}^2t_1 + v^{-4}z^4t_{-1} + 3v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-6}z^4t_{-1} + 2v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - 2v^{-8}z^2t_{-1} - 2v^{-8}t_{-1}$

$\mathcal{S}_3(5_{60}) = -v^3zt_{-1}^2 - z^2t_{-1}t_1^2 - v^2z^2t_1 + v^{-2}t_1$

$\mathcal{S}_3(\overline{5_{60}}) = -z^2t_{-1}^2t_1 - v^{-2}z^2t_{-1}^2t_1 + v^2t_{-1} + v^{-2}z^2t_{-1} + v^{-1}zt_1^2$

$\mathcal{S}_3(5_{61}) = v^4z^2t_{-1}^3 - vz^3t_{-1}^2t_1^2 + 2v^3zt_{-1}t_1 + 2vz^3t_{-1}t_1 - v^5z^{-1} + v^3z^3 + v^3z^{-1} - vz^3$

$\mathcal{S}_3(\overline{5_{61}}) = 2v^{-1}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + v^2z^2t_1^3 + vz^3 - v^{-1}z^3 - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{62}) = v^{-3}z^3t_{-1}^2t_1^2 + v^{-1}z^5t_{-1}t_1 + 4v^{-1}z^3t_{-1}t_1 + 2v^{-1}zt_{-1}t_1 - 4v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + z^4t_1^3 - vz - vz^{-1} - 2v^{-1}z^3 + v^{-1}z^{-1} + 2v^{-3}z^3 + v^{-3}z$

$\mathcal{S}_3(\overline{5_{62}}) = -v^{-1}z^3t_{-1}t_1 - v^{-1}zt_{-1}t_1 + 2v^{-3}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 + z^2t_1^3 + 2v^{-1}z^3 + v^{-1}z - 2v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{63}) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} + v^2t_{-1} + z^2t_{-1} - vz^3t_1^2 + v^{-1}zt_1^2$

$\mathcal{S}_3(\overline{5_{63}}) = v^{-1}z^3t_{-1}^3t_1 - vzt_{-1}^2 - v^{-1}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-2}z^2t_{-1}t_1^2 - z^2t_1 + v^{-2}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(5_{64}) = vz^3t_{-1}^3t_1 + v^3z^3t_{-1}^2 - 3vz^3t_{-1}^2 - vzt_{-1}^2 + z^4t_{-1}t_1^2 - z^2t_{-1}t_1^2 + v^2z^4t_1 - v^2z^2t_1 - v^2t_1 - z^4t_1 + 2z^2t_1 + 2t_1$

$\mathcal{S}_3(\overline{5_{64}}) = z^4t_{-1}^2t_1 + z^2t_{-1}^2t_1 - v^{-2}z^2t_{-1}^2t_1 + v^2t_{-1} - z^4t_{-1} - 2z^2t_{-1} + v^{-2}z^2t_{-1} + v^{-1}zt_1^2$

$\mathcal{S}_3(5_{65}) = v^{-8}z^2t_{-1}^3 - 4v^{-9}z^5t_{-1}t_1 - 9v^{-9}z^3t_{-1}t_1 - 4v^{-9}zt_{-1}t_1 + v^{-6}z^6t_1^3 + 2v^{-6}z^4t_1^3 + v^{-6}z^2t_1^3 - 4v^{-7}z^5 - 12v^{-7}z^3 - 8v^{-7}z - v^{-7}z^{-1} + 4v^{-9}z^5 + 12v^{-9}z^3 + 8v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{65}}) = v^{-8}z^8t_{-1}^3 + 5v^{-8}z^6t_{-1}^3 + 8v^{-8}z^4t_{-1}^3 + 4v^{-8}z^2t_{-1}^3 - v^{-13}z^3t_{-1}^2t_1^2 - 3v^{-11}z^7t_{-1}t_1 - 14v^{-11}z^5t_{-1}t_1 - 19v^{-11}z^3t_{-1}t_1 - 6v^{-11}zt_{-1}t_1 + 4v^{-13}z^3t_{-1}t_1 + 2v^{-13}zt_{-1}t_1 - 3v^{-9}z^7 - 15v^{-9}z^5 - 22v^{-9}z^3 - 11v^{-9}z - 2v^{-9}z^{-1} + 3v^{-11}z^7 + 15v^{-11}z^5 + 24v^{-11}z^3 + 13v^{-11}z + 3v^{-11}z^{-1} - 2v^{-13}z^3 - 2v^{-13}z - v^{-13}z^{-1}$

$\mathcal{S}_3(5_{66}) = v^{-2}z^2t_{-1} + v^{-4}z^2t_{-1} + v^{-4}t_{-1} - v^{-3}zt_1^2 - v^{-5}zt_1^2$

$\mathcal{S}_3(\overline{5_{66}}) = v^{-5}zt_{-1}^2 + v^{-4}z^2t_{-1}t_1^2 - 2v^{-6}z^2t_{-1}t_1^2 + v^{-1}z^3t_1^4 + v^{-2}z^4t_1 + 4v^{-2}z^2t_1 + 3v^{-2}t_1 - 2v^{-4}z^4t_1 - 7v^{-4}z^2t_1 - 2v^{-4}t_1 + 4v^{-6}z^2t_1$

$\mathcal{S}_3(\overline{5_{67}}) = -v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + 2v^{-8}z^2t_{-1}^2t_1 + 3v^{-10}z^2t_{-1}^2t_1 + 2v^{-6}z^4t_{-1} + 5v^{-6}z^2t_{-1} + v^{-6}t_{-1} + 3v^{-8}z^4t_{-1} + 5v^{-8}z^2t_{-1} + 2v^{-8}t_{-1} - 6v^{-10}z^2t_{-1} - 2v^{-10}t_{-1} - v^{-9}zt_1^2$

$\mathcal{S}_3(5_{68}) = -v^{-2}z^4t_{-1}^4t_1 + z^2t_{-1}^3 + 3v^{-2}z^4t_{-1}^3 + 3v^{-5}z^3t_{-1}^2t_1^2 + 3v^{-3}z^5t_{-1}t_1 + 9v^{-3}z^3t_{-1}t_1 + v^{-3}zt_{-1}t_1 - 10v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-2}z^2t_1^3 - 2v^{-1}z - v^{-1}z^{-1} - 4v^{-3}z^3 + v^{-3}z + v^{-3}z^{-1} + 4v^{-5}z^3 + v^{-5}z$

$\mathcal{S}_3(5_{69}) = -3v^{-7}z^3t_{-1}t_1 - 3v^{-7}zt_{-1}t_1 + v^{-4}z^4t_1^3 + v^{-4}z^2t_1^3 - 3v^{-5}z^3 - 5v^{-5}z - v^{-5}z^{-1} + 3v^{-7}z^3 + 5v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{69}}) = v^{-12}z^4t_{-1}^4t_1 + v^{-10}z^{10}t_{-1}^3 + 8v^{-10}z^8t_{-1}^3 + 22v^{-10}z^6t_{-1}^3 + 24v^{-10}z^4t_{-1}^3 + 7v^{-10}z^2t_{-1}^3 - 4v^{-12}z^4t_{-1}^3 - 2v^{-12}z^2t_{-1}^3 - 2v^{-15}z^3t_{-1}^2t_1^2 - 2v^{-13}z^9t_{-1}t_1 - 14v^{-13}z^7t_{-1}t_1 - 34v^{-13}z^5t_{-1}t_1 - 34v^{-13}z^3t_{-1}t_1 - 9v^{-13}zt_{-1}t_1 + 8v^{-15}z^3t_{-1}t_1 + 4v^{-15}zt_{-1}t_1 - 2v^{-11}z^9 - 15v^{-11}z^7 - 38v^{-11}z^5 - 40v^{-11}z^3 - 16v^{-11}z - 2v^{-11}z^{-1} + 2v^{-13}z^9 + 15v^{-13}z^7 + 38v^{-13}z^5 + 44v^{-13}z^3 + 20v^{-13}z + 3v^{-13}z^{-1} - 4v^{-15}z^3 - 4v^{-15}z - v^{-15}z^{-1}$

$\mathcal{S}_3(5_{70}) = -v^3z^3t_{-1}^4 - v^{-2}z^4t_{-1}^2t_1 + 2v^{-2}z^2t_{-1}^2t_1 - z^6t_{-1} - 3z^4t_{-1} + t_{-1} + 2v^{-2}z^4t_{-1} - 2v^{-2}z^2t_{-1} + v^{-1}z^3t_1^2$

$\mathcal{S}_3(\overline{5_{70}}) = v^5zt_{-1}^2 - v^3z^3t_{-1}^2 - v^3zt_{-1}^2 - v^4t_1 + 2v^2z^2t_1 + 2v^2t_1$

$\mathcal{S}_3(\overline{5_{71}}) = -v^4z^2t_{-1}^3 + vz^3t_{-1}^2t_1^2 - 4vz^3t_{-1}t_1 - v^4z^2t_1^3 - 2v^3z^3 - v^3z + 2vz^3 - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{72}) = v^5z^3t_{-1}^2 + v^5zt_{-1}^2 - v^6t_1 + v^4z^4t_1 + 3v^4z^2t_1 + 2v^4t_1$

$\mathcal{S}_3(\overline{5_{72}}) = -vz^3t_{-1}^4 + v^{-4}z^4t_{-1}^3t_1^2 - 5v^{-4}z^4t_{-1}^2t_1 + 3v^{-4}z^2t_{-1}^2t_1 - 2v^{-3}z^3t_{-1}t_1^3 - v^{-2}z^6t_{-1} - 5v^{-2}z^4t_{-1} + v^{-2}t_{-1} + 5v^{-4}z^4t_{-1} - 3v^{-4}z^2t_{-1} - 2v^{-1}z^3t_1^2 - v^{-1}zt_1^2 + 5v^{-3}z^3t_1^2$

$\mathcal{S}_3(5_{73}) = v^{-8}z^8t_{-1}^3 + 6v^{-8}z^6t_{-1}^3 + 10v^{-8}z^4t_{-1}^3 + 4v^{-8}z^2t_{-1}^3 - v^{-10}z^2t_{-1}^3 - 2v^{-11}z^7t_{-1}t_1 - 11v^{-11}z^5t_{-1}t_1 - 16v^{-11}z^3t_{-1}t_1 - 6v^{-11}zt_{-1}t_1 + 2v^{-13}zt_{-1}t_1 - 2v^{-9}z^7 - 12v^{-9}z^5 - 21v^{-9}z^3 - 12v^{-9}z - 2v^{-9}z^{-1} + 2v^{-11}z^7 + 12v^{-11}z^5 + 21v^{-11}z^3 + 14v^{-11}z + 3v^{-11}z^{-1} - 2v^{-13}z - v^{-13}z^{-1}$

$\mathcal{S}_3(\overline{5_{73}}) = -2v^{-9}z^5t_{-1}t_1 - 7v^{-9}z^3t_{-1}t_1 - 4v^{-9}zt_{-1}t_1 + v^{-6}z^6t_1^3 + 4v^{-6}z^4t_1^3 + 3v^{-6}z^2t_1^3 - 2v^{-7}z^5 - 8v^{-7}z^3 - 7v^{-7}z - v^{-7}z^{-1} + 2v^{-9}z^5 + 8v^{-9}z^3 + 7v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{74}}) = -v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + 3v^{-10}z^4t_{-1}^2t_1 + 5v^{-10}z^2t_{-1}^2t_1 + 3v^{-8}z^6t_{-1} + 14v^{-8}z^4t_{-1} + 17v^{-8}z^2t_{-1} + 4v^{-8}t_{-1} - 6v^{-10}z^4t_{-1} - 12v^{-10}z^2t_{-1} - 3v^{-10}t_{-1} - v^{-9}z^3t_1^2 - v^{-9}zt_1^2$

$\mathcal{S}_3(\overline{5_{75}}) = -2z^2t_{-1}^2t_1 + vz^3t_{-1}t_1^3 - 2v^2z^2t_{-1} - v^2t_{-1} + 4z^2t_{-1} + 2t_{-1} - 3vz^3t_1^2 - vzt_1^2$

$\mathcal{S}_3(\overline{5_{76}}) = z^2t_{-1}^3 - v^{-2}z^4t_{-1}^3 - v^{-2}z^2t_{-1}^3 - 2v^{-3}zt_{-1}t_1 + 2v^{-5}z^3t_{-1}t_1 + v^{-5}zt_{-1}t_1 - 2v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z^3 + 4v^{-3}z + v^{-3}z^{-1} - 2v^{-5}z^3 - 2v^{-5}z$

$\mathcal{S}_3(\overline{5_{77}}) = -v^{-1}z^3t_{-1}^4 - v^{-6}z^4t_{-1}^2t_1 + 2v^{-6}z^2t_{-1}^2t_1 - v^{-4}z^6t_{-1} - 2v^{-4}z^4t_{-1} + 4v^{-4}z^2t_{-1} + 3v^{-4}t_{-1} + 2v^{-6}z^4t_{-1} - 3v^{-6}z^2t_{-1} - 2v^{-6}t_{-1} + v^{-5}z^3t_1^2$

$\mathcal{S}_3(\overline{5_{78}}) = z^2t_{-1}^2t_1 - 2z^2t_{-1} + v^{-2}t_{-1} - vzt_1^2$

$\mathcal{S}_3(5_{79}) = -v^6t_{-1} + v^4z^2t_{-1} + v^4t_{-1} + v^2z^2t_{-1} + v^2t_{-1}$

$\mathcal{S}_3(\overline{5_{79}}) = -v^{-5}z^3t_{-1}^3t_1 - v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + 2v^{-5}z^3t_{-1}^2 + 2v^{-6}z^2t_{-1}t_1^2 + 3v^{-4}z^2t_1 + 2v^{-4}t_1 - 3v^{-6}z^2t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{80}) = v^{-4}z^4t_{-1} + 4v^{-4}z^2t_{-1} + 3v^{-4}t_{-1} - v^{-6}z^2t_{-1} - 2v^{-6}t_{-1}$

$\mathcal{S}_3(\overline{5_{80}}) = -v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-4}z^4t_{-1}t_1^2 - v^{-4}z^2t_{-1}t_1^2 + vz^3t_1^4 + v^{-2}z^6t_1 + 3v^{-2}z^4t_1 + v^{-2}z^2t_1 + v^{-2}t_1 - 2v^{-4}z^4t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(\overline{5_{81}}) = -v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + 3v^{-10}z^4t_{-1}^2t_1 + 5v^{-10}z^2t_{-1}^2t_1 + 3v^{-8}z^6t_{-1} + 14v^{-8}z^4t_{-1} + 17v^{-8}z^2t_{-1} + 3v^{-8}t_{-1} - 6v^{-10}z^4t_{-1} - 12v^{-10}z^2t_{-1} - 2v^{-10}t_{-1} - v^{-9}z^3t_1^2 - 2v^{-9}zt_1^2$

$\mathcal{S}_3(\overline{5_{82}}) = -v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + 2v^{-8}z^2t_{-1}^2t_1 + 3v^{-10}z^2t_{-1}^2t_1 + 2v^{-6}z^4t_{-1} + 5v^{-6}z^2t_{-1} + 3v^{-8}z^4t_{-1} + 5v^{-8}z^2t_{-1} + 3v^{-8}t_{-1} - 6v^{-10}z^2t_{-1} - 2v^{-10}t_{-1} - v^{-7}zt_1^2 - v^{-9}zt_1^2$

$\mathcal{S}_3(5_{83}) = -vz^3t_{-1}^2 + z^4t_{-1}t_1^2 - 2z^2t_{-1}t_1^2 + v^5z^3t_1^4 + v^2z^6t_1 + 3v^2z^4t_1 - 2z^4t_1 + 2z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{83}}) = v^2t_{-1} - 2z^2t_{-1} - t_{-1} + v^{-2}t_{-1} - v^3zt_1^2 + vz^3t_1^2 + vzt_1^2$

$\mathcal{S}_3(5_{84}) = -3v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 + v^{-4}z^4t_1^3 + v^{-4}z^2t_1^3 - 3v^{-5}z^3 - 5v^{-5}z - 2v^{-5}z^{-1} + 3v^{-7}z^3 + 5v^{-7}z + 3v^{-7}z^{-1} - v^{-9}z^{-1}$

$\mathcal{S}_3(\overline{5_{84}}) = v^{-12}z^4t_{-1}^4t_1 + v^{-10}z^{10}t_{-1}^3 + 8v^{-10}z^8t_{-1}^3 + 22v^{-10}z^6t_{-1}^3 + 24v^{-10}z^4t_{-1}^3 + 8v^{-10}z^2t_{-1}^3 - 4v^{-12}z^4t_{-1}^3 - 2v^{-12}z^2t_{-1}^3 - 2v^{-15}z^3t_{-1}^2t_1^2 - 2v^{-13}z^9t_{-1}t_1 - 14v^{-13}z^7t_{-1}t_1 - 34v^{-13}z^5t_{-1}t_1 - 33v^{-13}z^3t_{-1}t_1 - 8v^{-13}zt_{-1}t_1 + 8v^{-15}z^3t_{-1}t_1 + 2v^{-15}zt_{-1}t_1 + v^{-12}z^2t_1^3 - 2v^{-11}z^9 - 15v^{-11}z^7 - 38v^{-11}z^5 - 39v^{-11}z^3 - 14v^{-11}z - v^{-11}z^{-1} + 2v^{-13}z^9 + 15v^{-13}z^7 + 38v^{-13}z^5 + 43v^{-13}z^3 + 16v^{-13}z + v^{-13}z^{-1} - 4v^{-15}z^3 - 2v^{-15}z$

$\mathcal{S}_3(\overline{5_{85}}) = -v^{-5}z^5t_{-1}^6 + 5v^{-14}z^4t_{-1}^4t_1 + 5v^{-12}z^{10}t_{-1}^3 + 30v^{-12}z^8t_{-1}^3 + 64v^{-12}z^6t_{-1}^3 + 55v^{-12}z^4t_{-1}^3 + 10v^{-12}z^2t_{-1}^3 - 20v^{-14}z^4t_{-1}^3 - 4v^{-14}z^2t_{-1}^3 - 6v^{-17}z^3t_{-1}^2t_1^2 - 10v^{-15}z^9t_{-1}t_1 - 46v^{-15}z^7t_{-1}t_1 - 76v^{-15}z^5t_{-1}t_1 - 64v^{-15}z^3t_{-1}t_1 - 12v^{-15}zt_{-1}t_1 + 24v^{-17}z^3t_{-1}t_1 + 6v^{-17}zt_{-1}t_1 + v^{-14}z^2t_1^3 - 10v^{-13}z^9 - 51v^{-13}z^7 - 86v^{-13}z^5 - 70v^{-13}z^3 - 21v^{-13}z - 2v^{-13}z^{-1} + 10v^{-15}z^9 + 51v^{-15}z^7 + 86v^{-15}z^5 + 82v^{-15}z^3 + 27v^{-15}z + 3v^{-15}z^{-1} - 12v^{-17}z^3 - 6v^{-17}z - v^{-17}z^{-1}$

$\mathcal{S}_3(\overline{5_{87}}) = -v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-5}z^3t_{-1}^2 - 2v^{-5}zt_{-1}^2 + v^{-4}z^2t_1 + v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_{88}}) = -v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-5}z^3t_{-1}^2 - v^{-5}zt_{-1}^2 + v^{-4}z^2t_1 + v^{-4}t_1 + v^{-6}z^2t_1$
$\mathcal{S}_3(5_{92}) = v^5zt_{-1}^2 - v^6t_1 + 2v^4z^2t_1 + 2v^4t_1 + v^2z^2t_1$
$\mathcal{S}_3(\overline{5_{92}}) = v^{-4}z^2t_{-1}^2t_1 + v^{-2}z^2t_{-1} + 2v^{-4}t_{-1} - v^{-6}t_{-1} - v^{-3}zt_1^2$
$\mathcal{S}_3(\overline{5_{93}}) = -v^{-5}z^5t_{-1}^2 - 4v^{-5}z^3t_{-1}^2 - 3v^{-5}zt_{-1}^2 + v^{-6}z^4t_1 + 3v^{-6}z^2t_1 + v^{-6}t_1$

## Knots in $L(4,1)$

$\mathcal{S}_3(2_2) = v^{-2}t_{-1} - v^{-1}zt_1t_2$
$\mathcal{S}_3(2_3) = t_{-1}$
$\mathcal{S}_3(\overline{2_3}) = v^{-2}z^2t_{-1}^3 - 2v^{-5}zt_{-1}t_2 + 2v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(3_2) = v^{-6}z^2t_{-1}^2t_2 - 2v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-3}zt_2^2 - v^{-3}z^3 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{3_2}) = -v^{-3}zt_{-1}t_1 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(3_5) = -v^{-1}zt_2^2 - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{3_5}) = v^{-6}z^2t_{-1}^2t_2 - 2v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 2v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{3_6}) = -v^{-3}z^3t_{-1}^4 + 3v^{-8}z^2t_{-1}^2t_2 - 6v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-5}zt_2^2 - 3v^{-5}z^3 - 2v^{-5}z - v^{-5}z^{-1} + 3v^{-7}z^3 + 2v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{4_1}) = -v^{-3}z^3t_{-1}^3t_1 + 2v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + 2v^{-4}z^2t_{-1}t_1t_2 - 2v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(4_2) = -v^{-9}z^3t_{-1}^3t_2 + 3v^{-8}z^4t_{-1}^2t_1 + v^{-8}z^2t_{-1}^2t_1 + 2v^{-8}z^2t_{-1}t_2^2 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 5v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - 3v^{-8}z^4t_{-1} - 2v^{-8}z^2t_{-1} - v^{-8}t_{-1} - 4v^{-7}z^3t_1t_2 - 2v^{-7}zt_1t_2$
$\mathcal{S}_3(\overline{4_2}) = v^{-2}z^2t_1 + 2v^{-2}t_1 - v^{-4}t_1$
$\mathcal{S}_3(4_4) = vzt_{-1}^2 - z^2t_{-1}t_1t_2 - vzt_1^2 + z^2t_2 + t_2$
$\mathcal{S}_3(\overline{4_4}) = v^{-1}z^3t_{-1}^2 + v^{-2}z^2t_{-1}t_1t_2 - v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - v^{-2}z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(4_5) = -2v^{-7}zt_{-1}t_2 + v^{-2}z^2t_1t_2^2 + 2v^{-4}z^2t_1 + v^{-4}t_1 + 2v^{-6}z^2t_1$
$\mathcal{S}_3(\overline{4_5}) = v^{-6}z^2t_{-1}t_1^2 + v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 3v^{-4}t_1 - 2v^{-6}z^2t_1 - 2v^{-6}t_1$
$\mathcal{S}_3(4_8) = -v^{-5}z^3t_{-1}t_2 - 2v^{-5}zt_{-1}t_2 + v^{-4}z^4t_1 + 3v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(\overline{4_8}) = z^2t_{-1} + t_{-1} - v^2z^2t_1^3 - vz^3t_1t_2$
$\mathcal{S}_3(4_{11}) = t_{-1} - z^2t_1^3 - vzt_1t_2 + v^{-1}zt_1t_2$
$\mathcal{S}_3(\overline{4_{11}}) = -v^{-3}zt_{-1}t_2 - v^{-5}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(4_{12}) = vzt_{-1}^2 - z^2t_{-1}t_1t_2 + v^2t_2 + z^2t_2$
$\mathcal{S}_3(\overline{4_{12}}) = v^{-1}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-2}z^2t_{-1}t_1t_2 - v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - v^{-2}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(4_{14}) = -v^{-7}zt_{-1}t_2 + v^{-2}z^2t_1t_2^2 + 2v^{-4}z^2t_1 + 2v^{-4}t_1 + v^{-6}z^2t_1 - v^{-6}t_1$
$\mathcal{S}_3(4_{15}) = -vz^3t_{-1}t_1 + v^{-1}zt_{-1}t_1 - v^3zt_2^2 + 2v^3z - 2vz - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(\overline{4_{15}}) = v^{-1}z^3t_{-1}^2t_1^2 - v^{-2}z^2t_{-1}^2t_2 - vzt_{-1}t_1 - v^{-1}z^3t_{-1}t_1 - v^2z^2t_1^2t_2 + vzt_2^2 - vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(4_{16}) = v^2t_{-1} + vzt_1t_2$
$\mathcal{S}_3(\overline{4_{16}}) = v^{-2}z^2t_{-1}^3 + v^{-4}z^2t_{-1}^3 - v^{-5}zt_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + v^{-4}z^2t_1 + 2v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(4_{17}) = v^{-1}zt_{-1}^2 + v^{-1}z^3t_{-1}t_1^3 - 2v^{-2}z^2t_{-1}t_1t_2 - vzt_1^2 + t_2 + v^{-2}z^2t_2$
$\mathcal{S}_3(\overline{4_{17}}) = -vzt_{-1}^2 - z^2t_2 + v^{-2}t_2$
$\mathcal{S}_3(4_{18}) = -v^{-5}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}z^2t_1$
$\mathcal{S}_3(\overline{4_{18}}) = -v^{-9}z^3t_{-1}^3t_2 + 3v^{-8}z^4t_{-1}^2t_1 + 2v^{-8}z^2t_{-1}^2t_1 + v^{-8}z^2t_{-1}t_2^2 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 7v^{-6}z^2t_{-1} + 3v^{-6}t_{-1} - 3v^{-8}z^4t_{-1} - 4v^{-8}z^2t_{-1} - 2v^{-8}t_{-1} - 2v^{-7}z^3t_1t_2 - v^{-7}zt_1t_2$
$\mathcal{S}_3(4_{19}) = v^{-2}z^2t_{-1}^2t_2 - v^{-1}z^3t_{-1}t_1 - vzt_2^2 + 2vz - 2v^{-1}z - v^{-1}z^{-1} + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{4_{19}}) = -vz^3t_{-1}t_1 - v^4z^2t_1^2t_2 + v^3zt_2^2 - v^3z^3 - 2v^3z - v^3z^{-1} + vz^3 + 2vz + vz^{-1}$
$\mathcal{S}_3(4_{20}) = v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 + v^5zt_2^2 - 2v^5z - v^5z^{-1} + 2v^3z + v^3z^{-1}$
$\mathcal{S}_3(\overline{4_{20}}) = -v^{-3}z^3t_{-1}^2t_1^2 + v^{-4}z^2t_{-1}^2t_2 + v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 + z^2t_1^2t_2 - v^{-1}zt_2^2 + v^{-1}z^3 + 2v^{-1}z - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(4_{21}) = v^{-6}z^2t_{-1}t_1^2 - v^{-7}zt_{-1}t_2 + v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-6}z^2t_1 - v^{-6}t_1$
$\mathcal{S}_3(\overline{4_{22}}) = v^{-4}z^4t_{-1}^3 + 2v^{-4}z^2t_{-1}^3 - 2v^{-7}z^3t_{-1}t_2 - 3v^{-7}zt_{-1}t_2 + 2v^{-6}z^4t_1 + 4v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{4_{24}}) = v^{-4}z^4t_{-1}^5 - 4v^{-11}z^3t_{-1}^3t_2 + 12v^{-10}z^4t_{-1}^2t_1 + 3v^{-10}z^2t_{-1}^2t_1 + 3v^{-10}z^2t_{-1}t_2^2 + 4v^{-8}z^6t_{-1} + 18v^{-8}z^4t_{-1} + 10v^{-8}z^2t_{-1} + 3v^{-8}t_{-1} - 12v^{-10}z^4t_{-1} - 6v^{-10}z^2t_{-1} - 2v^{-10}t_{-1} - 6v^{-9}z^3t_1t_2 - 2v^{-9}zt_1t_2$
$\mathcal{S}_3(4_{27}) = -v^{-2}z^2t_{-1} + v^{-2}t_{-1} + z^2t_1^3 + v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2$
$\mathcal{S}_3(5_2) = v^3zt_{-1}^2 - v^2z^2t_{-1}t_1t_2 + 2v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{5_2}) = -v^{-3}zt_{-1}^2 + v^{-4}z^2t_{-1}t_1t_2 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-4}t_2$
$\mathcal{S}_3(5_4) = v^{-8}z^4t_{-1}^2t_2 + 2v^{-8}z^2t_{-1}^2t_2 - 2v^{-7}z^5t_{-1}t_1 - 5v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-5}z^3t_2^2 - v^{-5}zt_2^2 - v^{-5}z^5 - 3v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + v^{-7}z^5 + 3v^{-7}z^3 + 3v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{5_4}) = -v^{-1}z^3t_{-1}t_1 - v^2z^2t_1^2t_2 - vz^3 - vz - vz^{-1} + v^{-1}z^3 + v^{-1}z + v^{-1}z^{-1}$
$\mathcal{S}_3(5_5) = -v^4t_{-1} + v^2z^2t_{-1} + 2v^2t_{-1}$
$\mathcal{S}_3(\overline{5_5}) = v^{-4}z^4t_{-1}^4t_1 - 3v^{-4}z^4t_{-1}^3 + v^{-4}z^2t_{-1}^3 - 3v^{-7}z^3t_{-1}^2t_1t_2 + 6v^{-6}z^4t_{-1}t_1^2 + 2v^{-6}z^2t_{-1}t_1^2 + 4v^{-7}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + v^{-2}z^2t_1t_2^2 + 2v^{-4}z^4t_1 + 2v^{-4}z^2t_1 + v^{-4}t_1 - 7v^{-6}z^4t_1 - v^{-6}z^2t_1$
$\mathcal{S}_3(5_6) = v^{-12}z^4t_{-1}^4t_2 - 4v^{-11}z^5t_{-1}^3t_1 - v^{-11}z^3t_{-1}^3t_1 - 3v^{-13}z^3t_{-1}^2t_2^2 - v^{-9}z^9t_{-1}^2 - 8v^{-9}z^7t_{-1}^2 - 21v^{-9}z^5t_{-1}^2 - 13v^{-9}z^3t_{-1}^2 - 3v^{-9}zt_{-1}^2 + 6v^{-11}z^5t_{-1}^2 + 3v^{-11}z^3t_{-1}^2 + v^{-11}zt_{-1}^2 + 12v^{-12}z^4t_{-1}t_1t_2 + 4v^{-12}z^2t_{-1}t_1t_2 + v^{-9}z^5t_1^2 - 6v^{-11}z^5t_1^2 - 4v^{-11}z^3t_1^2 - v^{-11}zt_1^2 + v^{-6}z^2t_2^3 + v^{-10}z^8t_2 + 7v^{-10}z^6t_2 + 15v^{-10}z^4t_2 + 11v^{-10}z^2t_2 + 3v^{-10}t_2 - 6v^{-12}z^4t_2 - 4v^{-12}z^2t_2 - 2v^{-12}t_2$
$\mathcal{S}_3(\overline{5_7}) = -v^{-3}z^3t_{-1}^2t_1^2 + v^{-4}z^2t_{-1}^2t_2 - 2v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$
$\mathcal{S}_3(5_8) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} - v^2t_{-1} + 2z^2t_{-1} + 2t_{-1} - vz^3t_1t_2 - vzt_1t_2$
$\mathcal{S}_3(\overline{5_8}) = -v^{-2}z^4t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1t_2 + 2v^{-4}z^4t_{-1}t_1^2 + v^{-4}z^2t_{-1}t_1^2 + 3v^{-5}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 + z^2t_1t_2^2 + v^{-2}z^2t_1 + v^{-2}t_1 - 4v^{-4}z^4t_1 - v^{-4}z^2t_1$
$\mathcal{S}_3(5_9) = -v^{-11}z^3t_{-1}^2t_2^2 - v^{-7}z^7t_{-1}^2 - 5v^{-7}z^5t_{-1}^2 - 5v^{-7}z^3t_{-1}^2 - v^{-7}zt_{-1}^2 + 4v^{-10}z^4t_{-1}t_1t_2 + 2v^{-10}z^2t_{-1}t_1t_2 + v^{-7}z^5t_1^2 - 2v^{-9}z^5t_1^2 - 2v^{-9}z^3t_1^2 - v^{-9}zt_1^2 + v^{-4}z^2t_2^3 + 2v^{-8}z^6t_2 + 6v^{-8}z^4t_2 + 8v^{-8}z^2t_2 + 3v^{-8}t_2 - 2v^{-10}z^4t_2 - 2v^{-10}z^2t_2 - 2v^{-10}t_2$
$\mathcal{S}_3(\overline{5_9}) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 4v^{-5}zt_1^2 + v^{-7}zt_1^2 + v^{-6}z^4t_2 + 2v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{12}) = -z^2t_{-1}t_2^2 - v^2t_{-1} + z^2t_{-1} + 2t_{-1} - vzt_1t_2$
$\mathcal{S}_3(\overline{5_{12}}) = -v^{-2}z^4t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1t_2 + 2v^{-4}z^4t_{-1}t_1^2 + 2v^{-4}z^2t_{-1}t_1^2 + 2v^{-5}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 + v^{-2}z^4t_1 + 2v^{-2}z^2t_1 + v^{-2}t_1 - 3v^{-4}z^4t_1 - 2v^{-4}z^2t_1$
$\mathcal{S}_3(\overline{5_{13}}) = v^{-1}z^3t_{-1}^2t_1^2 - v^{-2}z^2t_{-1}^2t_2 - vzt_{-1}t_1 + v^{-1}zt_{-1}t_1 - vz^{-1} + v^{-1}z^{-1}$
$\mathcal{S}_3(5_{14}) = -v^{-2}z^2t_{-1}^2t_1 + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_2^2 - z^2t_{-1} + 2v^{-2}z^2t_{-1} + v^{-2}t_{-1} - vzt_1t_2 - v^{-1}z^3t_1t_2$
$\mathcal{S}_3(\overline{5_{14}}) = v^{-2}z^2t_{-1}t_1^2 + v^{-3}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 + z^2t_1 + t_1 - v^{-2}z^4t_1 - 2v^{-2}z^2t_1 - v^{-2}t_1 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(5_{15}) = -v^{-11}z^3t_{-1}^2t_2^2 - v^{-7}z^7t_{-1}^2 - 5v^{-7}z^5t_{-1}^2 - 6v^{-7}z^3t_{-1}^2 - 2v^{-7}zt_{-1}^2 + 4v^{-10}z^4t_{-1}t_1t_2 + 3v^{-10}z^2t_{-1}t_1t_2 - 2v^{-9}z^5t_1^2 - 3v^{-9}z^3t_1^2 - v^{-9}zt_1^2 + 2v^{-8}z^6t_2 + 8v^{-8}z^4t_2 + 8v^{-8}z^2t_2 + 2v^{-8}t_2 - 2v^{-10}z^4t_2 - 3v^{-10}z^2t_2 - v^{-10}t_2$
$\mathcal{S}_3(\overline{5_{15}}) = v^{-8}z^2t_{-1}t_1t_2 - v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - v^{-7}z^3t_1^2 + 2v^{-6}z^4t_2 + 4v^{-6}z^2t_2 + 2v^{-6}t_2 - v^{-8}z^2t_2 - v^{-8}t_2$
$\mathcal{S}_3(5_{16}) = z^4t_{-1}^2t_1 + z^2t_{-1}t_2^2 - 2v^2z^2t_{-1} - z^4t_{-1} + v^{-2}t_{-1} - 2vz^3t_1t_2 - vzt_1t_2$
$\mathcal{S}_3(\overline{5_{16}}) = -v^2z^2t_{-1}^3 + v^{-1}z^3t_{-1}^2t_1t_2 - z^4t_{-1}t_1^2 - 2v^{-1}z^3t_{-1}t_2 + v^{-1}zt_{-1}t_2 - v^4z^2t_1t_2^2 + v^2z^4t_1 + 2v^2z^2t_1 + v^2t_1 + 2z^4t_1 - z^2t_1$
$\mathcal{S}_3(5_{17}) = v^{-6}z^2t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 - 2v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - 2v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-5}zt_2^2 - v^{-3}z^3 - v^{-3}z - v^{-5}z^{-1} + v^{-7}z^3 + v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{5_{17}}) = -v^{-1}zt_{-1}t_1 + v^{-3}zt_{-1}t_1 - z^2t_1^2t_2 - vz - vz^{-1} + 2v^{-1}z + v^{-1}z^{-1} - v^{-3}z$
$\mathcal{S}_3(5_{19}) = -v^{-2}z^2t_{-1}^2t_1 + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_2^2 - z^2t_{-1} - t_{-1} + 2v^{-2}z^2t_{-1} + 2v^{-2}t_{-1} + z^2t_1^3 - v^{-1}z^3t_1t_2 -$

$2v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{19}}) = v^{-2}z^2t_{-1}t_1^2 + v^{-3}z^3t_{-1}t_2 + z^2t_1 + t_1 - v^{-2}z^4t_1 - 2v^{-2}z^2t_1$

$\mathcal{S}_3(5_{20}) = -v^{-7}zt_{-1}^2 + v^{-8}z^2t_{-1}t_1t_2 - v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-7}z^3t_1^2 + 2v^{-6}z^4t_2 + 5v^{-6}z^2t_2 + 2v^{-6}t_2 - v^{-8}z^2t_2 - v^{-8}t_2$

$\mathcal{S}_3(\overline{5_{20}}) = -v^{-9}z^3t_{-1}^3t_1 - v^{-7}z^7t_{-1}^2 - 6v^{-7}z^5t_{-1}^2 - 11v^{-7}z^3t_{-1}^2 - 5v^{-7}zt_{-1}^2 + 3v^{-9}z^3t_{-1}^2 + 2v^{-9}zt_{-1}^2 + v^{-10}z^2t_{-1}t_1t_2 - v^{-9}z^3t_1^2 + v^{-8}z^6t_2 + 4v^{-8}z^4t_2 + 5v^{-8}z^2t_2 + 2v^{-8}t_2 - v^{-10}z^2t_2 - v^{-10}t_2$

$\mathcal{S}_3(5_{21}) = -v^{-2}z^4t_{-1}^2t_1 - v^{-1}z^3t_{-1}t_1^2t_2 + v^{-2}z^2t_{-1}t_2^2 - z^4t_{-1} - 2z^2t_{-1} + v^{-2}z^4t_{-1} - v^{-2}z^2t_{-1} + v^{-4}t_{-1} + z^4t_1^3 + z^2t_1^3 - 2v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{21}}) = -v^2z^4t_{-1}t_1^2 - v^2z^2t_{-1}t_1^2 + 2vzt_{-1}t_2 - v^6z^2t_1t_2^2 + v^4z^4t_1 + 2v^4z^2t_1 + v^4t_1 + v^2z^4t_1 - v^2z^2t_1$

$\mathcal{S}_3(5_{22}) = -vzt_{-1}^2 + v^{-1}zt_1^2 - z^2t_2 + t_2$

$\mathcal{S}_3(\overline{5_{22}}) = v^{-1}z^3t_{-1}^3t_1 - vzt_{-1}^2 - 2v^{-1}z^3t_{-1}^2 - v^{-1}zt_{-1}^2 - 2v^{-2}z^2t_{-1}t_1t_2 + 2v^{-1}z^3t_1^2 + v^{-1}zt_1^2 + v^{-2}z^2t_2 + v^{-2}t_2$

$\mathcal{S}_3(5_{24}) = v^{-1}zt_{-1}t_1 - vzt_2^2 + vz - vz^{-1} - v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{24}}) = -v^{-3}z^3t_{-1}^4 + 3v^{-8}z^2t_{-1}^2t_2 - 6v^{-7}z^3t_{-1}t_1 - 3v^{-7}zt_{-1}t_1 - 3v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + 3v^{-7}z^3 + 3v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{25}) = v^5zt_{-1}^2 + v^3z^3t_1^2 + v^3zt_1^2 + 2v^4z^2t_2 + v^4t_2$

$\mathcal{S}_3(\overline{5_{25}}) = -v^{-5}z^3t_{-1}^3t_1 - v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + v^{-5}z^3t_{-1}^2 - v^{-5}zt_{-1}^2 + 2v^{-6}z^2t_{-1}t_1t_2 - 2v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-6}t_2$

$\mathcal{S}_3(5_{28}) = -v^{-5}zt_{-1}t_1 - v^{-3}z^3t_2^2 - v^{-3}zt_2^2 - 2v^{-3}z - v^{-3}z^{-1} + 2v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{28}}) = -2v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3 - 2v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z^3 + 2v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(5_{32}) = t_{-1} - v^{-2}z^2t_{-1} - vzt_1t_2 + v^{-1}z^3t_1t_2 + v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{32}}) = z^2t_{-1}^3 + v^{-3}z^3t_{-1}t_2 - v^{-3}zt_{-1}t_2 - v^{-5}zt_{-1}t_2 - v^{-2}z^4t_1 + v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(5_{34}) = -v^{-3}zt_{-1}t_1 + z^2t_1^2t_2 + v^{-1}z^3t_2^2 - v^{-1}z^{-1} + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{34}}) = -v^{-1}z^3t_{-1}^4 - v^{-6}z^4t_{-1}^2t_2 + 2v^{-6}z^2t_{-1}^2t_2 + 2v^{-5}z^5t_{-1}t_1 - 2v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-3}zt_2^2 + v^{-3}z^5 - v^{-3}z^{-1} - v^{-5}z^5 + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{35}) = v^5z^3t_{-1}t_1 + v^3z^3t_{-1}t_1 + v^3zt_{-1}t_1 + v^7zt_2^2 - 2v^7z + v^5z - v^5z^{-1} + v^3z + v^3z^{-1}$

$\mathcal{S}_3(\overline{5_{35}}) = -v^{-5}z^3t_{-1}^2t_1^2 + v^{-6}z^2t_{-1}^2t_2 - v^{-3}z^3t_{-1}t_1 - v^{-3}zt_{-1}t_1 + v^{-5}z^3t_{-1}t_1 + v^{-2}z^2t_1^2t_2 - v^{-3}zt_2^2 + v^{-3}z^3 + v^{-3}z - v^{-3}z^{-1} - v^{-5}z^3 - v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{36}) = z^2t_{-1} + t_{-1} + v^3zt_1t_2 - vz^3t_1t_2 - vzt_1t_2$

$\mathcal{S}_3(\overline{5_{36}}) = v^{-4}z^2t_{-1}^3 - v^{-5}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 - v^{-7}zt_{-1}t_2 + v^{-4}z^4t_1 + 2v^{-4}z^2t_1 + v^{-4}t_1 + v^{-6}z^2t_1$

$\mathcal{S}_3(5_{38}) = -v^3z^3t_{-1}t_1 - v^6z^2t_1^2t_2 + v^5zt_2^2 - v^5z^3 - 2v^5z + v^3z^3 + v^3z - v^3z^{-1} + vz + vz^{-1}$

$\mathcal{S}_3(\overline{5_{38}}) = v^{-4}z^2t_{-1}^2t_2 - v^{-3}z^3t_{-1}t_1 - v^{-1}zt_2^2 + v^{-1}z - v^{-1}z^{-1} - v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{39}}) = v^{-2}z^2t_{-1} + v^{-2}t_{-1} - v^2z^2t_1^3 + z^4t_1^3 + z^2t_1^3 + vzt_1t_2 - 2v^{-1}z^3t_1t_2 - 2v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{41}) = v^{-8}z^4t_{-1}^2t_2 + 2v^{-8}z^2t_{-1}^2t_2 - 2v^{-7}z^5t_{-1}t_1 - 5v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-5}z^3t_2^2 - 2v^{-5}zt_2^2 - v^{-5}z^5 - 3v^{-5}z^3 - 2v^{-5}z - v^{-5}z^{-1} + v^{-7}z^5 + 3v^{-7}z^3 + 2v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{41}}) = -v^{-1}z^3t_{-1}t_1 - v^{-1}zt_{-1}t_1 - v^2z^2t_1^2t_2 + vzt_2^2 - vz^3 - 2vz - vz^{-1} + v^{-1}z^3 + 2v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{43}) = -v^{-3}zt_{-1}t_1 + v^{-5}zt_{-1}t_1 + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - v^{-1}zt_2^2 - v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z + v^{-3}z^{-1} - v^{-5}z$

$\mathcal{S}_3(\overline{5_{43}}) = -v^{-3}zt_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-1}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{44}) = -z^2t_{-1}^2t_1 - v^2z^2t_{-1} + 2z^2t_{-1} + t_{-1} - vz^3t_1t_2$

$\mathcal{S}_3(\overline{5_{44}}) = -v^{-2}z^4t_{-1}^3 + v^{-4}z^2t_{-1}^3 - v^{-5}z^3t_{-1}^2t_1t_2 + 2v^{-4}z^4t_{-1}t_1^2 + v^{-4}z^2t_{-1}t_1^2 + 3v^{-5}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + z^2t_1t_2^2 + v^{-2}z^2t_1 + v^{-2}t_1 - 4v^{-4}z^4t_1 - 2v^{-4}z^2t_1 - v^{-4}t_1 + 2v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(5_{46}) = -v^2z^4t_{-1}t_1^2 + vzt_{-1}t_2 - v^6z^2t_1t_2^2 + v^4z^4t_1 + 2v^4z^2t_1 + v^2z^4t_1 - v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(\overline{5_{46}}) = -v^{-2}z^4t_{-1}^2t_1 - v^{-1}z^3t_{-1}t_1^2t_2 + v^{-2}z^2t_{-1}t_2^2 - z^4t_{-1} - 2z^2t_{-1} + v^{-2}z^4t_{-1} + v^{-2}t_{-1} + z^4t_1^3 + z^2t_1^3 - v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{47}) = -v^{-11}z^3t_{-1}^2t_2^2 - v^{-7}z^7t_{-1}^2 - 5v^{-7}z^5t_{-1}^2 - 6v^{-7}z^3t_{-1}^2 - 3v^{-7}zt_{-1}^2 + v^{-9}zt_{-1}^2 + 4v^{-10}z^4t_{-1}t_1t_2 + v^{-10}z^2t_{-1}t_1t_2 + v^{-7}z^5t_1^2 - 2v^{-9}z^5t_1^2 - v^{-9}z^3t_1^2 + v^{-4}z^2t_2^3 + 2v^{-8}z^6t_2 + 5v^{-8}z^4t_2 + 6v^{-8}z^2t_2 + 3v^{-8}t_2 - 2v^{-10}z^4t_2 - v^{-10}z^2t_2 - 2v^{-10}t_2$

$\mathcal{S}_3(\overline{5_{47}}) = -v^{-7}zt_{-1}^2 + v^{-8}z^2t_{-1}t_1t_2 - v^{-5}z^5t_1^2 - 3v^{-5}z^3t_1^2 - 2v^{-5}zt_1^2 - v^{-7}z^3t_1^2 + 2v^{-6}z^4t_2 + 4v^{-6}z^2t_2 + v^{-6}t_2 - v^{-8}z^2t_2$

$\mathcal{S}_3(5_{48}) = -vz^3t_{-1}t_2 - vzt_{-1}t_2 + v^{-1}zt_{-1}t_2 - v^2z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{48}}) = -v^{-2}z^4t_{-1}^3t_1^2 + v^{-3}z^3t_{-1}^3t_2 + z^2t_{-1}^2t_1 + 2v^{-2}z^4t_{-1}^2t_1 + 2v^{-1}z^3t_{-1}t_1^2t_2 - 2v^{-2}z^2t_{-1}t_2^2 + 2z^4t_{-1} + 3z^2t_{-1} + t_{-1} - 2v^{-2}z^4t_{-1} - 2v^{-2}z^2t_{-1} - 2z^4t_1^3 - z^2t_1^3 - vzt_1t_2 + v^{-1}z^3t_1t_2 + v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{49}) = v^{-6}z^2t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 - 2v^{-5}z^3t_{-1}t_1 - 2v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-3}zt_2^2 - v^{-5}zt_2^2 - v^{-3}z^3 - v^{-5}z - v^{-5}z^{-1} + v^{-7}z^3 + v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{49}}) = -v^{-1}zt_{-1}t_1 - z^2t_1^2t_2 + v^{-1}zt_2^2 - vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(5_{53}) = v^{-1}zt_{-1}t_1 - v^2z^2t_1^2t_2 - vz^3t_2^2 - vzt_2^2 - vz^{-1} + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{53}}) = v^{-8}z^4t_{-1}^2t_2 + 2v^{-8}z^2t_{-1}^2t_2 - 2v^{-7}z^5t_{-1}t_1 - 6v^{-7}z^3t_{-1}t_1 - 3v^{-7}zt_{-1}t_1 - v^{-5}z^5 - 4v^{-5}z^3 - 4v^{-5}z - v^{-5}z^{-1} + v^{-7}z^5 + 4v^{-7}z^3 + 4v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{55}) = v^{-3}z^3t_{-1}^2 + v^{-4}z^2t_{-1}t_1t_2 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + v^{-2}z^2t_2 + v^{-2}t_2 - v^{-4}z^2t_2$

$\mathcal{S}_3(\overline{5_{55}}) = v^3zt_{-1}^2 - v^2z^2t_{-1}t_1t_2 - vz^3t_1^2 - vzt_1^2 + v^2z^2t_2 + z^2t_2 + t_2$

$\mathcal{S}_3(5_{56}) = -v^{-3}z^3t_2^2 - 2v^{-3}zt_2^2 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{56}}) = -2v^{-3}z^3t_{-1}t_1 - 2v^{-3}zt_{-1}t_1 + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 + v^{-1}zt_2^2 - 2v^{-1}z^3 - 3v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z^3 + 3v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(5_{59}) = v^{-3}zt_{-1}t_1 - z^2t_1^2t_2 - vzt_2^2 - vz^{-1} + v^{-1}z + v^{-1}z^{-1} - v^{-3}z$

$\mathcal{S}_3(\overline{5_{59}}) = v^{-6}z^2t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 - 2v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - 2v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-3}z^3 - v^{-3}z - v^{-5}z - v^{-5}z^{-1} + v^{-7}z^3 + 2v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{61}}) = v^{-2}z^2t_{-1}^2t_2 - 2v^{-3}zt_{-1}t_1 - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{62}) = -2v^{-2}z^2t_{-1}^2t_1 + v^{-1}z^3t_{-1}t_1^2t_2 - 2z^2t_{-1} + 3v^{-2}z^2t_{-1} + v^{-2}t_{-1} - vzt_1t_2 - 2v^{-1}z^3t_1t_2$

$\mathcal{S}_3(\overline{5_{62}}) = 2v^{-3}z^3t_{-1}t_2 - v^{-5}zt_{-1}t_2 + v^2z^2t_1t_2^2 - z^4t_1 + t_1 - 2v^{-2}z^4t_1 - v^{-2}z^2t_1 - v^{-2}t_1 + v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(5_{64}) = -v^2z^2t_{-1}^3 + v^{-1}z^3t_{-1}^2t_1t_2 - z^4t_{-1}t_1^2 - vzt_{-1}t_2 - 2v^{-1}z^3t_{-1}t_2 + v^{-1}zt_{-1}t_2 - v^4z^2t_1t_2^2 + v^2z^4t_1 + 2v^2z^2t_1 + 2z^4t_1 - z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{64}}) = z^4t_{-1}^2t_1 - v^{-2}z^2t_{-1}^2t_1 + z^2t_{-1}t_2^2 - 2v^2z^2t_{-1} - z^4t_{-1} + t_{-1} + v^{-2}z^2t_{-1} - 2vz^3t_1t_2 - vzt_1t_2 + v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{65}) = -2v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-2}z^2t_2^3 + 4v^{-6}z^2t_2 + 3v^{-6}t_2 - 2v^{-8}t_2$

$\mathcal{S}_3(\overline{5_{65}}) = -v^{-11}z^3t_{-1}^2t_2^2 - v^{-7}z^7t_{-1}^2 - 5v^{-7}z^5t_{-1}^2 - 7v^{-7}z^3t_{-1}^2 - 4v^{-7}zt_{-1}^2 + v^{-9}zt_{-1}^2 + 4v^{-10}z^4t_{-1}t_1t_2 + 2v^{-10}z^2t_{-1}t_1t_2 - 2v^{-9}z^5t_1^2 - 2v^{-9}z^3t_1^2 - v^{-9}zt_1^2 + 2v^{-8}z^6t_2 + 7v^{-8}z^4t_2 + 6v^{-8}z^2t_2 + v^{-8}t_2 - 2v^{-10}z^4t_2 - 2v^{-10}z^2t_2$

$\mathcal{S}_3(5_{66}) = -v^{-1}zt_2^2 - v^{-3}zt_2^2 - v^{-3}z^{-1} + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{66}}) = -2v^{-3}zt_{-1}t_1 + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 + v^{-3}zt_2^2 - 2v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{5_{67}}) = -v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + 2v^{-8}z^2t_{-1}^2t_2 + 3v^{-10}z^2t_{-1}^2t_2 - 4v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - 6v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - v^{-7}zt_2^2 - 2v^{-5}z^3 - v^{-5}z - v^{-7}z^3 - v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^3 + 2v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(5_{68}) = -v^{-2}z^4t_{-1}^4t_1 + z^2t_{-1}^3 + 3v^{-2}z^4t_{-1}^3 + 3v^{-5}z^3t_{-1}^2t_1t_2 - 6v^{-4}z^4t_{-1}t_1^2 - 2v^{-4}z^2t_{-1}t_1^2 - 2v^{-3}zt_{-1}t_2 - 4v^{-5}z^3t_{-1}t_2 - z^2t_1t_2^2 - 2v^{-2}z^4t_1 + 7v^{-4}z^4t_1 + 3v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{68}}) = v^2t_{-1} - z^2t_{-1} - t_{-1} + v^{-2}t_{-1}$

$\mathcal{S}_3(5_{69}) = -v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + 2v^{-4}z^2t_2 + 2v^{-4}t_2 - v^{-6}t_2$

$\mathcal{S}_3(\overline{5_{69}}) = v^{-12}z^4t_{-1}^4t_2 - 4v^{-11}z^5t_{-1}^3t_1 - 2v^{-11}z^3t_{-1}^3t_1 - 2v^{-13}z^3t_{-1}^2t_2^2 - v^{-9}z^9t_{-1}^2 - 8v^{-9}z^7t_{-1}^2 - 22v^{-9}z^5t_{-1}^2 - 18v^{-9}z^3t_{-1}^2 - 5v^{-9}zt_{-1}^2 + 6v^{-11}z^5t_{-1}^2 + 6v^{-11}z^3t_{-1}^2 + 2v^{-11}zt_{-1}^2 + 8v^{-12}z^4t_{-1}t_1t_2 + 4v^{-12}z^2t_{-1}t_1t_2 - 4v^{-11}z^5t_1^2 - 4v^{-11}z^3t_1^2 - v^{-11}zt_1^2 + v^{-10}z^8t_2 + 6v^{-10}z^6t_2 + 14v^{-10}z^4t_2 + 10v^{-10}z^2t_2 + 2v^{-10}t_2 - 4v^{-12}z^4t_2 - 4v^{-12}z^2t_2 - v^{-12}t_2$

$\mathcal{S}_3(5_{70}) = -v^{-3}z^3t_{-1}^3t_2 + 2v^{-2}z^4t_{-1}^2t_1 + 2v^{-2}z^2t_{-1}t_2^2 - 4z^2t_{-1} - v^{-2}z^4t_{-1} + 2v^{-2}z^2t_{-1} + v^{-2}t_{-1} - 3v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{70}}) = -v^4z^2t_{-1}^3 - vz^3t_{-1}t_2 + vzt_{-1}t_2 + v^2z^4t_1 + v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(\overline{5_{71}}) = vz^3t_{-1}^2t_1^2 - z^2t_{-1}^2t_2 - 2vz^3t_{-1}t_1 - vzt_{-1}t_1 - v^4z^2t_1^2t_2 + v^3zt_2^2 - v^3z^3 - 2v^3z + vz^3 + vz - vz^{-1} + v^{-1}z + v^{-1}z^{-1}$

$\mathcal{S}_3(\overline{5_{72}}) = v^{-4}z^4t_{-1}^3t_1^2 - v^{-5}z^3t_{-1}^3t_2 - 2v^{-4}z^4t_{-1}^2t_1 + v^{-4}z^2t_{-1}^2t_1 - 2v^{-3}z^3t_{-1}t_1^2t_2 + 2v^{-4}z^2t_{-1}t_2^2 - 2v^{-2}z^4t_{-1} - 4v^{-2}z^2t_{-1} + 2v^{-4}z^4t_{-1} + v^{-4}z^2t_{-1} + v^{-4}t_{-1} + 2v^{-2}z^4t_1^3 + v^{-2}z^2t_1^3 - v^{-3}z^3t_1t_2 - 2v^{-3}zt_1t_2$

$\mathcal{S}_3(5_{73}) = -v^{-9}z^3t_{-1}^3t_1 - v^{-7}z^7t_{-1}^2 - 6v^{-7}z^5t_{-1}^2 - 10v^{-7}z^3t_{-1}^2 - 3v^{-7}zt_{-1}^2 + 3v^{-9}z^3t_{-1}^2 + v^{-9}zt_{-1}^2 + 2v^{-10}z^2t_{-1}t_1t_2 - 2v^{-9}z^3t_1^2 - v^{-9}zt_1^2 + v^{-8}z^6t_2 + 5v^{-8}z^4t_2 + 7v^{-8}z^2t_2 + 2v^{-8}t_2 - 2v^{-10}z^2t_2 - v^{-10}t_2$

$\mathcal{S}_3(\overline{5_{73}}) = -v^{-5}z^5t_1^2 - 4v^{-5}z^3t_1^2 - 3v^{-5}zt_1^2 + v^{-7}zt_1^2 + v^{-6}z^4t_2 + 3v^{-6}z^2t_2 + 2v^{-6}t_2 - v^{-8}t_2$
$\mathcal{S}_3(\overline{5_{74}}) = -v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + 3v^{-10}z^4t_{-1}^2t_2 + 5v^{-10}z^2t_{-1}^2t_2 - 6v^{-9}z^5t_{-1}t_1 - 12v^{-9}z^3t_{-1}t_1 - 3v^{-9}zt_{-1}t_1 - v^{-7}z^3t_2^2 - v^{-7}zt_2^2 - 3v^{-7}z^5 - 7v^{-7}z^3 - 4v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^5 + 7v^{-9}z^3 + 4v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(\overline{5_{75}}) = vzt_{-1}^2 + vz^3t_{-1}t_1^3 - 2z^2t_{-1}t_1t_2 - vz^3t_1^2 - vzt_1^2 + 2z^2t_2 + t_2$
$\mathcal{S}_3(\overline{5_{76}}) = z^2t_{-1}^3 - v^{-2}z^4t_{-1}^3 - v^{-2}z^2t_{-1}^3 - 2v^{-3}zt_{-1}t_2 + 2v^{-5}z^3t_{-1}t_2 + v^{-5}zt_{-1}t_2 + 2v^{-2}z^2t_1 + v^{-2}t_1 - 2v^{-4}z^4t_1 - 2v^{-4}z^2t_1$
$\mathcal{S}_3(\overline{5_{77}}) = -v^{-1}z^3t_{-1}^4 - v^{-6}z^4t_{-1}^2t_2 + 2v^{-6}z^2t_{-1}^2t_2 + 2v^{-5}z^5t_{-1}t_1 - 3v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 + v^{-3}z^3t_2^2 + v^{-3}z^5 - v^{-3}z^3 - v^{-3}z - v^{-3}z^{-1} - v^{-5}z^5 + v^{-5}z^3 + v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(5_{79}) = v^5zt_{-1}^2 + v^4z^2t_2 + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{5_{79}}) = -v^{-5}z^3t_{-1}^3t_1 - v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 + 2v^{-5}z^3t_{-1}^2 + 2v^{-6}z^2t_{-1}t_1t_2 - 2v^{-5}z^3t_1^2 - v^{-5}zt_1^2 + v^{-4}z^2t_2 + v^{-4}t_2 - v^{-6}z^2t_2$
$\mathcal{S}_3(5_{80}) = -v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{5_{80}}) = -v^{-3}z^3t_{-1}t_1 + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - v^{-1}z^3t_2^2 - v^{-1}zt_2^2 - v^{-1}z^3 - v^{-1}z - v^{-1}z^{-1} + v^{-3}z^3 + v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(5_{81}) = v^3zt_2^2 - v^3z - v^3z^{-1} + vz + vz^{-1}$
$\mathcal{S}_3(\overline{5_{81}}) = -v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + 3v^{-10}z^4t_{-1}^2t_2 + 5v^{-10}z^2t_{-1}^2t_2 - 6v^{-9}z^5t_{-1}t_1 - 12v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - v^{-7}z^3t_2^2 - 2v^{-7}zt_2^2 - 3v^{-7}z^5 - 7v^{-7}z^3 - 3v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^5 + 7v^{-9}z^3 + 3v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(5_{82}) = vzt_2^2 - v^3z^{-1} + vz^{-1}$
$\mathcal{S}_3(\overline{5_{82}}) = -v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + 2v^{-8}z^2t_{-1}^2t_2 + 3v^{-10}z^2t_{-1}^2t_2 - 4v^{-7}z^3t_{-1}t_1 - 6v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - v^{-5}zt_2^2 - v^{-7}zt_2^2 - 2v^{-5}z^3 - v^{-7}z^3 - 2v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^3 + 2v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(5_{83}) = z^4t_{-1}t_1^2 - v^{-1}z^3t_{-1}t_2 + v^{-1}zt_{-1}t_2 + v^5z^3t_1^3t_2 - 2v^4z^2t_1t_2^2 + v^2z^6t_1 + 5v^2z^4t_1 + 4v^2z^2t_1 + v^2t_1 - z^4t_1 - 3z^2t_1$
$\mathcal{S}_3(\overline{5_{83}}) = -2z^2t_{-1} + v^{-2}t_{-1} + v^2z^2t_1^3 + vz^3t_1t_2 - vzt_1t_2$
$\mathcal{S}_3(5_{84}) = -v^{-5}zt_{-1}^2 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 + 2v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{5_{84}}) = v^{-12}z^4t_{-1}^4t_2 - 4v^{-11}z^5t_{-1}^3t_1 - 2v^{-11}z^3t_{-1}^3t_1 - 2v^{-13}z^3t_{-1}^2t_2^2 - v^{-9}z^9t_{-1}^2 - 8v^{-9}z^7t_{-1}^2 - 22v^{-9}z^5t_{-1}^2 - 18v^{-9}z^3t_{-1}^2 - 6v^{-9}zt_{-1}^2 + 6v^{-11}z^5t_{-1}^2 + 6v^{-11}z^3t_{-1}^2 + 3v^{-11}zt_{-1}^2 + 8v^{-12}z^4t_{-1}t_1t_2 + 2v^{-12}z^2t_{-1}t_1t_2 + v^{-9}z^5t_1^2 - 4v^{-11}z^5t_1^2 - 2v^{-11}z^3t_1^2 + v^{-6}z^2t_2^3 + v^{-10}z^8t_2 + 6v^{-10}z^6t_2 + 11v^{-10}z^4t_2 + 8v^{-10}z^2t_2 + 3v^{-10}t_2 - 4v^{-12}z^4t_2 - 2v^{-12}z^2t_2 - 2v^{-12}t_2$
$\mathcal{S}_3(\overline{5_{85}}) = -v^{-5}z^5t_{-1}^6 + 5v^{-14}z^4t_{-1}^4t_2 - 20v^{-13}z^5t_{-1}^3t_1 - 4v^{-13}z^3t_{-1}^3t_1 - 6v^{-15}z^3t_{-1}^2t_2^2 - 5v^{-11}z^9t_{-1}^2 - 30v^{-11}z^7t_{-1}^2 - 64v^{-11}z^5t_{-1}^2 - 29v^{-11}z^3t_{-1}^2 - 6v^{-11}zt_{-1}^2 + 30v^{-13}z^5t_{-1}^2 + 12v^{-13}z^3t_{-1}^2 + 3v^{-13}zt_{-1}^2 + 24v^{-14}z^4t_{-1}t_1t_2 + 6v^{-14}z^2t_{-1}t_1t_2 + v^{-11}z^5t_1^2 - 12v^{-13}z^5t_1^2 - 6v^{-13}z^3t_1^2 - v^{-13}zt_1^2 + v^{-8}z^2t_2^3 + 5v^{-12}z^8t_2 + 16v^{-12}z^6t_2 + 27v^{-12}z^4t_2 + 14v^{-12}z^2t_2 + 3v^{-12}t_2 - 12v^{-14}z^4t_2 - 6v^{-14}z^2t_2 - 2v^{-14}t_2$

## Knots in $L(5,1)$

$\mathcal{S}_3(2_2) = v^{-2}t_{-2} - v^{-1}zt_1t_2$
$\mathcal{S}_3(\overline{2_3}) = -2v^{-3}zt_{-2}t_{-1} + v^{-2}z^2t_{-1}^3 + v^{-4}t_2$
$\mathcal{S}_3(3_2) = -v^{-3}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 - v^{-5}zt_{-1}t_2 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(\overline{3_2}) = -v^{-3}zt_{-2}t_1 + v^{-2}z^2t_{-1} + v^{-2}t_{-1}$
$\mathcal{S}_3(3_5) = v^{-2}t_{-1} - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{3_5}) = v^{-2}z^2t_{-2}t_{-1}^2 - 2v^{-5}zt_{-1}t_2 + 2v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(3_6) = t_{-1}$
$\mathcal{S}_3(\overline{3_6}) = -v^{-5}zt_{-2}^2 + 3v^{-4}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - 2v^{-7}zt_{-1}t_2 + 2v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(4_1) = v^3zt_{-2}t_{-1} + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{4_1}) = 2v^{-4}z^2t_{-2}t_{-1}t_1 - v^{-4}z^2t_{-2} + v^{-4}t_{-2} - v^{-3}z^3t_{-1}^3t_1 - v^{-3}zt_{-1}^2 - v^{-3}zt_1t_2$

$\mathcal{S}_3(4_2) = 2v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - 2v^{-9}zt_{-2}t_2 + v^{-8}z^2t_{-1}^2t_2 - v^{-7}zt_{-1}t_1 - v^{-5}z^3 - 5v^{-5}z - v^{-5}z^{-1} + v^{-7}z^3 + 5v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(4_4) = vzt_{-2}t_{-1} - z^2t_{-1}t_1t_2 - vzt_1^2 + z^2t_2 + t_2$

$\mathcal{S}_3(\overline{4_4}) = v^{-2}z^2t_{-2}t_{-1}t_1 - v^{-2}z^2t_{-2} + v^{-2}t_{-2} - v^{-1}zt_1t_2$

$\mathcal{S}_3(4_5) = -2v^{-7}zt_{-2}t_2 + 2v^{-5}z^3t_{-1}t_1 + v^{-2}z^2t_1t_2^2 - 4v^{-3}z - v^{-3}z^{-1} + 4v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{4_5}) = v^{-6}z^2t_{-2}t_1^2 - 2v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 2v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(4_8) = -v^{-3}z^3t_{-2}t_{-1} - 2v^{-3}zt_{-2}t_{-1} + v^{-4}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(\overline{4_8}) = z^2t_{-2} + t_{-2} - v^2z^2t_1^3 - vz^3t_1t_2$

$\mathcal{S}_3(4_{11}) = t_{-2} - z^2t_1^3 - vzt_1t_2 + v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{4_{11}}) = -v^{-1}zt_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + v^{-4}t_2$

$\mathcal{S}_3(4_{12}) = vzt_{-2}t_{-1} - z^2t_{-1}t_1t_2 + v^2t_2 + z^2t_2$

$\mathcal{S}_3(\overline{4_{12}}) = v^{-2}z^2t_{-2}t_{-1}t_1 - v^{-2}z^2t_{-2} + v^{-4}t_{-2} - v^{-3}zt_{-1}^2 - v^{-1}zt_1t_2$

$\mathcal{S}_3(4_{14}) = -v^{-7}zt_{-2}t_2 + v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 + v^{-2}z^2t_1t_2^2 - 3v^{-3}z - v^{-3}z^{-1} + 3v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(4_{16}) = v^2t_{-2} + vzt_1t_2$

$\mathcal{S}_3(\overline{4_{16}}) = -v^{-3}zt_{-2}t_{-1} - 2v^{-5}zt_{-2}t_{-1} + v^{-2}z^2t_{-1}^3 + v^{-4}z^2t_{-1}^3 + v^{-6}t_2$

$\mathcal{S}_3(4_{17}) = v^{-1}zt_{-2}t_{-1} + v^{-1}z^3t_{-1}t_1^3 - 2v^{-2}z^2t_{-1}t_1t_2 - vzt_1^2 + t_2 + v^{-2}z^2t_2$

$\mathcal{S}_3(\overline{4_{17}}) = -vzt_{-2}t_{-1} - z^2t_2 + v^{-2}t_2$

$\mathcal{S}_3(4_{18}) = -v^{-5}zt_{-2}t_2 + v^{-3}z^3t_{-1}t_1 - 2v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z + v^{-3}z^{-1}$

$\mathcal{S}_3(\overline{4_{18}}) = v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - v^{-9}zt_{-2}t_2 + 2v^{-8}z^2t_{-1}^2t_2 - 3v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - 2v^{-5}z^3 - 4v^{-5}z - v^{-5}z^{-1} + 2v^{-7}z^3 + 4v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(4_{21}) = v^{-6}z^2t_{-2}t_1^2 - v^{-7}zt_{-2}t_2 - v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{4_{22}}) = -2v^{-5}z^3t_{-2}t_{-1} - 3v^{-5}zt_{-2}t_{-1} + v^{-4}z^4t_{-1}^3 + 2v^{-4}z^2t_{-1}^3 + v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(\overline{4_{24}}) = 3v^{-6}z^2t_{-2}^2t_{-1} - 4v^{-5}z^3t_{-2}t_{-1}^3 - 2v^{-11}zt_{-2}t_2 + v^{-4}z^4t_{-1}^5 + 3v^{-10}z^2t_{-1}^2t_2 - 4v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - 3v^{-7}z^3 - 6v^{-7}z - v^{-7}z^{-1} + 3v^{-9}z^3 + 6v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(4_{27}) = -v^{-2}z^2t_{-2} + v^{-2}t_{-2} + z^2t_1^3 + v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2$

$\mathcal{S}_3(5_2) = v^3zt_{-2}t_{-1} - v^2z^2t_{-1}t_1t_2 + 2v^2z^2t_2 + v^2t_2$

$\mathcal{S}_3(\overline{5_2}) = v^{-4}z^2t_{-2}t_{-1}t_1 + v^{-4}t_{-2} - v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-3}zt_1t_2$

$\mathcal{S}_3(5_4) = -v^{-5}z^3t_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 - v^{-7}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + v^{-6}z^4t_1 + 3v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(\overline{5_4}) = -v^{-1}z^3t_{-2}t_1 + z^4t_{-1} + z^2t_{-1} + t_{-1} - v^2z^2t_1^2t_2$

$\mathcal{S}_3(\overline{5_5}) = v^{-6}z^2t_{-2}^2t_1 - 3v^{-5}z^3t_{-2}t_{-1}^2t_1 + 2v^{-5}z^3t_{-2}t_{-1} - 2v^{-5}zt_{-2}t_{-1} + v^{-4}z^4t_{-1}^4t_1 + v^{-4}z^2t_{-1}^3 + 2v^{-6}z^2t_{-1}t_1t_2 - 2v^{-5}z^3t_1^2 - v^{-5}zt_1^2 - v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(5_6) = v^{-6}z^2t_{-2}^3 - 3v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 4v^{-12}z^2t_{-2}t_{-1}t_2 - 4v^{-11}z^3t_{-2}t_1 - 2v^{-11}zt_{-2}t_1 - v^{-11}z^3t_{-1}^3t_2 - v^{-10}z^4t_{-1}^2t_1 + v^{-10}z^2t_{-1}^2t_1 + v^{-8}z^6t_{-1} + 8v^{-8}z^4t_{-1} + 14v^{-8}z^2t_{-1} + 2v^{-8}t_{-1} + v^{-10}z^4t_{-1} - 8v^{-10}z^2t_{-1} - v^{-10}t_{-1} - v^{-9}zt_2^2$

$\mathcal{S}_3(\overline{5_6}) = v^{-2}z^2t_1 + 2v^{-2}t_1 - v^{-4}t_1$

$\mathcal{S}_3(5_8) = -z^2t_{-2}t_{-1}t_1 + z^2t_{-2} + t_{-2} + vz^3t_{-1}^2 + vzt_{-1}^2 - vz^3t_1t_2 - vzt_1t_2$

$\mathcal{S}_3(\overline{5_8}) = v^{-4}z^2t_{-2}^2t_1 - v^{-3}z^3t_{-2}t_{-1}^2t_1 + v^{-3}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + v^{-4}z^2t_{-1}t_1t_2 - v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-4}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_9) = v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 + 2v^{-10}z^2t_{-2}t_{-1}t_2 - 2v^{-9}z^3t_{-2}t_1 - 2v^{-9}zt_{-2}t_1 - 2v^{-8}z^4t_{-1}^2t_1 + 2v^{-6}z^4t_{-1} + 7v^{-6}z^2t_{-1} + v^{-6}t_{-1} + 2v^{-8}z^4t_{-1} - 2v^{-8}z^2t_{-1} - v^{-7}zt_2^2$

$\mathcal{S}_3(\overline{5_9}) = v^{-6}z^2t_{-1}t_1^2 + v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 3v^{-4}t_1 - 2v^{-6}z^2t_1 - 2v^{-6}t_1$

$\mathcal{S}_3(5_{12}) = z^2t_{-2} + t_{-2} + vzt_{-1}^2 - z^2t_{-1}t_2^2 - vzt_1t_2$

$\mathcal{S}_3(\overline{5_{12}}) = -v^{-3}z^3t_{-2}t_{-1}^2t_1 + 2v^{-3}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + 2v^{-4}z^2t_{-1}t_1t_2 - 2v^{-3}z^3t_1^2 - v^{-3}zt_1^2 - v^{-4}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_{14}) = -v^{-2}z^2t_{-2}t_{-1}t_1 + t_{-2} + v^{-2}z^2t_{-2} + v^{-1}z^3t_{-1}^2 + v^{-1}zt_{-1}^2 + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_2^2 - vzt_1t_2 - v^{-1}z^3t_1t_2$

$\mathcal{S}_3(\overline{5_{14}}) = v^{-1}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + v^{-2}z^2t_{-1}t_1t_2 - v^{-1}z^3t_1^2 - v^{-1}zt_1^2 - v^{-2}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_{15}) = -v^{-3}z^3t_{-2}^2t_{-1}^2+3v^{-10}z^2t_{-2}t_{-1}t_2-3v^{-9}z^3t_{-2}t_1-v^{-9}zt_{-2}t_1-3v^{-8}z^4t_{-1}^2t_1+3v^{-6}z^4t_{-1}+9v^{-6}z^2t_{-1}+2v^{-6}t_{-1}+3v^{-8}z^4t_{-1}-5v^{-8}z^2t_{-1}-v^{-8}t_{-1}-v^{-7}zt_2^2$

$\mathcal{S}_3(\overline{5_{15}}) = v^{-8}z^2t_{-2}t_1t_2-v^{-6}z^4t_{-1}t_1^2-v^{-7}z^3t_{-1}t_2-v^{-7}zt_{-1}t_2+v^{-4}z^4t_1+4v^{-4}z^2t_1+2v^{-4}t_1+v^{-6}z^4t_1-v^{-6}z^2t_1-v^{-6}t_1$

$\mathcal{S}_3(5_{16}) = v^{-2}z^2t_{-2}t_{-1}t_2-v^{-1}z^3t_{-2}t_1-2z^2t_{-1}+v^{-2}t_{-1}-vzt_2^2$

$\mathcal{S}_3(\overline{5_{16}}) = vzt_{-2}^2-v^2z^2t_{-2}t_{-1}^2-v^{-2}z^2t_{-2}t_1t_2+v^{-1}z^3t_{-1}^2t_1t_2-v^{-1}z^3t_{-1}t_2+v^2t_1+z^4t_1+2z^2t_1$

$\mathcal{S}_3(5_{17}) = -v^{-5}zt_{-2}^2+v^{-2}z^2t_{-2}t_{-1}^2+v^{-4}z^2t_{-2}t_{-1}^2-v^{-5}zt_{-1}t_2-v^{-7}zt_{-1}t_2+v^{-4}z^2t_1+v^{-6}z^2t_1+v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{17}}) = -v^{-1}zt_{-2}t_1+v^{-3}zt_{-2}t_1+z^2t_{-1}+t_{-1}-v^{-2}z^2t_{-1}-z^2t_1^2t_2$

$\mathcal{S}_3(5_{19}) = -v^{-2}z^2t_{-2}t_{-1}t_1+v^{-2}z^2t_{-2}+v^{-2}t_{-2}+v^{-1}z^3t_{-1}^2+v^{-1}zt_{-1}^2+v^{-1}z^3t_{-1}t_1^2t_2-v^{-2}z^2t_{-1}t_2^2+z^2t_1^3-v^{-1}z^3t_1t_2-2v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{19}}) = v^{-1}z^3t_{-2}t_{-1}+v^{-2}z^2t_{-1}t_1t_2-v^{-1}z^3t_1^2-v^{-1}zt_1^2-v^{-2}z^2t_2+v^{-2}t_2$

$\mathcal{S}_3(5_{20}) = -v^{-5}zt_{-2}^2+v^{-8}z^2t_{-2}t_1t_2-v^{-6}z^4t_{-1}t_1^2-v^{-7}z^3t_{-1}t_2-v^{-7}zt_{-1}t_2+v^{-4}z^4t_1+4v^{-4}z^2t_1+v^{-4}t_1+v^{-6}z^4t_1-v^{-6}z^2t_1$

$\mathcal{S}_3(\overline{5_{20}}) = v^{-10}z^2t_{-2}t_{-1}t_2-v^{-9}z^3t_{-2}t_1-v^{-9}zt_{-2}t_1-v^{-9}z^3t_{-1}^3t_2+2v^{-8}z^4t_{-1}^2t_1+2v^{-8}z^2t_{-1}^2t_1+v^{-6}z^6t_{-1}+6v^{-6}z^4t_{-1}+9v^{-6}z^2t_{-1}+3v^{-6}t_{-1}-2v^{-8}z^4t_{-1}-5v^{-8}z^2t_{-1}-2v^{-8}t_{-1}$

$\mathcal{S}_3(5_{21}) = -v^{-3}z^3t_{-2}t_{-1}t_1^2+v^{-4}z^2t_{-2}t_{-1}t_2+v^{-3}z^3t_{-2}t_1-v^{-3}zt_{-2}t_1-v^{-2}z^4t_{-1}-2v^{-2}z^2t_{-1}+v^{-4}t_{-1}+z^2t_1^2t_2-v^{-1}zt_2^2$

$\mathcal{S}_3(\overline{5_{21}}) = v^3zt_{-2}^2-z^2t_{-2}t_1t_2-v^2z^2t_{-1}t_1^2+vz^3t_{-1}t_2+vzt_{-1}t_2+v^4t_1+2v^2z^2t_1$

$\mathcal{S}_3(5_{22}) = -vzt_{-2}t_{-1}+v^{-1}zt_1^2-z^2t_2+t_2$

$\mathcal{S}_3(\overline{5_{22}}) = -2v^{-2}z^2t_{-2}t_{-1}t_1+v^{-2}z^2t_{-2}+v^{-2}t_{-2}+v^{-1}z^3t_{-1}^3t_1-vzt_{-1}^2-v^{-1}zt_{-1}^2+v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{24}) = v^{-1}zt_{-2}t_1-z^2t_{-1}+t_{-1}-vzt_2^2$

$\mathcal{S}_3(\overline{5_{24}}) = 3v^{-4}z^2t_{-2}t_{-1}^2-v^{-3}z^3t_{-1}^4-3v^{-7}zt_{-1}t_2+3v^{-6}z^2t_1+v^{-6}t_1$

$\mathcal{S}_3(5_{25}) = v^5zt_{-2}t_{-1}+v^3z^3t_1^2+v^3zt_1^2+2v^4z^2t_2+v^4t_2$

$\mathcal{S}_3(\overline{5_{25}}) = 2v^{-6}z^2t_{-2}t_{-1}t_1+v^{-6}t_{-2}-v^{-5}z^3t_{-1}^3t_1-v^{-3}z^3t_{-1}^2-v^{-3}zt_{-1}^2-v^{-5}z^3t_{-1}^2-v^{-5}zt_{-1}^2-v^{-5}zt_1t_2$

$\mathcal{S}_3(5_{28}) = -v^{-3}z^3t_{-2}^2-v^{-3}zt_{-2}^2-v^{-5}zt_{-1}t_2+2v^{-4}z^2t_1+v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{28}}) = -2v^{-3}z^3t_{-2}t_1-v^{-3}zt_{-2}t_1+2v^{-2}z^4t_{-1}+2v^{-2}z^2t_{-1}+v^{-2}t_{-1}+vz^3t_1^4+z^4t_1^2t_2-z^2t_1^2t_2$

$\mathcal{S}_3(5_{32}) = t_{-2}-v^{-2}z^2t_{-2}-vzt_1t_2+v^{-1}z^3t_1t_2+v^{-1}zt_1t_2$

$\mathcal{S}_3(\overline{5_{32}}) = v^{-1}z^3t_{-2}t_{-1}-v^{-1}zt_{-2}t_{-1}-v^{-3}zt_{-2}t_{-1}+z^2t_{-1}^3-v^{-2}z^2t_2+v^{-4}t_2$

$\mathcal{S}_3(5_{34}) = -v^{-3}zt_{-2}t_1+v^{-2}t_{-1}+z^2t_1^2t_2+v^{-1}z^3t_2^2$

$\mathcal{S}_3(\overline{5_{34}}) = -v^{-3}zt_{-2}^2-v^{-2}z^4t_{-2}t_{-1}^2+2v^{-2}z^2t_{-2}t_{-1}^2-v^{-1}z^3t_{-1}^4+2v^{-5}z^3t_{-1}t_2-v^{-5}zt_{-1}t_2-2v^{-4}z^4t_1+v^{-4}t_1$

$\mathcal{S}_3(5_{36}) = z^2t_{-2}+t_{-2}+v^3zt_1t_2-vz^3t_1t_2-vzt_1t_2$

$\mathcal{S}_3(\overline{5_{36}}) = -v^{-3}z^3t_{-2}t_{-1}-v^{-3}zt_{-2}t_{-1}-v^{-5}zt_{-2}t_{-1}+v^{-4}z^2t_{-1}^3+v^{-4}z^2t_2+v^{-4}t_2$

$\mathcal{S}_3(5_{39}) = -v^{-3}zt_{-2}t_{-1}+v^{-2}z^2t_2+v^{-2}t_2$

$\mathcal{S}_3(\overline{5_{39}}) = v^{-2}z^2t_{-2}+v^{-2}t_{-2}-v^2z^2t_1^3+z^4t_1^3+z^2t_1^3+vzt_1t_2-2v^{-1}z^3t_1t_2-2v^{-1}zt_1t_2$

$\mathcal{S}_3(5_{41}) = -v^{-5}z^3t_{-2}^2-2v^{-5}zt_{-2}^2+v^{-4}z^4t_{-2}t_{-1}^2+2v^{-4}z^2t_{-2}t_{-1}^2-v^{-7}z^3t_{-1}t_2-v^{-7}zt_{-1}t_2+v^{-6}z^4t_1+2v^{-6}z^2t_1+v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{41}}) = -v^{-1}z^3t_{-2}t_1-v^{-1}zt_{-2}t_1+z^4t_{-1}+2z^2t_{-1}+t_{-1}-v^2z^2t_1^2t_2+vzt_2^2$

$\mathcal{S}_3(5_{43}) = -v^{-3}zt_{-2}t_1+v^{-5}zt_{-2}t_1+v^{-2}z^2t_{-1}+v^{-2}t_{-1}-v^{-4}z^2t_{-1}+v^{-1}z^3t_1^4+z^2t_1^2t_2-2v^{-2}z^2t_1^2t_2-v^{-1}zt_2^2$

$\mathcal{S}_3(\overline{5_{43}}) = -v^{-3}zt_{-1}t_2-v^{-5}zt_{-1}t_2+v^{-2}z^2t_1+v^{-4}z^2t_1+v^{-4}t_1$

$\mathcal{S}_3(5_{44}) = -z^2t_{-2}t_{-1}t_1+v^2t_{-2}+z^2t_{-2}+vz^3t_{-1}^2+vzt_{-1}^2-vz^3t_1t_2$

$\mathcal{S}_3(\overline{5_{44}}) = v^{-4}z^2t_{-2}^2t_1-v^{-3}z^3t_{-2}t_{-1}^2t_1+v^{-3}z^3t_{-2}t_{-1}-2v^{-5}zt_{-2}t_{-1}+v^{-4}z^2t_{-1}^3+v^{-4}z^2t_{-1}t_1t_2-v^{-3}z^3t_1^2-v^{-3}zt_1^2-v^{-4}z^2t_2+v^{-6}t_2$

$\mathcal{S}_3(5_{46}) = v^3zt_{-2}^2-z^2t_{-2}t_1t_2+vz^3t_{-1}t_2+2v^2z^2t_1+v^2t_1$

$\mathcal{S}_3(\overline{5_{46}}) = -v^{-3}z^3t_{-2}t_{-1}t_1^2+v^{-4}z^2t_{-2}t_{-1}t_2+v^{-3}z^3t_{-2}t_1-v^{-2}z^4t_{-1}-2v^{-2}z^2t_{-1}+v^{-2}t_{-1}+z^2t_1^2t_2-v^{-1}zt_2^2$

$\mathcal{S}_3(5_{47}) = v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2+v^{-10}z^2t_{-2}t_{-1}t_2-v^{-9}z^3t_{-2}t_1-2v^{-9}zt_{-2}t_1-v^{-8}z^4t_{-1}^2t_1+v^{-8}z^2t_{-1}^2t_1+$

$2v^{-6}z^4t_{-1} + 7v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} + v^{-8}z^4t_{-1} - 2v^{-8}z^2t_{-1} - v^{-8}t_{-1}$

$\mathcal{S}_3(\overline{5_{47}}) = -v^{-5}zt_{-2}^2 + v^{-8}z^2t_{-2}t_1t_2 - v^{-6}z^4t_{-1}t_1^2 - v^{-7}z^3t_{-1}t_2 + v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 2v^{-4}t_1 + v^{-6}z^4t_1 - 2v^{-6}z^2t_1 - v^{-6}t_1$

$\mathcal{S}_3(5_{48}) = -v^3zt_{-2}^2 - vz^3t_{-1}t_2 + v^{-1}zt_{-1}t_2 - 2v^2z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{48}}) = 2v^{-3}z^3t_{-2}t_{-1}t_1^2 - 2v^{-4}z^2t_{-2}t_{-1}t_2 - v^{-1}zt_{-2}t_1 - v^{-3}z^3t_{-2}t_1 - v^{-2}z^4t_{-1}^3t_1^2 + v^{-3}z^3t_{-1}^3t_2 + z^2t_{-1}^2t_1 + t_{-1} + v^{-2}z^4t_{-1} + 2v^{-2}z^2t_{-1} - z^2t_1^2t_2 + v^{-1}zt_2^2$

$\mathcal{S}_3(5_{49}) = -v^{-3}zt_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 - v^{-7}zt_{-1}t_2 + v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{49}}) = -v^{-1}zt_{-2}t_1 + z^2t_{-1} + t_{-1} - z^2t_1^2t_2 + v^{-1}zt_2^2$

$\mathcal{S}_3(5_{53}) = v^{-1}zt_{-2}t_1 + t_{-1} - v^2z^2t_1^2t_2 - vz^3t_2^2 - vzt_2^2$

$\mathcal{S}_3(\overline{5_{53}}) = v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 - 2v^{-7}z^3t_{-1}t_2 - 3v^{-7}zt_{-1}t_2 + 2v^{-6}z^4t_1 + 4v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(5_{55}) = v^{-4}z^2t_{-2}t_{-1}t_1 + v^{-2}z^2t_{-2} + v^{-2}t_{-2} - v^{-4}z^2t_{-2} - v^{-3}zt_1t_2$

$\mathcal{S}_3(\overline{5_{55}}) = v^3zt_{-2}t_{-1} - v^2z^2t_{-1}t_1t_2 - vz^3t_1^2 - vzt_1^2 + v^2z^2t_2 + z^2t_2 + t_2$

$\mathcal{S}_3(5_{56}) = -v^{-3}z^3t_{-2}^2 - 2v^{-3}zt_{-2}^2 + v^{-4}z^2t_1 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{56}}) = -2v^{-3}z^3t_{-2}t_1 - 2v^{-3}zt_{-2}t_1 + 2v^{-2}z^4t_{-1} + 3v^{-2}z^2t_{-1} + v^{-2}t_{-1} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 + v^{-1}zt_2^2$

$\mathcal{S}_3(5_{59}) = v^{-3}zt_{-2}t_1 + t_{-1} - v^{-2}z^2t_{-1} - z^2t_1^2t_2 - vzt_2^2$

$\mathcal{S}_3(\overline{5_{59}}) = v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 - v^{-5}zt_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + v^{-4}z^2t_1 + 2v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(5_{62}) = -2v^{-2}z^2t_{-2}t_{-1}t_1 + t_{-2} + v^{-2}z^2t_{-2} + 2v^{-1}z^3t_{-1}^2 + v^{-1}zt_{-1}^2 + v^{-1}z^3t_{-1}t_1^2t_2 - vzt_1t_2 - 2v^{-1}z^3t_1t_2$

$\mathcal{S}_3(\overline{5_{62}}) = v^{-2}z^2t_{-2}^2t_1 - v^{-3}zt_{-2}t_{-1} - v^{-1}zt_1^2 - v^{-2}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_{64}) = vzt_{-2}^2 - v^2z^2t_{-2}t_{-1}^2 - v^{-2}z^2t_{-2}t_1t_2 + v^{-1}z^3t_{-1}^2t_1t_2 - vzt_{-1}t_2 - v^{-1}z^3t_{-1}t_2 + z^4t_1 + 2z^2t_1 + t_1$

$\mathcal{S}_3(\overline{5_{64}}) = v^{-2}z^2t_{-2}t_{-1}t_2 - v^{-1}z^3t_{-2}t_1 + v^{-3}zt_{-2}t_1 - v^{-2}z^2t_{-1}^2t_1 - 2z^2t_{-1} + t_{-1} - vzt_2^2$

$\mathcal{S}_3(5_{65}) = -2v^{-7}zt_{-1}t_2 + 2v^{-4}z^2t_1 + v^{-4}t_1 + 2v^{-6}z^2t_1 + v^{-2}z^2t_2^3$

$\mathcal{S}_3(\overline{5_{65}}) = -v^{-3}z^3t_{-2}^2t_{-1}^2 + 2v^{-10}z^2t_{-2}t_{-1}t_2 - 2v^{-9}z^3t_{-2}t_1 - 2v^{-8}z^4t_{-1}^2t_1 + v^{-8}z^2t_{-1}^2t_1 + 3v^{-6}z^4t_{-1} + 9v^{-6}z^2t_{-1} + 3v^{-6}t_{-1} + 2v^{-8}z^4t_{-1} - 6v^{-8}z^2t_{-1} - 2v^{-8}t_{-1} - v^{-7}zt_2^2$

$\mathcal{S}_3(5_{66}) = -v^{-1}zt_{-2}^2 - v^{-3}zt_{-2}^2 + v^{-4}t_1$

$\mathcal{S}_3(\overline{5_{66}}) = -2v^{-3}zt_{-2}t_1 + 2v^{-2}z^2t_{-1} + v^{-2}t_{-1} + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 + v^{-3}zt_2^2$

$\mathcal{S}_3(5_{67}) = v^{-1}zt_{-2}t_1 + v^2t_{-1} - z^2t_{-1}$

$\mathcal{S}_3(\overline{5_{67}}) = -v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 - v^{-7}zt_{-1}t_2 - 2v^{-9}zt_{-1}t_2 + v^{-6}z^2t_1 + 2v^{-8}z^2t_1 + v^{-8}t_1$

$\mathcal{S}_3(5_{68}) = -v^{-4}z^2t_{-2}^2t_1 + 3v^{-3}z^3t_{-2}t_{-1}^2t_1 - 2v^{-1}zt_{-2}t_{-1} - 2v^{-3}z^3t_{-2}t_{-1} - v^{-2}z^4t_{-1}^4t_1 + z^2t_{-1}^3 - 2v^{-4}z^2t_{-1}t_1t_2 + 2v^{-3}z^3t_1^2 + v^{-3}zt_1^2 + v^{-2}t_2 + v^{-4}z^2t_2$

$\mathcal{S}_3(5_{69}) = -v^{-5}zt_{-1}t_2 + v^{-2}z^2t_1 + v^{-2}t_1 + v^{-4}z^2t_1$

$\mathcal{S}_3(\overline{5_{69}}) = -2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 4v^{-12}z^2t_{-2}t_{-1}t_2 - 4v^{-11}z^3t_{-2}t_1 - v^{-11}zt_{-2}t_1 - 2v^{-11}z^3t_{-1}^3t_2 + 2v^{-10}z^4t_{-1}^2t_1 + 2v^{-10}z^2t_{-1}^2t_1 + 2v^{-8}z^6t_{-1} + 12v^{-8}z^4t_{-1} + 16v^{-8}z^2t_{-1} + 3v^{-8}t_{-1} - 2v^{-10}z^4t_{-1} - 11v^{-10}z^2t_{-1} - 2v^{-10}t_{-1} - v^{-9}zt_2^2$

$\mathcal{S}_3(5_{70}) = 2v^{-4}z^2t_{-2}t_{-1}t_2 - v^{-3}z^3t_{-2}t_1 - v^{-3}z^3t_{-1}^3t_2 - 2v^{-2}z^2t_{-1} + v^{-2}t_{-1} - v^{-1}zt_2^2$

$\mathcal{S}_3(\overline{5_{70}}) = v^3zt_{-2}^2 - v^4z^2t_{-2}t_{-1}^2 - vz^3t_{-1}t_2 + v^2z^4t_1 + 2v^2z^2t_1 + v^2t_1$

$\mathcal{S}_3(5_{72}) = v^5zt_{-2}^2 + v^3z^3t_{-1}t_2 + v^3zt_{-1}t_2 + 2v^4z^2t_1 + v^4t_1$

$\mathcal{S}_3(\overline{5_{72}}) = -2v^{-5}z^3t_{-2}t_{-1}t_1^2 + 2v^{-6}z^2t_{-2}t_{-1}t_2 + v^{-5}z^3t_{-2}t_1 - v^{-5}zt_{-2}t_1 + v^{-4}z^4t_{-1}^3t_1^2 - v^{-5}z^3t_{-1}^3t_2 + v^{-4}z^2t_{-1}^2t_1 - v^{-4}z^4t_{-1} - 2v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1^2t_2 - v^{-3}zt_2^2$

$\mathcal{S}_3(5_{73}) = 2v^{-10}z^2t_{-2}t_{-1}t_2 - 2v^{-9}z^3t_{-2}t_1 - v^{-9}zt_{-2}t_1 - v^{-9}z^3t_{-1}^3t_2 + v^{-8}z^4t_{-1}^2t_1 + v^{-8}z^2t_{-1}^2t_1 + v^{-6}z^6t_{-1} + 6v^{-6}z^4t_{-1} + 9v^{-6}z^2t_{-1} + 2v^{-6}t_{-1} - v^{-8}z^4t_{-1} - 5v^{-8}z^2t_{-1} - v^{-8}t_{-1} - v^{-7}zt_2^2$

$\mathcal{S}_3(\overline{5_{73}}) = v^{-6}z^2t_{-1}t_1^2 - v^{-7}zt_{-1}t_2 + v^{-4}z^4t_1 + 4v^{-4}z^2t_1 + 2v^{-4}t_1 - v^{-6}z^2t_1 - v^{-6}t_1$

$\mathcal{S}_3(\overline{5_{74}}) = -v^{-7}z^3t_{-2}^2 - v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 - 2v^{-9}z^3t_{-1}t_2 - 3v^{-9}zt_{-1}t_2 + 2v^{-8}z^4t_1 + 4v^{-8}z^2t_1 + v^{-8}t_1$

$\mathcal{S}_3(\overline{5_{75}}) = vzt_{-2}t_{-1} + vz^3t_{-1}t_1^3 - 2z^2t_{-1}t_1t_2 - vz^3t_1^2 - vzt_1^2 + 2z^2t_2 + t_2$

$\mathcal{S}_3(\overline{5_{76}}) = -2v^{-1}zt_{-2}t_{-1} + 2v^{-3}z^3t_{-2}t_{-1} + v^{-3}zt_{-2}t_{-1} + z^2t_{-1}^3 - v^{-2}z^4t_{-1}^3 - v^{-2}z^2t_{-1}^3 + v^{-2}t_2 - v^{-4}z^2t_2$
$\mathcal{S}_3(5_{77}) = v^{-3}z^3t_{-2}t_1 - v^{-2}z^4t_{-1} - v^{-2}z^2t_{-1} + v^{-2}t_{-1} + z^2t_1^2t_2 - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{5_{77}}) = v^{-3}z^3t_{-2}^2 - v^{-2}z^4t_{-2}t_{-1}^2 + 2v^{-2}z^2t_{-2}t_{-1}^2 - v^{-1}z^3t_{-1}^4 + v^{-5}z^3t_{-1}t_2 - 2v^{-5}zt_{-1}t_2 - v^{-4}z^4t_1 + v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(5_{79}) = v^5zt_{-2}t_{-1} + v^4z^2t_2 + v^2z^2t_2 + v^2t_2$
$\mathcal{S}_3(\overline{5_{79}}) = 2v^{-6}z^2t_{-2}t_{-1}t_1 + v^{-4}z^2t_{-2} + v^{-4}t_{-2} - v^{-6}z^2t_{-2} - v^{-5}z^3t_{-1}^3t_1 - v^{-3}z^3t_{-1}^2 - v^{-3}zt_{-1}^2 - v^{-5}zt_1t_2$
$\mathcal{S}_3(5_{80}) = -v^{-5}z^3t_{-1}t_2 - 2v^{-5}zt_{-1}t_2 + v^{-4}z^4t_1 + 3v^{-4}z^2t_1 + v^{-4}t_1$
$\mathcal{S}_3(\overline{5_{80}}) = -v^{-3}z^3t_{-2}t_1 + v^{-2}z^4t_{-1} + v^{-2}z^2t_{-1} + v^{-2}t_{-1} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - v^{-1}z^3t_2^2 - v^{-1}zt_2^2$
$\mathcal{S}_3(5_{81}) = v^2z^2t_{-1} + v^2t_{-1} + v^3zt_2^2$
$\mathcal{S}_3(\overline{5_{81}}) = -v^{-7}z^3t_{-2}^2 - 2v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 - 2v^{-9}z^3t_{-1}t_2 - 2v^{-9}zt_{-1}t_2 + 2v^{-8}z^4t_1 + 3v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_{82}) = v^2t_{-1} + vzt_2^2$
$\mathcal{S}_3(\overline{5_{82}}) = -v^{-5}zt_{-2}^2 - v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 - 2v^{-9}zt_{-1}t_2 + 2v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_{83}) = vzt_{-2}^2 + v^{-1}z^3t_{-2}t_1^3 - 2v^{-2}z^2t_{-2}t_1t_2 + v^{-1}z^3t_{-1}t_2 + v^2t_1 + 2z^2t_1$
$\mathcal{S}_3(\overline{5_{83}}) = v^{-1}z^3t_{-2}t_1 - z^4t_{-1} - 2z^2t_{-1} + v^{-2}t_{-1} + v^2z^2t_1^2t_2 - vzt_2^2$
$\mathcal{S}_3(5_{84}) = -v^{-3}zt_{-2}^2 + v^{-2}z^2t_1 + v^{-2}t_1$
$\mathcal{S}_3(\overline{5_{84}}) = v^{-6}z^2t_{-2}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 2v^{-12}z^2t_{-2}t_{-1}t_2 - 2v^{-11}z^3t_{-2}t_1 - 2v^{-11}zt_{-2}t_1 - 2v^{-11}z^3t_{-1}^3t_2 + 4v^{-10}z^4t_{-1}^2t_1 + 3v^{-10}z^2t_{-1}^2t_1 + 2v^{-8}z^6t_{-1} + 11v^{-8}z^4t_{-1} + 14v^{-8}z^2t_{-1} + 3v^{-8}t_{-1} - 4v^{-10}z^4t_{-1} - 8v^{-10}z^2t_{-1} - 2v^{-10}t_{-1}$
$\mathcal{S}_3(\overline{5_{85}}) = v^{-8}z^2t_{-2}^3 - 6v^{-7}z^3t_{-2}^2t_{-1}^2 + 5v^{-6}z^4t_{-2}t_{-1}^4 + 6v^{-14}z^2t_{-2}t_{-1}t_2 - 6v^{-13}z^3t_{-2}t_1 - 2v^{-13}zt_{-2}t_1 - v^{-5}z^5t_{-1}^6 - 4v^{-13}z^3t_{-1}^3t_2 + 6v^{-12}z^4t_{-1}^2t_1 + 3v^{-12}z^2t_{-1}^2t_1 + 4v^{-10}z^6t_{-1} + 21v^{-10}z^4t_{-1} + 23v^{-10}z^2t_{-1} + 3v^{-10}t_{-1} - 6v^{-12}z^4t_{-1} - 16v^{-12}z^2t_{-1} - 2v^{-12}t_{-1} - v^{-11}zt_2^2$

## Knots in $L(6,1)$

$\mathcal{S}_3(3_2) = -v^{-3}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 - v^{-5}zt_{-1}t_3 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{3_2}) = v^{-2}t_{-2} - v^{-1}zt_1t_3$
$\mathcal{S}_3(3_5) = v^{-2}t_{-2} - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{3_5}) = v^{-2}z^2t_{-2}t_{-1}^2 - 2v^{-5}zt_{-1}t_3 + 2v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{3_6}) = -v^{-5}zt_{-2}^2 + 3v^{-4}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - 2v^{-7}zt_{-1}t_3 + 2v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(4_2) = 2v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - 2v^{-9}zt_{-2}t_3 + v^{-8}z^2t_{-1}^2t_3 - v^{-7}zt_{-1}t_2 + v^{-6}z^4t_1 + 5v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{4_2}) = -v^{-3}zt_{-2}t_1 + v^{-2}z^2t_{-1} + v^{-2}t_{-1}$
$\mathcal{S}_3(4_5) = v^{-4}t_{-1} + v^{-2}z^2t_1t_2^2 - 2v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_5}) = -2v^{-5}zt_{-2}t_1 + 2v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1^2t_3$
$\mathcal{S}_3(4_{14}) = -v^{-5}zt_{-2}t_1 + v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1t_2^2 - v^{-3}zt_2t_3$
$\mathcal{S}_3(4_{18}) = v^{-2}t_{-1} - v^{-1}zt_2t_3$
$\mathcal{S}_3(\overline{4_{18}}) = v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - v^{-9}zt_{-2}t_3 + 2v^{-8}z^2t_{-1}^2t_3 - 3v^{-7}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + 2v^{-6}z^4t_1 + 4v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(4_{21}) = -v^{-5}zt_{-2}t_1 + v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1^2t_3 - v^{-3}zt_2t_3$
$\mathcal{S}_3(4_{24}) = t_{-1}$
$\mathcal{S}_3(\overline{4_{24}}) = 3v^{-6}z^2t_{-2}^2t_{-1} - 4v^{-5}z^3t_{-2}t_{-1}^3 - 2v^{-11}zt_{-2}t_3 + v^{-4}z^4t_{-1}^5 + 3v^{-10}z^2t_{-1}^2t_3 - 4v^{-9}z^3t_{-1}t_2 - 2v^{-9}zt_{-1}t_2 + 3v^{-8}z^4t_1 + 6v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_4) = -v^{-5}z^3t_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 - v^{-7}z^3t_{-1}t_3 - 2v^{-7}zt_{-1}t_3 + v^{-6}z^4t_2 + 3v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(\overline{5_4}) = z^2t_{-2} + t_{-2} - v^2z^2t_1^2t_2 - vz^3t_1t_3$

$\mathcal{S}_3(\overline{5_5}) = v^{-6}z^2t_{-2}^2t_1 - 3v^{-5}z^3t_{-2}t_{-1}^2t_1 + 2v^{-5}z^3t_{-2}t_{-1} - 2v^{-5}zt_{-2}t_{-1} + v^{-4}z^4t_{-1}^4t_1 + v^{-4}z^2t_{-1}^3 + 2v^{-6}z^2t_{-1}t_1t_3 - 2v^{-5}z^3t_1t_2 - v^{-5}zt_1t_2 - v^{-6}z^2t_3 + v^{-6}t_3$

$\mathcal{S}_3(5_6) = v^{-6}z^2t_{-2}^3 - 3v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 4v^{-12}z^2t_{-2}t_{-1}t_3 - 4v^{-11}z^3t_{-2}t_2 - 2v^{-11}zt_{-2}t_2 - v^{-11}z^3t_{-1}^3t_3 - v^{-10}z^4t_{-1}^2t_2 + v^{-10}z^2t_{-1}^2t_2 + v^{-9}z^5t_{-1}t_1 - 8v^{-9}z^3t_{-1}t_1 - v^{-9}zt_{-1}t_1 - v^{-7}zt_3^2 - v^{-7}z^5 - 9v^{-7}z^3 - 5v^{-7}z - v^{-7}z^{-1} + v^{-9}z^5 + 9v^{-9}z^3 + 5v^{-9}z + v^{-9}z^{-1}$

$\mathcal{S}_3(5_8) = vzt_{-2}t_{-1} - z^2t_{-1}t_1t_3 - vzt_1t_2 + z^2t_3 + t_3$

$\mathcal{S}_3(\overline{5_8}) = v^{-4}z^2t_{-2}^2t_1 - v^{-3}z^3t_{-2}t_{-1}^2t_1 + v^{-3}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + v^{-4}z^2t_{-1}t_1t_3 - v^{-3}z^3t_1t_2 - v^{-3}zt_1t_2 - v^{-4}z^2t_3 + v^{-4}t_3$

$\mathcal{S}_3(5_9) = v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 + 2v^{-10}z^2t_{-2}t_{-1}t_3 - 2v^{-9}z^3t_{-2}t_2 - 2v^{-9}zt_{-2}t_2 - 2v^{-8}z^4t_{-1}^2t_2 + 2v^{-7}z^5t_{-1}t_1 - 2v^{-7}z^3t_{-1}t_1 - v^{-5}zt_3^2 - 4v^{-5}z^3 - 4v^{-5}z - v^{-5}z^{-1} + 4v^{-7}z^3 + 4v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_9}) = v^{-6}z^2t_{-2}t_1^2 - 2v^{-5}z^3t_{-1}t_1 - 2v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 2v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 2v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{12}) = vzt_{-2}t_{-1} - z^2t_{-1}t_2^2 - vzt_1t_2 + z^2t_3 + t_3$

$\mathcal{S}_3(\overline{5_{12}}) = -v^{-3}z^3t_{-2}t_{-1}^2t_1 + 2v^{-3}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + 2v^{-4}z^2t_{-1}t_1t_3 - 2v^{-3}z^3t_1t_2 - v^{-3}zt_1t_2 - v^{-4}z^2t_3 + v^{-4}t_3$

$\mathcal{S}_3(5_{14}) = v^{-1}zt_{-2}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 - vzt_1t_2 + t_3 + v^{-2}z^2t_3$

$\mathcal{S}_3(\overline{5_{14}}) = v^{-1}z^3t_{-2}t_{-1} - v^{-3}zt_{-2}t_{-1} + v^{-2}z^2t_{-1}t_1t_3 - v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2 - v^{-2}z^2t_3 + v^{-4}t_3$

$\mathcal{S}_3(5_{15}) = -v^{-3}z^3t_{-2}^2t_{-1}^2 + 3v^{-10}z^2t_{-2}t_{-1}t_3 - 3v^{-9}z^3t_{-2}t_2 - v^{-9}zt_{-2}t_2 - 3v^{-8}z^4t_{-1}^2t_2 + 3v^{-7}z^5t_{-1}t_1 - 5v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-5}zt_3^2 - 6v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + 6v^{-7}z^3 + 3v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{15}}) = -v^{-7}zt_{-2}t_2 + v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 + v^{-2}z^2t_1t_2t_3 - 3v^{-3}z - v^{-3}z^{-1} + 3v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{17}) = -v^{-5}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 - v^{-5}zt_{-1}t_3 - v^{-7}zt_{-1}t_3 + v^{-4}z^2t_2 + v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(\overline{5_{17}}) = t_{-2} - z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_1t_3$

$\mathcal{S}_3(5_{19}) = v^{-1}zt_{-2}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 + z^2t_1^3 - 2v^{-1}zt_1t_2 + v^{-2}z^2t_3 + v^{-2}t_3$

$\mathcal{S}_3(\overline{5_{19}}) = v^{-1}z^3t_{-2}t_{-1} + v^{-2}z^2t_{-1}t_1t_3 - v^{-1}z^3t_1t_2 - v^{-1}zt_1t_2 - v^{-2}z^2t_3 + v^{-2}t_3$

$\mathcal{S}_3(5_{20}) = -v^{-7}zt_{-2}t_2 + v^{-5}z^3t_{-1}t_1 + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_3^2 - 2v^{-3}z - v^{-3}z^{-1} + 2v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(\overline{5_{20}}) = v^{-10}z^2t_{-2}t_{-1}t_3 - v^{-9}z^3t_{-2}t_2 - v^{-9}zt_{-2}t_2 - v^{-9}z^3t_{-1}^3t_3 + 2v^{-8}z^4t_{-1}^2t_2 + 2v^{-8}z^2t_{-1}^2t_2 - 2v^{-7}z^5t_{-1}t_1 - 5v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-5}z^5 - 4v^{-5}z^3 - 4v^{-5}z - v^{-5}z^{-1} + v^{-7}z^5 + 4v^{-7}z^3 + 4v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(5_{24}) = t_{-2} + vzt_1t_3 - vzt_2^2$

$\mathcal{S}_3(\overline{5_{24}}) = 3v^{-4}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - 3v^{-7}zt_{-1}t_3 + 3v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(5_{28}) = -v^{-3}z^3t_{-2}^2 - v^{-3}zt_{-2}^2 - v^{-5}zt_{-1}t_3 + 2v^{-4}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(\overline{5_{28}}) = v^{-2}z^2t_{-2} + v^{-2}t_{-2} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - v^{-1}zt_1t_3$

$\mathcal{S}_3(5_{34}) = -v^{-2}z^2t_{-2} + v^{-2}t_{-2} + z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-1}z^3t_2^2$

$\mathcal{S}_3(\overline{5_{34}}) = -v^{-3}zt_{-2}^2 - v^{-2}z^4t_{-2}t_{-1}^2 + 2v^{-2}z^2t_{-2}t_{-1}^2 - v^{-1}z^3t_{-1}^4 + 2v^{-5}z^3t_{-1}t_3 - v^{-5}zt_{-1}t_3 - 2v^{-4}z^4t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_{41}) = -v^{-5}z^3t_{-2}^2 - 2v^{-5}zt_{-2}^2 + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 - v^{-7}z^3t_{-1}t_3 - v^{-7}zt_{-1}t_3 + v^{-6}z^4t_2 + 2v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(\overline{5_{41}}) = z^2t_{-2} + t_{-2} - v^2z^2t_1^2t_2 - vz^3t_1t_3 - vzt_1t_3 + vzt_2^2$

$\mathcal{S}_3(5_{43}) = v^{-2}t_{-2} + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-3}zt_1t_3 - v^{-1}zt_2^2$

$\mathcal{S}_3(\overline{5_{43}}) = -v^{-3}zt_{-1}t_3 - v^{-5}zt_{-1}t_3 + v^{-2}z^2t_2 + v^{-4}z^2t_2 + v^{-4}t_2$

$\mathcal{S}_3(5_{44}) = vzt_{-2}t_{-1} - z^2t_{-1}t_1t_3 + v^2t_3 + z^2t_3$

$\mathcal{S}_3(\overline{5_{44}}) = v^{-4}z^2t_{-2}^2t_1 - v^{-3}z^3t_{-2}t_{-1}^2t_1 + v^{-3}z^3t_{-2}t_{-1} - 2v^{-5}zt_{-2}t_{-1} + v^{-4}z^2t_{-1}^3 + v^{-4}z^2t_{-1}t_1t_3 - v^{-3}z^3t_1t_2 - v^{-3}zt_1t_2 - v^{-4}z^2t_3 + v^{-6}t_3$

$\mathcal{S}_3(5_{47}) = v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 + v^{-10}z^2t_{-2}t_{-1}t_3 - v^{-9}z^3t_{-2}t_2 - 2v^{-9}zt_{-2}t_2 - v^{-8}z^4t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 + v^{-7}z^5t_{-1}t_1 - 2v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - 3v^{-5}z^3 - 5v^{-5}z - v^{-5}z^{-1} + 3v^{-7}z^3 + 5v^{-7}z + v^{-7}z^{-1}$

$\mathcal{S}_3(\overline{5_{47}}) = -v^{-5}zt_{-1}t_1 + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_3^2 - v^{-3}z - v^{-3}z^{-1} + v^{-5}z + v^{-5}z^{-1}$

$\mathcal{S}_3(5_{49}) = -v^{-3}zt_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 - v^{-7}zt_{-1}t_3 + v^{-6}z^2t_2 + v^{-6}t_2$

$\mathcal{S}_3(\overline{5_{49}}) = t_{-2} - z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_2^2$

$\mathcal{S}_3(5_{53}) = z^2t_{-2} + t_{-2} - v^2z^2t_1^2t_2 + vzt_1t_3 - vz^3t_2^2 - vzt_2^2$
$\mathcal{S}_3(\overline{5_{53}}) = v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 - 2v^{-7}z^3t_{-1}t_3 - 3v^{-7}zt_{-1}t_3 + 2v^{-6}z^4t_2 + 4v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{56}) = -v^{-3}z^3t_{-2}^2 - 2v^{-3}zt_{-2}^2 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{5_{56}}) = v^{-2}z^2t_{-2} + v^{-2}t_{-2} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - 2v^{-1}zt_1t_3 + v^{-1}zt_2^2$
$\mathcal{S}_3(5_{59}) = t_{-2} - z^2t_1^2t_2 + v^{-1}zt_1t_3 - vzt_2^2$
$\mathcal{S}_3(\overline{5_{59}}) = v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 - v^{-5}zt_{-1}t_3 - 2v^{-7}zt_{-1}t_3 + v^{-4}z^2t_2 + 2v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{62}) = v^{-1}zt_{-2}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - 2v^{-2}z^2t_{-1}t_1t_3 - vzt_1t_2 + t_3 + v^{-2}z^2t_3$
$\mathcal{S}_3(\overline{5_{62}}) = v^{-2}z^2t_{-2}^2t_1 - v^{-3}zt_{-2}t_{-1} - v^{-1}zt_1t_2 - v^{-2}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(5_{65}) = -2v^{-7}zt_{-2}t_2 + 2v^{-5}z^3t_{-1}t_1 + v^{-2}z^2t_2^3 - 4v^{-3}z - v^{-3}z^{-1} + 4v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{5_{65}}) = -v^{-3}z^3t_{-2}^2t_{-1}^2 + 2v^{-10}z^2t_{-2}t_{-1}t_3 - 2v^{-9}z^3t_{-2}t_2 - 2v^{-8}z^4t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 + 2v^{-7}z^5t_{-1}t_1 - 6v^{-7}z^3t_{-1}t_1 - 2v^{-7}zt_{-1}t_1 - v^{-5}zt_3^2 - 5v^{-5}z^3 - 2v^{-5}z - v^{-5}z^{-1} + 5v^{-7}z^3 + 2v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(5_{66}) = -v^{-1}zt_{-2}^2 - v^{-3}zt_{-2}^2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{5_{66}}) = v^{-2}t_{-2} + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - 2v^{-1}zt_1t_3 + v^{-3}zt_2^2$
$\mathcal{S}_3(5_{67}) = v^2t_{-2} + vzt_1t_3$
$\mathcal{S}_3(\overline{5_{67}}) = -v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 - v^{-7}zt_{-1}t_3 - 2v^{-9}zt_{-1}t_3 + v^{-6}z^2t_2 + 2v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(5_{68}) = -v^{-4}z^2t_{-2}^2t_1 + 3v^{-3}z^3t_{-2}t_{-1}^2t_1 - 2v^{-1}zt_{-2}t_{-1} - 2v^{-3}z^3t_{-2}t_{-1} - v^{-2}z^4t_{-1}^4t_1 + z^2t_{-1}^3 - 2v^{-4}z^2t_{-1}t_1t_3 + 2v^{-3}z^3t_1t_2 + v^{-3}zt_1t_2 + v^{-2}t_3 + v^{-4}z^2t_3$
$\mathcal{S}_3(\overline{5_{68}}) = -vzt_{-2}t_{-1} - z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(5_{69}) = -v^{-5}zt_{-2}t_2 + v^{-3}z^3t_{-1}t_1 - 2v^{-1}z - v^{-1}z^{-1} + 2v^{-3}z + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{5_{69}}) = -2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 4v^{-12}z^2t_{-2}t_{-1}t_3 - 4v^{-11}z^3t_{-2}t_2 - v^{-11}zt_{-2}t_2 - 2v^{-11}z^3t_{-1}^3t_3 + 2v^{-10}z^4t_{-1}^2t_2 + 2v^{-10}z^2t_{-1}^2t_2 - 2v^{-9}z^5t_{-1}t_1 - 11v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - v^{-7}zt_3^2 - 2v^{-7}z^5 - 10v^{-7}z^3 - 4v^{-7}z - v^{-7}z^{-1} + 2v^{-9}z^5 + 10v^{-9}z^3 + 4v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(5_{73}) = 2v^{-10}z^2t_{-2}t_{-1}t_3 - 2v^{-9}z^3t_{-2}t_2 - v^{-9}zt_{-2}t_2 - v^{-9}z^3t_{-1}^3t_3 + v^{-8}z^4t_{-1}^2t_2 + v^{-8}z^2t_{-1}^2t_2 - v^{-7}z^5t_{-1}t_1 - 5v^{-7}z^3t_{-1}t_1 - v^{-7}zt_{-1}t_1 - v^{-5}zt_3^2 - v^{-5}z^5 - 5v^{-5}z^3 - 3v^{-5}z - v^{-5}z^{-1} + v^{-7}z^5 + 5v^{-7}z^3 + 3v^{-7}z + v^{-7}z^{-1}$
$\mathcal{S}_3(\overline{5_{73}}) = v^{-6}z^2t_{-2}t_1^2 - v^{-7}zt_{-2}t_2 - v^{-5}z^3t_{-1}t_1 - v^{-5}zt_{-1}t_1 - v^{-3}z^3 - 3v^{-3}z - v^{-3}z^{-1} + v^{-5}z^3 + 3v^{-5}z + v^{-5}z^{-1}$
$\mathcal{S}_3(\overline{5_{74}}) = -v^{-7}z^3t_{-2}^2 - v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 - 2v^{-9}z^3t_{-1}t_3 - 3v^{-9}zt_{-1}t_3 + 2v^{-8}z^4t_2 + 4v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(5_{77}) = -v^{-2}z^2t_{-2} + v^{-2}t_{-2} + z^2t_1^2t_2 + v^{-1}z^3t_1t_3 - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{5_{77}}) = v^{-3}z^3t_{-2}^2 - v^{-2}z^4t_{-2}t_{-1}^2 + 2v^{-2}z^2t_{-2}t_{-1}^2 - v^{-1}z^3t_{-1}^4 + v^{-5}z^3t_{-1}t_3 - 2v^{-5}zt_{-1}t_3 - v^{-4}z^4t_2 + v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(5_{80}) = -v^{-5}z^3t_{-1}t_3 - 2v^{-5}zt_{-1}t_3 + v^{-4}z^4t_2 + 3v^{-4}z^2t_2 + v^{-4}t_2$
$\mathcal{S}_3(\overline{5_{80}}) = v^{-2}z^2t_{-2} + v^{-2}t_{-2} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - v^{-1}z^3t_1t_3 - v^{-1}z^3t_2^2 - v^{-1}zt_2^2$
$\mathcal{S}_3(5_{81}) = v^2z^2t_{-2} + v^2t_{-2} + v^3zt_2^2$
$\mathcal{S}_3(\overline{5_{81}}) = -v^{-7}z^3t_{-2}^2 - 2v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 - 2v^{-9}z^3t_{-1}t_3 - 2v^{-9}zt_{-1}t_3 + 2v^{-8}z^4t_2 + 3v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(5_{82}) = v^2t_{-2} + vzt_2^2$
$\mathcal{S}_3(\overline{5_{82}}) = -v^{-5}zt_{-2}^2 - v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 - 2v^{-9}zt_{-1}t_3 + 2v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(5_{84}) = -v^{-1}zt_3^2 - v^{-1}z^{-1} + v^{-3}z^{-1}$
$\mathcal{S}_3(\overline{5_{84}}) = v^{-6}z^2t_{-2}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + 2v^{-12}z^2t_{-2}t_{-1}t_3 - 2v^{-11}z^3t_{-2}t_2 - 2v^{-11}zt_{-2}t_2 - 2v^{-11}z^3t_{-1}^3t_3 + 4v^{-10}z^4t_{-1}^2t_2 + 3v^{-10}z^2t_{-1}^2t_2 - 4v^{-9}z^5t_{-1}t_1 - 8v^{-9}z^3t_{-1}t_1 - 2v^{-9}zt_{-1}t_1 - 2v^{-7}z^5 - 7v^{-7}z^3 - 6v^{-7}z - v^{-7}z^{-1} + 2v^{-9}z^5 + 7v^{-9}z^3 + 6v^{-9}z + v^{-9}z^{-1}$
$\mathcal{S}_3(\overline{5_{85}}) = v^{-8}z^2t_{-2}^3 - 6v^{-7}z^3t_{-2}^2t_{-1}^2 + 5v^{-6}z^4t_{-2}t_{-1}^4 + 6v^{-14}z^2t_{-2}t_{-1}t_3 - 6v^{-13}z^3t_{-2}t_2 - 2v^{-13}zt_{-2}t_2 - v^{-5}z^5t_{-1}^6 - 4v^{-13}z^3t_{-1}^3t_3 + 6v^{-12}z^4t_{-1}^2t_2 + 3v^{-12}z^2t_{-1}^2t_2 - 6v^{-11}z^5t_{-1}t_1 - 16v^{-11}z^3t_{-1}t_1 - 2v^{-11}zt_{-1}t_1 - v^{-9}zt_3^2 - 4v^{-9}z^5 -$

$15v^{-9}z^3 - 6v^{-9}z - v^{-9}z^{-1} + 4v^{-11}z^5 + 15v^{-11}z^3 + 6v^{-11}z + v^{-11}z^{-1}$

## Knots in $L(7,1)$

$\mathcal{S}_3(3_2) = -v^{-3}zt_{-3}t_{-1} - v^{-3}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}t_3$
$\mathcal{S}_3(\overline{3_2}) = v^{-2}t_{-3} - v^{-1}zt_1t_3$
$\mathcal{S}_3(3_5) = v^{-2}t_{-3} - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{3_5}) = -2v^{-3}zt_{-3}t_{-1} + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}t_3$
$\mathcal{S}_3(\overline{3_6}) = -2v^{-5}zt_{-3}t_{-1} - v^{-5}zt_{-2}^2 + 3v^{-4}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 + v^{-6}t_3$
$\mathcal{S}_3(4_2) = -2v^{-5}zt_{-3}t_{-2} + v^{-4}z^2t_{-3}t_{-1}^2 + 2v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - v^{-7}zt_{-1}t_3 + v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(\overline{4_2}) = -v^{-3}zt_{-3}t_1 + v^{-2}z^2t_{-2} + v^{-2}t_{-2}$
$\mathcal{S}_3(4_5) = v^{-4}t_{-2} + v^{-2}z^2t_1t_2^2 - 2v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_5}) = -2v^{-5}zt_{-3}t_1 + 2v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1^2t_3$
$\mathcal{S}_3(4_{14}) = -v^{-5}zt_{-3}t_1 + v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1t_2^2 - v^{-3}zt_2t_3$
$\mathcal{S}_3(4_{18}) = v^{-2}t_{-2} - v^{-1}zt_2t_3$
$\mathcal{S}_3(\overline{4_{18}}) = -v^{-5}zt_{-3}t_{-2} + 2v^{-4}z^2t_{-3}t_{-1}^2 + v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - 2v^{-7}zt_{-1}t_3 + 2v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(4_{21}) = -v^{-5}zt_{-3}t_1 + v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1^2t_3 - v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_{24}}) = -2v^{-7}zt_{-3}t_{-2} + 3v^{-6}z^2t_{-3}t_{-1}^2 + 3v^{-6}z^2t_{-2}^2t_{-1} - 4v^{-5}z^3t_{-2}t_{-1}^3 + v^{-4}z^4t_{-1}^5 - 2v^{-9}zt_{-1}t_3 + 2v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(5_4) = -v^{-5}z^3t_{-3}t_{-1} - 2v^{-5}zt_{-3}t_{-1} - v^{-5}z^3t_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(\overline{5_4}) = z^2t_{-3} + t_{-3} - v^2z^2t_1^2t_2 - vz^3t_1t_3$
$\mathcal{S}_3(5_5) = v^3zt_{-3}t_{-1} + v^2z^2t_3 + v^2t_3$
$\mathcal{S}_3(\overline{5_5}) = 2v^{-6}z^2t_{-3}t_{-1}t_1 - v^{-6}z^2t_{-3} + v^{-6}t_{-3} + v^{-6}z^2t_{-2}^2t_1 - 3v^{-5}z^3t_{-2}t_{-1}^2t_1 - 2v^{-5}zt_{-2}t_{-1} + v^{-4}z^4t_{-1}^4t_1 + v^{-4}z^2t_{-1}^3 - v^{-5}zt_1t_3$
$\mathcal{S}_3(5_6) = -v^{-7}zt_{-3}^2 + 4v^{-6}z^2t_{-3}t_{-2}t_{-1} - v^{-5}z^3t_{-3}t_{-1}^3 + v^{-6}z^2t_{-2}^3 - 3v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 - 2v^{-11}zt_{-2}t_3 + v^{-10}z^2t_{-1}^2t_3 - v^{-9}zt_{-1}t_2 + v^{-8}z^4t_1 + 5v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_8) = vzt_{-3}t_{-1} - z^2t_{-1}t_1t_3 - vzt_1t_2 + z^2t_3 + t_3$
$\mathcal{S}_3(\overline{5_8}) = v^{-4}z^2t_{-3}t_{-1}t_1 - v^{-4}z^2t_{-3} + v^{-4}t_{-3} + v^{-4}z^2t_{-2}^2t_1 - v^{-3}z^3t_{-2}t_{-1}^2t_1 - v^{-3}zt_{-2}t_{-1} - v^{-3}zt_1t_3$
$\mathcal{S}_3(5_9) = -v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} + v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 - 2v^{-9}zt_{-2}t_3 + 2v^{-7}z^3t_{-1}t_2 + 4v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_9}) = v^{-6}z^2t_{-3}t_1^2 - 2v^{-5}z^3t_{-2}t_1 - 2v^{-5}zt_{-2}t_1 + v^{-4}z^4t_{-1} + 2v^{-4}z^2t_{-1} + v^{-4}t_{-1}$
$\mathcal{S}_3(5_{12}) = vzt_{-3}t_{-1} - z^2t_{-1}t_2^2 - vzt_1t_2 + z^2t_3 + t_3$
$\mathcal{S}_3(\overline{5_{12}}) = 2v^{-4}z^2t_{-3}t_{-1}t_1 - v^{-4}z^2t_{-3} + v^{-4}t_{-3} - v^{-3}z^3t_{-2}t_{-1}^2t_1 - v^{-3}zt_{-2}t_{-1} - v^{-3}zt_1t_3$
$\mathcal{S}_3(5_{14}) = v^{-1}zt_{-3}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 - vzt_1t_2 + t_3 + v^{-2}z^2t_3$
$\mathcal{S}_3(\overline{5_{14}}) = v^{-2}z^2t_{-3}t_{-1}t_1 - v^{-2}z^2t_{-3} + v^{-4}t_{-3} - v^{-3}zt_{-2}t_{-1} - v^{-1}zt_1t_3$
$\mathcal{S}_3(5_{15}) = -v^{-5}zt_{-3}^2 + 3v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-2}^2t_{-1}^2 - v^{-9}zt_{-2}t_3 + v^{-7}z^3t_{-1}t_2 - v^{-7}zt_{-1}t_2 + 3v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_{15}}) = -v^{-7}zt_{-3}t_2 + v^{-5}z^3t_{-2}t_1 - v^{-5}zt_{-2}t_1 + 3v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1t_2t_3$
$\mathcal{S}_3(5_{17}) = -v^{-3}zt_{-3}t_{-1} - v^{-5}zt_{-3}t_{-1} - v^{-5}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{17}}) = t_{-3} - z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_1t_3$
$\mathcal{S}_3(5_{19}) = v^{-1}zt_{-3}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - v^{-2}z^2t_{-1}t_1t_3 - v^{-2}z^2t_{-1}t_2^2 + z^2t_1^3 - 2v^{-1}zt_1t_2 + v^{-2}z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(\overline{5_{19}}) = v^{-2}z^2t_{-3}t_{-1}t_1 - v^{-2}z^2t_{-3} + v^{-2}t_{-3} - v^{-1}zt_1t_3$
$\mathcal{S}_3(5_{20}) = -v^{-7}zt_{-3}t_2 + v^{-5}z^3t_{-2}t_1 + 2v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_3^2$
$\mathcal{S}_3(\overline{5_{20}}) = v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-3}t_{-1}^3 - v^{-9}zt_{-2}t_3 + 2v^{-8}z^2t_{-1}^2t_3 - 3v^{-7}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + 2v^{-6}z^4t_1 + 4v^{-6}z^2t_1 + v^{-6}t_1$

$\mathcal{S}_3(5_{24}) = t_{-3} + vzt_1t_3 - vzt_2^2$
$\mathcal{S}_3(\overline{5_{24}}) = -3v^{-5}zt_{-3}t_{-1} + 3v^{-4}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 + v^{-6}t_3$
$\mathcal{S}_3(5_{28}) = -v^{-3}zt_{-3}t_{-1} - v^{-3}z^3t_{-2}^2 - v^{-3}zt_{-2}^2 + v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(\overline{5_{28}}) = v^{-2}z^2t_{-3} + v^{-2}t_{-3} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - v^{-1}zt_1t_3$
$\mathcal{S}_3(5_{34}) = -v^{-2}z^2t_{-3} + v^{-2}t_{-3} + z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-1}z^3t_2^2$
$\mathcal{S}_3(\overline{5_{34}}) = 2v^{-3}z^3t_{-3}t_{-1} - v^{-3}zt_{-3}t_{-1} - v^{-3}zt_{-2}^2 - v^{-2}z^4t_{-2}t_{-1}^2 + 2v^{-2}z^2t_{-2}t_{-1}^2 - v^{-1}z^3t_{-1}^4 - v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(5_{41}) = -v^{-5}z^3t_{-3}t_{-1} - v^{-5}zt_{-3}t_{-1} - v^{-5}z^3t_{-2}^2 - 2v^{-5}zt_{-2}^2 + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{41}}) = z^2t_{-3} + t_{-3} - v^2z^2t_1^2t_2 - vz^3t_1t_3 - vzt_1t_3 + vzt_2^2$
$\mathcal{S}_3(5_{43}) = v^{-2}t_{-3} + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - v^{-1}zt_1t_3 + v^{-3}zt_1t_3 - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{5_{43}}) = -v^{-1}zt_{-3}t_{-1} - v^{-3}zt_{-3}t_{-1} + v^{-4}t_3$
$\mathcal{S}_3(5_{44}) = vzt_{-3}t_{-1} - z^2t_{-1}t_1t_3 + v^2t_3 + z^2t_3$
$\mathcal{S}_3(\overline{5_{44}}) = v^{-4}z^2t_{-3}t_{-1}t_1 - v^{-4}z^2t_{-3} + v^{-6}t_{-3} + v^{-4}z^2t_{-2}^2t_1 - v^{-3}z^3t_{-2}t_{-1}^2t_1 - 2v^{-5}zt_{-2}t_{-1} + v^{-4}z^2t_{-1}^3 - v^{-3}zt_1t_3$
$\mathcal{S}_3(5_{47}) = v^{-4}z^2t_{-3}t_{-2}t_{-1} + v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 - 2v^{-9}zt_{-2}t_3 + v^{-8}z^2t_{-1}^2t_3 - v^{-7}zt_{-1}t_2 + v^{-6}z^4t_1 + 5v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_{47}}) = -v^{-5}zt_{-2}t_1 + v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_3^2$
$\mathcal{S}_3(5_{49}) = -v^{-5}zt_{-3}t_{-1} - v^{-3}zt_{-2}^2 - v^{-5}zt_{-2}^2 + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{49}}) = t_{-3} - z^2t_1^2t_2 - vzt_1t_3 + v^{-1}zt_2^2$
$\mathcal{S}_3(5_{53}) = z^2t_{-3} + t_{-3} - v^2z^2t_1^2t_2 + vzt_1t_3 - vz^3t_2^2 - vzt_2^2$
$\mathcal{S}_3(\overline{5_{53}}) = -2v^{-5}z^3t_{-3}t_{-1} - 3v^{-5}zt_{-3}t_{-1} + v^{-4}z^4t_{-2}t_{-1}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(5_{56}) = -v^{-3}z^3t_{-2}^2 - 2v^{-3}zt_{-2}^2 + v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(\overline{5_{56}}) = v^{-2}z^2t_{-3} + v^{-2}t_{-3} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - 2v^{-1}z^3t_1t_3 - 2v^{-1}zt_1t_3 + v^{-1}zt_2^2$
$\mathcal{S}_3(5_{59}) = t_{-3} - z^2t_1^2t_2 + v^{-1}zt_1t_3 - vzt_2^2$
$\mathcal{S}_3(\overline{5_{59}}) = -v^{-3}zt_{-3}t_{-1} - 2v^{-5}zt_{-3}t_{-1} + v^{-2}z^2t_{-2}t_{-1}^2 + v^{-4}z^2t_{-2}t_{-1}^2 + v^{-6}t_3$
$\mathcal{S}_3(5_{62}) = v^{-1}zt_{-3}t_{-1} + v^{-1}z^3t_{-1}t_1^2t_2 - 2v^{-2}z^2t_{-1}t_1t_3 - vzt_1t_2 + t_3 + v^{-2}z^2t_3$
$\mathcal{S}_3(\overline{5_{62}}) = -v^{-2}z^2t_{-3} + v^{-4}t_{-3} + v^{-2}z^2t_{-2}^2t_1 - v^{-3}zt_{-2}t_{-1} - v^{-1}zt_1t_3$
$\mathcal{S}_3(5_{65}) = -2v^{-7}zt_{-3}t_2 + 2v^{-5}z^3t_{-2}t_1 + 4v^{-4}z^2t_{-1} + v^{-4}t_{-1} + v^{-2}z^2t_2^3$
$\mathcal{S}_3(\overline{5_{65}}) = -v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-2}^2t_{-1}^2 + v^{-8}z^2t_{-1}^2t_3 - 2v^{-7}z^3t_{-1}t_2 - 2v^{-7}zt_{-1}t_2 + v^{-6}z^4t_1 + 2v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(5_{66}) = -v^{-1}zt_{-2}^2 - v^{-3}zt_{-2}^2 + v^{-4}t_3$
$\mathcal{S}_3(\overline{5_{66}}) = v^{-2}t_{-3} + v^{-1}z^3t_1^4 + z^2t_1^2t_2 - 2v^{-2}z^2t_1^2t_2 - 2v^{-1}zt_1t_3 + v^{-3}zt_2^2$
$\mathcal{S}_3(5_{67}) = v^2t_{-3} + vzt_1t_3$
$\mathcal{S}_3(\overline{5_{67}}) = -v^{-5}zt_{-3}t_{-1} - 2v^{-7}zt_{-3}t_{-1} - v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + v^{-8}t_3$
$\mathcal{S}_3(5_{68}) = -2v^{-4}z^2t_{-3}t_{-1}t_1 + v^{-2}t_{-3} + v^{-4}z^2t_{-3} - v^{-4}z^2t_{-2}^2t_1 + 3v^{-3}z^3t_{-2}t_{-1}^2t_1 - 2v^{-1}zt_{-2}t_{-1} - v^{-2}z^4t_{-1}^4t_1 + z^2t_{-1}^3 + v^{-3}zt_1t_3$
$\mathcal{S}_3(\overline{5_{68}}) = -vzt_{-3}t_{-1} - z^2t_3 + v^{-2}t_3$
$\mathcal{S}_3(5_{69}) = -v^{-5}zt_{-3}t_2 + v^{-3}z^3t_{-2}t_1 + 2v^{-2}z^2t_{-1} + v^{-2}t_{-1}$
$\mathcal{S}_3(\overline{5_{69}}) = -v^{-7}zt_{-3}^2 + 4v^{-6}z^2t_{-3}t_{-2}t_{-1} - 2v^{-5}z^3t_{-3}t_{-1}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 - v^{-11}zt_{-2}t_3 + 2v^{-10}z^2t_{-1}^2t_3 - 3v^{-9}z^3t_{-1}t_2 - 2v^{-9}zt_{-1}t_2 + 2v^{-8}z^4t_1 + 4v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_{73}) = -v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-3}t_{-1}^3 - v^{-9}zt_{-2}t_3 + v^{-8}z^2t_{-1}^2t_3 - v^{-7}z^3t_{-1}t_2 - v^{-7}zt_{-1}t_2 + v^{-6}z^4t_1 + 3v^{-6}z^2t_1 + v^{-6}t_1$
$\mathcal{S}_3(\overline{5_{73}}) = v^{-6}z^2t_{-3}t_1^2 - v^{-7}zt_{-3}t_2 - v^{-5}z^3t_{-2}t_1 - v^{-5}zt_{-2}t_1 + v^{-4}z^4t_{-1} + 3v^{-4}z^2t_{-1} + v^{-4}t_{-1}$
$\mathcal{S}_3(\overline{5_{74}}) = -2v^{-7}z^3t_{-3}t_{-1} - 3v^{-7}zt_{-3}t_{-1} - v^{-7}z^3t_{-2}^2 - v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + v^{-8}z^2t_3 + v^{-8}t_3$
$\mathcal{S}_3(5_{77}) = -v^{-2}z^2t_{-3} + v^{-2}t_{-3} + z^2t_1^2t_2 + v^{-1}z^3t_1t_3 - v^{-1}zt_2^2$
$\mathcal{S}_3(\overline{5_{77}}) = v^{-3}z^3t_{-3}t_{-1} - 2v^{-3}zt_{-3}t_{-1} + v^{-3}z^3t_{-2}^2 - v^{-2}z^4t_{-2}t_{-1}^2 + 2v^{-2}z^2t_{-2}t_{-1}^2 - v^{-1}z^3t_{-1}^4 - v^{-4}z^2t_3 + v^{-4}t_3$

$\mathcal{S}_3(5_{80}) = -v^{-3}z^3t_{-3}t_{-1} - 2v^{-3}zt_{-3}t_{-1} + v^{-4}z^2t_3 + v^{-4}t_3$
$\mathcal{S}_3(\overline{5_{80}}) = v^{-2}z^2t_{-3} + v^{-2}t_{-3} + vz^3t_1^4 + z^4t_1^2t_2 - z^2t_1^2t_2 - v^{-1}z^3t_1t_3 - v^{-1}z^3t_2^2 - v^{-1}zt_2^2$
$\mathcal{S}_3(5_{81}) = v^2z^2t_{-3} + v^2t_{-3} + v^3zt_2^2$
$\mathcal{S}_3(\overline{5_{81}}) = -2v^{-7}z^3t_{-3}t_{-1} - 2v^{-7}zt_{-3}t_{-1} - v^{-7}z^3t_{-2}^2 - 2v^{-7}zt_{-2}^2 + 3v^{-6}z^4t_{-2}t_{-1}^2 + 5v^{-6}z^2t_{-2}t_{-1}^2 - v^{-5}z^5t_{-1}^4 - 2v^{-5}z^3t_{-1}^4 + v^{-8}z^2t_3 + v^{-8}t_3$
$\mathcal{S}_3(5_{82}) = v^2t_{-3} + vzt_2^2$
$\mathcal{S}_3(\overline{5_{82}}) = -2v^{-7}zt_{-3}t_{-1} - v^{-5}zt_{-2}^2 - v^{-7}zt_{-2}^2 + 2v^{-4}z^2t_{-2}t_{-1}^2 + 3v^{-6}z^2t_{-2}t_{-1}^2 - v^{-3}z^3t_{-1}^4 - v^{-5}z^3t_{-1}^4 + v^{-8}t_3$
$\mathcal{S}_3(5_{84}) = v^{-2}t_{-1} - v^{-1}zt_3^2$
$\mathcal{S}_3(\overline{5_{84}}) = 2v^{-6}z^2t_{-3}t_{-2}t_{-1} - 2v^{-5}z^3t_{-3}t_{-1}^3 + v^{-6}z^2t_{-2}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 - 2v^{-11}zt_{-2}t_3 + 3v^{-10}z^2t_{-1}^2t_3 - 4v^{-9}z^3t_{-1}t_2 - 2v^{-9}zt_{-1}t_2 + 3v^{-8}z^4t_1 + 6v^{-8}z^2t_1 + v^{-8}t_1$
$\mathcal{S}_3(5_{85}) = t_{-1}$
$\mathcal{S}_3(\overline{5_{85}}) = -v^{-9}zt_{-3}^2 + 6v^{-8}z^2t_{-3}t_{-2}t_{-1} - 4v^{-7}z^3t_{-3}t_{-1}^3 + v^{-8}z^2t_{-2}^3 - 6v^{-7}z^3t_{-2}^2t_{-1}^2 + 5v^{-6}z^4t_{-2}t_{-1}^4 - 2v^{-13}zt_{-2}t_3 - v^{-5}z^5t_{-1}^6 + 3v^{-12}z^2t_{-1}^2t_3 - 4v^{-11}z^3t_{-1}t_2 - 2v^{-11}zt_{-1}t_2 + 3v^{-10}z^4t_1 + 6v^{-10}z^2t_1 + v^{-10}t_1$

## Knots in $L(8,1)$

$\mathcal{S}_3(4_2) = -2v^{-5}zt_{-3}t_{-2} + v^{-4}z^2t_{-3}t_{-1}^2 + 2v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - v^{-7}zt_{-1}t_4 + v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(\overline{4_2}) = v^{-2}t_{-3} - v^{-1}zt_1t_4$
$\mathcal{S}_3(4_5) = v^{-4}t_{-3} + v^{-2}z^2t_1t_2^2 - 2v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_5}) = v^{-4}t_{-3} + v^{-2}z^2t_1^2t_3 - 2v^{-3}zt_1t_4$
$\mathcal{S}_3(4_{14}) = v^{-4}t_{-3} + v^{-2}z^2t_1t_2^2 - v^{-3}zt_1t_4 - v^{-3}zt_2t_3$
$\mathcal{S}_3(4_{18}) = v^{-2}t_{-3} - v^{-1}zt_2t_3$
$\mathcal{S}_3(\overline{4_{18}}) = -v^{-5}zt_{-3}t_{-2} + 2v^{-4}z^2t_{-3}t_{-1}^2 + v^{-4}z^2t_{-2}^2t_{-1} - v^{-3}z^3t_{-2}t_{-1}^3 - 2v^{-7}zt_{-1}t_4 + 2v^{-6}z^2t_3 + v^{-6}t_3$
$\mathcal{S}_3(4_{21}) = v^{-4}t_{-3} + v^{-2}z^2t_1^2t_3 - v^{-3}zt_1t_4 - v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_{24}}) = -2v^{-7}zt_{-3}t_{-2} + 3v^{-6}z^2t_{-3}t_{-1}^2 + 3v^{-6}z^2t_{-2}^2t_{-1} - 4v^{-5}z^3t_{-2}t_{-1}^3 + v^{-4}z^4t_{-1}^5 - 2v^{-9}zt_{-1}t_4 + 2v^{-8}z^2t_3 + v^{-8}t_3$
$\mathcal{S}_3(5_6) = -v^{-7}zt_{-3}^2 + 4v^{-6}z^2t_{-3}t_{-2}t_{-1} - v^{-5}z^3t_{-3}t_{-1}^3 + v^{-6}z^2t_{-2}^3 - 3v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 - 2v^{-11}zt_{-2}t_4 + v^{-10}z^2t_{-1}^2t_4 - v^{-9}zt_{-1}t_3 + v^{-8}z^4t_2 + 5v^{-8}z^2t_2 + v^{-8}t_2$
$\mathcal{S}_3(\overline{5_6}) = -v^{-3}zt_{-3}t_1 + v^{-2}z^2t_{-2} + v^{-2}t_{-2}$
$\mathcal{S}_3(5_9) = -v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} + v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 - 2v^{-9}zt_{-2}t_4 + 2v^{-7}z^3t_{-1}t_3 + 4v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(\overline{5_9}) = -2v^{-5}zt_{-3}t_1 + 2v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1^2t_4$
$\mathcal{S}_3(5_{15}) = -v^{-5}zt_{-3}^2 + 3v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-2}^2t_{-1}^2 - v^{-9}zt_{-2}t_4 + v^{-7}z^3t_{-1}t_3 - v^{-7}zt_{-1}t_3 + 3v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(\overline{5_{15}}) = -v^{-5}zt_{-3}t_1 + v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{20}) = v^{-4}t_{-2} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_2t_4 - v^{-3}zt_3^2$
$\mathcal{S}_3(\overline{5_{20}}) = v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-3}t_{-1}^3 - v^{-9}zt_{-2}t_4 + 2v^{-8}z^2t_{-1}^2t_4 - 3v^{-7}z^3t_{-1}t_3 - 2v^{-7}zt_{-1}t_3 + 2v^{-6}z^4t_2 + 4v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{47}) = v^{-4}z^2t_{-3}t_{-2}t_{-1} + v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 - 2v^{-9}zt_{-2}t_4 + v^{-8}z^2t_{-1}^2t_4 - v^{-7}zt_{-1}t_3 + v^{-6}z^4t_2 + 5v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(\overline{5_{47}}) = -v^{-5}zt_{-3}t_1 + v^{-4}z^2t_{-2} + v^{-4}t_{-2} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_3^2$
$\mathcal{S}_3(5_{65}) = v^{-4}t_{-2} + v^{-2}z^2t_2^3 - 2v^{-3}zt_2t_4$
$\mathcal{S}_3(\overline{5_{65}}) = -v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-2}^2t_{-1}^2 + v^{-8}z^2t_{-1}^2t_4 - 2v^{-7}z^3t_{-1}t_3 - 2v^{-7}zt_{-1}t_3 + v^{-6}z^4t_2 + 2v^{-6}z^2t_2 + v^{-6}t_2$
$\mathcal{S}_3(5_{69}) = v^{-2}t_{-2} - v^{-1}zt_2t_4$

$\mathcal{S}_3(\overline{5_{69}}) = -v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-v^{-11}zt_{-2}t_4+2v^{-10}z^2t_{-1}^2t_4-3v^{-9}z^3t_{-1}t_3-2v^{-9}zt_{-1}t_3+2v^{-8}z^4t_2+4v^{-8}z^2t_2+v^{-8}t_2$
$\mathcal{S}_3(5_{73}) = -v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3-v^{-9}zt_{-2}t_4+v^{-8}z^2t_{-1}^2t_4-v^{-7}z^3t_{-1}t_3-v^{-7}zt_{-1}t_3+v^{-6}z^4t_2+3v^{-6}z^2t_2+v^{-6}t_2$
$\mathcal{S}_3(\overline{5_{73}}) = -v^{-5}zt_{-3}t_1+v^{-4}z^2t_{-2}+v^{-4}t_{-2}+v^{-2}z^2t_1^2t_4-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{84}) = v^{-2}t_{-2}-v^{-1}zt_3^2$
$\mathcal{S}_3(\overline{5_{84}}) = 2v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-2v^{-11}zt_{-2}t_4+3v^{-10}z^2t_{-1}^2t_4-4v^{-9}z^3t_{-1}t_3-2v^{-9}zt_{-1}t_3+3v^{-8}z^4t_2+6v^{-8}z^2t_2+v^{-8}t_2$
$\mathcal{S}_3(\overline{5_{85}}) = -v^{-9}zt_{-3}^2+6v^{-8}z^2t_{-3}t_{-2}t_{-1}-4v^{-7}z^3t_{-3}t_{-1}^3+v^{-8}z^2t_{-2}^3-6v^{-7}z^3t_{-2}^2t_{-1}^2+5v^{-6}z^4t_{-2}t_{-1}^4-2v^{-13}zt_{-2}t_4-v^{-5}z^5t_{-1}^6+3v^{-12}z^2t_{-1}^2t_4-4v^{-11}z^3t_{-1}t_3-2v^{-11}zt_{-1}t_3+3v^{-10}z^4t_2+6v^{-10}z^2t_2+v^{-10}t_2$

## Knots in $L(9,1)$

$\mathcal{S}_3(4_2) = -v^{-5}zt_{-4}t_{-1}-2v^{-5}zt_{-3}t_{-2}+v^{-4}z^2t_{-3}t_{-1}^2+2v^{-4}z^2t_{-2}^2t_{-1}-v^{-3}z^3t_{-2}t_{-1}^3+v^{-6}t_4$
$\mathcal{S}_3(\overline{4_2}) = v^{-2}t_{-4}-v^{-1}zt_1t_4$
$\mathcal{S}_3(4_5) = v^{-4}t_{-4}+v^{-2}z^2t_1t_2^2-2v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_5}) = v^{-4}t_{-4}+v^{-2}z^2t_1^2t_3-2v^{-3}zt_1t_4$
$\mathcal{S}_3(4_{14}) = v^{-4}t_{-4}+v^{-2}z^2t_1t_2^2-v^{-3}zt_1t_4-v^{-3}zt_2t_3$
$\mathcal{S}_3(4_{18}) = v^{-2}t_{-4}-v^{-1}zt_2t_3$
$\mathcal{S}_3(\overline{4_{18}}) = -2v^{-5}zt_{-4}t_{-1}-v^{-5}zt_{-3}t_{-2}+2v^{-4}z^2t_{-3}t_{-1}^2+v^{-4}z^2t_{-2}^2t_{-1}-v^{-3}z^3t_{-2}t_{-1}^3+v^{-6}t_4$
$\mathcal{S}_3(4_{21}) = v^{-4}t_{-4}+v^{-2}z^2t_1^2t_3-v^{-3}zt_1t_4-v^{-3}zt_2t_3$
$\mathcal{S}_3(\overline{4_{24}}) = -2v^{-7}zt_{-4}t_{-1}-2v^{-7}zt_{-3}t_{-2}+3v^{-6}z^2t_{-3}t_{-1}^2+3v^{-6}z^2t_{-2}^2t_{-1}-4v^{-5}z^3t_{-2}t_{-1}^3+v^{-4}z^4t_{-1}^5+v^{-8}t_4$
$\mathcal{S}_3(5_6) = -2v^{-7}zt_{-4}t_{-2}+v^{-6}z^2t_{-4}t_{-1}^2-v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-3v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-v^{-9}zt_{-1}t_4+v^{-8}z^2t_3+v^{-8}t_3$
$\mathcal{S}_3(\overline{5_6}) = -v^{-3}zt_{-4}t_1+v^{-2}z^2t_{-3}+v^{-2}t_{-3}$
$\mathcal{S}_3(5_9) = -2v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}+v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2+v^{-6}t_3$
$\mathcal{S}_3(\overline{5_9}) = -2v^{-5}zt_{-4}t_1+2v^{-4}z^2t_{-3}+v^{-4}t_{-3}+v^{-2}z^2t_1^2t_4$
$\mathcal{S}_3(5_{15}) = -v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+3v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-2}^2t_{-1}^2-v^{-7}zt_{-1}t_4+v^{-6}z^2t_3+v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{15}}) = -v^{-5}zt_{-4}t_1+v^{-4}z^2t_{-3}+v^{-4}t_{-3}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{20}) = v^{-4}t_{-3}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_2t_4-v^{-3}zt_3^2$
$\mathcal{S}_3(\overline{5_{20}}) = -v^{-5}zt_{-4}t_{-2}+2v^{-4}z^2t_{-4}t_{-1}^2+v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3-2v^{-7}zt_{-1}t_4+2v^{-6}z^2t_3+v^{-6}t_3$
$\mathcal{S}_3(5_{47}) = -2v^{-5}zt_{-4}t_{-2}+v^{-4}z^2t_{-4}t_{-1}^2+v^{-4}z^2t_{-3}t_{-2}t_{-1}+v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2-v^{-7}zt_{-1}t_4+v^{-6}z^2t_3+v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{47}}) = -v^{-5}zt_{-4}t_1+v^{-4}z^2t_{-3}+v^{-4}t_{-3}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_3^2$
$\mathcal{S}_3(5_{65}) = v^{-4}t_{-3}+v^{-2}z^2t_2^3-2v^{-3}zt_2t_4$
$\mathcal{S}_3(\overline{5_{65}}) = v^{-4}z^2t_{-4}t_{-1}^2-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-2}^2t_{-1}^2-2v^{-7}zt_{-1}t_4+2v^{-6}z^2t_3+v^{-6}t_3$
$\mathcal{S}_3(5_{69}) = v^{-2}t_{-3}-v^{-1}zt_2t_4$
$\mathcal{S}_3(\overline{5_{69}}) = -v^{-7}zt_{-4}t_{-2}+2v^{-6}z^2t_{-4}t_{-1}^2-v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-2v^{-9}zt_{-1}t_4+2v^{-8}z^2t_3+v^{-8}t_3$
$\mathcal{S}_3(5_{73}) = -v^{-5}zt_{-4}t_{-2}+v^{-4}z^2t_{-4}t_{-1}^2-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3-v^{-7}zt_{-1}t_4+v^{-6}z^2t_3+v^{-6}t_3$
$\mathcal{S}_3(\overline{5_{73}}) = -v^{-5}zt_{-4}t_1+v^{-4}z^2t_{-3}+v^{-4}t_{-3}+v^{-2}z^2t_1^2t_4-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{84}) = v^{-2}t_{-3}-v^{-1}zt_3^2$
$\mathcal{S}_3(\overline{5_{84}}) = -2v^{-7}zt_{-4}t_{-2}+3v^{-6}z^2t_{-4}t_{-1}^2+2v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-2v^{-9}zt_{-1}t_4+2v^{-8}z^2t_3+v^{-8}t_3$

$\mathcal{S}_3(\overline{5_{85}}) = -2v^{-9}zt_{-4}t_{-2}+3v^{-8}z^2t_{-4}t_{-1}^2-v^{-9}zt_{-3}^2+6v^{-8}z^2t_{-3}t_{-2}t_{-1}-4v^{-7}z^3t_{-3}t_{-1}^3+v^{-8}z^2t_{-2}^3-6v^{-7}z^3t_{-2}^2t_{-1}^2+5v^{-6}z^4t_{-2}t_{-1}^4-v^{-5}z^5t_{-1}^6-2v^{-11}zt_{-1}t_4+2v^{-10}z^2t_3+v^{-10}t_3$

## Knots in $L(10,1)$

$\mathcal{S}_3(5_6) = -2v^{-7}zt_{-4}t_{-2}+v^{-6}z^2t_{-4}t_{-1}^2-v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-3v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-v^{-9}zt_{-1}t_5+v^{-8}z^2t_4+v^{-8}t_4$
$\mathcal{S}_3(\overline{5_6}) = v^{-2}t_{-4}-v^{-1}zt_1t_5$
$\mathcal{S}_3(5_9) = -2v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}+v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2+v^{-6}t_4$
$\mathcal{S}_3(\overline{5_9}) = v^{-4}t_{-4}+v^{-2}z^2t_1^2t_4-2v^{-3}zt_1t_5$
$\mathcal{S}_3(5_{15}) = -v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+3v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-2}^2t_{-1}^2-v^{-7}zt_{-1}t_5+v^{-6}z^2t_4+v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{15}}) = v^{-4}t_{-4}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_1t_5-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{20}) = v^{-4}t_{-4}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_2t_4-v^{-3}zt_3^2$
$\mathcal{S}_3(\overline{5_{20}}) = -v^{-5}zt_{-4}t_{-2}+2v^{-4}z^2t_{-4}t_{-1}^2+v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3-2v^{-7}zt_{-1}t_5+2v^{-6}z^2t_4+v^{-6}t_4$
$\mathcal{S}_3(5_{47}) = -2v^{-5}zt_{-4}t_{-2}+v^{-4}z^2t_{-4}t_{-1}^2+v^{-4}z^2t_{-3}t_{-2}t_{-1}+v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2-v^{-7}zt_{-1}t_5+v^{-6}z^2t_4+v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{47}}) = v^{-4}t_{-4}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_1t_5-v^{-3}zt_3^2$
$\mathcal{S}_3(5_{65}) = v^{-4}t_{-4}+v^{-2}z^2t_2^3-2v^{-3}zt_2t_4$
$\mathcal{S}_3(\overline{5_{65}}) = v^{-4}z^2t_{-4}t_{-1}^2-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-2}^2t_{-1}^2-2v^{-7}zt_{-1}t_5+2v^{-6}z^2t_4+v^{-6}t_4$
$\mathcal{S}_3(5_{69}) = v^{-2}t_{-4}-v^{-1}zt_2t_4$
$\mathcal{S}_3(\overline{5_{69}}) = -v^{-7}zt_{-4}t_{-2}+2v^{-6}z^2t_{-4}t_{-1}^2-v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-2v^{-9}zt_{-1}t_5+2v^{-8}z^2t_4+v^{-8}t_4$
$\mathcal{S}_3(5_{73}) = -v^{-5}zt_{-4}t_{-2}+v^{-4}z^2t_{-4}t_{-1}^2-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3-v^{-7}zt_{-1}t_5+v^{-6}z^2t_4+v^{-6}t_4$
$\mathcal{S}_3(\overline{5_{73}}) = v^{-4}t_{-4}+v^{-2}z^2t_1^2t_4-v^{-3}zt_1t_5-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{84}) = v^{-2}t_{-4}-v^{-1}zt_3^2$
$\mathcal{S}_3(\overline{5_{84}}) = -2v^{-7}zt_{-4}t_{-2}+3v^{-6}z^2t_{-4}t_{-1}^2+2v^{-6}z^2t_{-3}t_{-2}t_{-1}-2v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-2v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4-2v^{-9}zt_{-1}t_5+2v^{-8}z^2t_4+v^{-8}t_4$
$\mathcal{S}_3(\overline{5_{85}}) = -2v^{-9}zt_{-4}t_{-2}+3v^{-8}z^2t_{-4}t_{-1}^2-v^{-9}zt_{-3}^2+6v^{-8}z^2t_{-3}t_{-2}t_{-1}-4v^{-7}z^3t_{-3}t_{-1}^3+v^{-8}z^2t_{-2}^3-6v^{-7}z^3t_{-2}^2t_{-1}^2+5v^{-6}z^4t_{-2}t_{-1}^4-v^{-5}z^5t_{-1}^6-2v^{-11}zt_{-1}t_5+2v^{-10}z^2t_4+v^{-10}t_4$

## Knots in $L(11,1)$

$\mathcal{S}_3(5_6) = -v^{-7}zt_{-5}t_{-1}-2v^{-7}zt_{-4}t_{-2}+v^{-6}z^2t_{-4}t_{-1}^2-v^{-7}zt_{-3}^2+4v^{-6}z^2t_{-3}t_{-2}t_{-1}-v^{-5}z^3t_{-3}t_{-1}^3+v^{-6}z^2t_{-2}^3-3v^{-5}z^3t_{-2}^2t_{-1}^2+v^{-4}z^4t_{-2}t_{-1}^4+v^{-8}t_5$
$\mathcal{S}_3(\overline{5_6}) = v^{-2}t_{-5}-v^{-1}zt_1t_5$
$\mathcal{S}_3(5_9) = -2v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+2v^{-4}z^2t_{-3}t_{-2}t_{-1}+v^{-4}z^2t_{-2}^3-v^{-3}z^3t_{-2}^2t_{-1}^2+v^{-6}t_5$
$\mathcal{S}_3(\overline{5_9}) = v^{-4}t_{-5}+v^{-2}z^2t_1^2t_4-2v^{-3}zt_1t_5$
$\mathcal{S}_3(5_{15}) = -v^{-5}zt_{-5}t_{-1}-v^{-5}zt_{-4}t_{-2}-v^{-5}zt_{-3}^2+3v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-2}^2t_{-1}^2+v^{-6}t_5$
$\mathcal{S}_3(\overline{5_{15}}) = v^{-4}t_{-5}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_1t_5-v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{20}) = v^{-4}t_{-5}+v^{-2}z^2t_1t_2t_3-v^{-3}zt_2t_4-v^{-3}zt_3^2$
$\mathcal{S}_3(\overline{5_{20}}) = -2v^{-5}zt_{-5}t_{-1}-v^{-5}zt_{-4}t_{-2}+2v^{-4}z^2t_{-4}t_{-1}^2+v^{-4}z^2t_{-3}t_{-2}t_{-1}-v^{-3}z^3t_{-3}t_{-1}^3+v^{-6}t_5$

$\mathcal{S}_3(5_{47}) = -v^{-5}zt_{-5}t_{-1} - 2v^{-5}zt_{-4}t_{-2} + v^{-4}z^2t_{-4}t_{-1}^2 + v^{-4}z^2t_{-3}t_{-2}t_{-1} + v^{-4}z^2t_{-2}^3 - v^{-3}z^3t_{-2}^2t_{-1}^2 + v^{-6}t_5$
$\mathcal{S}_3(\overline{5_{47}}) = v^{-4}t_{-5} + v^{-2}z^2t_1t_2t_3 - v^{-3}zt_1t_5 - v^{-3}zt_3^2$
$\mathcal{S}_3(5_{65}) = v^{-4}t_{-5} + v^{-2}z^2t_2^3 - 2v^{-3}zt_2t_4$
$\mathcal{S}_3(\overline{5_{65}}) = -2v^{-5}zt_{-5}t_{-1} + v^{-4}z^2t_{-4}t_{-1}^2 - v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-2}^2t_{-1}^2 + v^{-6}t_5$
$\mathcal{S}_3(5_{69}) = v^{-2}t_{-5} - v^{-1}zt_2t_4$
$\mathcal{S}_3(\overline{5_{69}}) = -2v^{-7}zt_{-5}t_{-1} - v^{-7}zt_{-4}t_{-2} + 2v^{-6}z^2t_{-4}t_{-1}^2 - v^{-7}zt_{-3}^2 + 4v^{-6}z^2t_{-3}t_{-2}t_{-1} - 2v^{-5}z^3t_{-3}t_{-1}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + v^{-8}t_5$
$\mathcal{S}_3(5_{73}) = -v^{-5}zt_{-5}t_{-1} - v^{-5}zt_{-4}t_{-2} + v^{-4}z^2t_{-4}t_{-1}^2 - v^{-5}zt_{-3}^2 + 2v^{-4}z^2t_{-3}t_{-2}t_{-1} - v^{-3}z^3t_{-3}t_{-1}^3 + v^{-6}t_5$
$\mathcal{S}_3(\overline{5_{73}}) = v^{-4}t_{-5} + v^{-2}z^2t_1^2t_4 - v^{-3}zt_1t_5 - v^{-3}zt_2t_4$
$\mathcal{S}_3(5_{84}) = v^{-2}t_{-5} - v^{-1}zt_3^2$
$\mathcal{S}_3(\overline{5_{84}}) = -2v^{-7}zt_{-5}t_{-1} - 2v^{-7}zt_{-4}t_{-2} + 3v^{-6}z^2t_{-4}t_{-1}^2 + 2v^{-6}z^2t_{-3}t_{-2}t_{-1} - 2v^{-5}z^3t_{-3}t_{-1}^3 + v^{-6}z^2t_{-2}^3 - 2v^{-5}z^3t_{-2}^2t_{-1}^2 + v^{-4}z^4t_{-2}t_{-1}^4 + v^{-8}t_5$
$\mathcal{S}_3(\overline{5_{85}}) = -2v^{-9}zt_{-5}t_{-1} - 2v^{-9}zt_{-4}t_{-2} + 3v^{-8}z^2t_{-4}t_{-1}^2 - v^{-9}zt_{-3}^2 + 6v^{-8}z^2t_{-3}t_{-2}t_{-1} - 4v^{-7}z^3t_{-3}t_{-1}^3 + v^{-8}z^2t_{-2}^3 - 6v^{-7}z^3t_{-2}^2t_{-1}^2 + 5v^{-6}z^4t_{-2}t_{-1}^4 - v^{-5}z^5t_{-1}^6 + v^{-10}t_5$

# D

# The Kauffman bracket skein modules

The Kauffman bracket skein modules of the non-affine prime knots are expressed in the standard basis using the evaluation $x^0 = -A^2 - A^{-2}$ (or equivalently $[\varnothing] = 1$), see Theorem 2.3.4 and Theorem 2.3.5. For knots in $L(p,q)$ that are not listed below (and all knots in $L(12,1)$ and $L(12,5)$) the KBSM expressions equal the KBSM expressions of the corresponding knots in the solid torus (up to a change of framing).

## Knots in the solid torus

$\mathcal{S}_{2,\infty}(0_1) = x$
$\mathcal{S}_{2,\infty}(1_1) = -A^{-2}x^2 + A^{-2} + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{1_1}) = A^6 - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(2_1) = -A^{-2}x^2 + A^{-6}x^2 - A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{2_1}) = -A^{10} + A^6x^2 - A^6 - A^2x^2$
$\mathcal{S}_{2,\infty}(2_2) = -A^4x + x^3 - x - A^{-4}x$
$\mathcal{S}_{2,\infty}(2_3) = A^{-4}x^3 - 2A^{-4}x - A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^8x + A^4x^3 - 2A^4x$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x^3 - A^{-4}x - A^{-8}x^3 + A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^8x^3 + A^8x + A^4x^3 - A^4x$
$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^4 + 2A^2x^2 - A^2 + A^{-2}x^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-6}x^2$
$\mathcal{S}_{2,\infty}(3_3) = -A^{-2}x^2 + A^{-6}x^2 - A^{-10}x^2 + A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{3_3}) = A^{14} - A^{10}x^2 + A^{10} + A^6x^2 - A^2x^2$
$\mathcal{S}_{2,\infty}(3_4) = -A^4x + x^3 - A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^8x - A^4x^3 + A^4x + x^3 - A^{-4}x$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - A^2x^4 + A^2x^2 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(3_6) = -A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} + A^{-10}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} - A^6x^4 + 3A^6x^2 - A^6$
$\mathcal{S}_{2,\infty}(3_7) = -A^{-6}x^2 + A^{-6} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{3_7}) = A^{18} - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(3_8) = -A^{-2} - A^{-10}x^2 + A^{-14}x^2 - A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{3_8}) = A^{14}x^2 - A^{14} - A^{10}x^2 - A^2$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 2A^{-10}x^2 + A^{-10} - A^{-14}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^4 - 2A^{10}x^2 + A^{10} - A^6x^4 + 2A^6x^2$
$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^5 - 3A^4x^3 + 2A^4x - x^3 + 2x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^5 - 3A^{-4}x^3 + 2A^{-4}x - A^{-8}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_3) = A^{-4}x^3 - 2A^{-8}x^3 + 2A^{-8}x + A^{-12}x^3 - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + A^{12}x^3 - 2A^8x^3 + 2A^8x + A^4x^3$
$\mathcal{S}_{2,\infty}(4_4) = A^2x^2 - A^{-2}x^4 + A^{-2} + A^{-6}x^4 - A^{-6}x^2 + A^{-6} - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + A^6x^4 - A^6x^2 + A^6 - A^2x^4 + A^2 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + x^5 - 2x^3 + 2x - 2A^{-4}x^3 + 2A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + x^5 - 2x^3 + 2x - A^{-4}x^3$
$\mathcal{S}_{2,\infty}(4_6) = -A^{-6}x^2 + A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{4_6}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + A^{10}x^2 - A^{10} - A^6x^2$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + A^{-12}x^3 - A^{-16}x^3 + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - A^{16}x^3 + A^{12}x^3 - A^8x + A^4x$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^8x + A^4x^3 - A^4x - x$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x^3 - A^{-4}x - A^{-8}x + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(4_9) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + A^6x^2 - A^2x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{4_9}) = A^2 - A^{-2}x^2 + A^{-6}x^2 - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(4_{10}) = -A^{-2}x^2 + A^{-6}x^2 - A^{-10}x^2 + A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{10}}) = -A^{18} + A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^4x + x^3 - A^{-4}x^3 + A^{-8}x^3 - A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^8x^3 - A^8x - A^4x^3 + x^3 - A^{-4}x$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - A^{-2}x^4 + A^{-6}x^4 - 2A^{-10}x^2 + A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + A^6x^4 - A^2x^4 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(4_{13}) = A^8x - A^4x^3 + 2x^3 - x - A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + x^5 - x^3 + x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + A^2x^4 - A^2x^2 - A^{-2}x^4 + 2A^{-6}x^2 - A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - A^2x^4 + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(4_{16}) = A^{-4}x^3 - A^{-4}x - A^{-8}x^3 + A^{-12}x^3 - A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x^3 - A^{12}x - A^8x^3 + A^4x^3 - A^4x$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - A^2x^4 + A^2x^2 + A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^4 - 2A^2x^2 + A^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^2$
$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + A^{-4}x^5 - 2A^{-4}x^3 + A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + A^4x^5 - 2A^4x^3 + A^4x - x^3 + x$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + A^6x^4 - A^6x^2 - A^2x^4 + A^2x^2 - A^2 + A^{-2}x^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-6}x^2 - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-10}x^4 - A^{-10}x^2 - A^{-10} - A^{-14}x^2$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + A^{10}x^4 - A^{10}x^2 - A^{10} - A^6x^4 + 2A^6x^2 - A^6$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^5 - 3x^3 + 3x - A^{-4}x^3 + A^{-4}x$
$\mathcal{S}_{2,\infty}(4_{22}) = A^{-8}x^3 - 2A^{-8}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_{22}}) = -A^{16}x + A^8x^3 - 2A^8x$
$\mathcal{S}_{2,\infty}(4_{23}) = -A^{12}x^3 + 2A^{12}x + A^8x^3 - A^8x$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x^3 - A^{-8}x - A^{-12}x^3 + 2A^{-12}x$
$\mathcal{S}_{2,\infty}(4_{24}) = A^{-8}x^5 - 4A^{-8}x^3 + 3A^{-8}x - A^{-12}x^3 + 2A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^8x^5 - 4A^8x^3 + 3A^8x$
$\mathcal{S}_{2,\infty}(4_{25}) = -A^6x^2 + A^6 + A^2x^2 - A^2 - A^{-2} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{25}}) = -A^{10} - A^2 + A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(4_{26}) = -A^6x^2 + A^6 + A^2x^2 - A^{-2}x^2 + A^{-2} + A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{26}}) = -A^{10} + A^6 - A^2x^2 + A^2 + A^{-2}x^2 - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(4_{27}) = A^8x - 2A^4x + x^3 - x - 2A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x^3 - A^{-8}x - A^{-12}x^3 + 2A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = -A^{12}x^3 + 2A^{12}x + A^8x^3 - A^8x$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - A^{-10}x^4 + A^{-10}x^2 + A^{-14}x^4 - A^{-14}x^2 - A^{-18}x^2 + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + A^{14}x^4 - A^{14}x^2 - A^{10}x^4 + A^{10}x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_3) = -A^{12}x^3 + A^{12}x + 2A^8x^3 - 2A^8x - A^4x^3 + A^4x + x$
$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x^3 + A^{-4}x + 2A^{-8}x^3 - 2A^{-8}x - A^{-12}x^3 + A^{-12}x$
$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^4 + 2A^6x^2 + A^2x^2 - A^2$
$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^5 - 3A^{-8}x^3 + A^{-8}x - A^{-12}x^5 + 3A^{-12}x^3 - 2A^{-12}x + A^{-16}x^3 - 2A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^5 + 3A^{12}x^3 - 2A^{12}x + A^8x^5 - 3A^8x^3 + A^8x$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^6 + 4A^6x^4 - 4A^6x^2 + A^6 + A^2x^4 - 3A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^6 + 4A^{-6}x^4 - 4A^{-6}x^2 + A^{-6} + A^{-10}x^4 - 2A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_7) = -A^{-6}x^4 + A^{-6}x^2 + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-14}x^4 + A^{-14}x^2 - A^{-14} + A^{-18}x^2 - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - A^{14}x^4 + A^{14}x^2 - A^{14} + 2A^{10}x^4 - 3A^{10}x^2 - A^6x^4 + A^6x^2$
$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + A^{-4}x^5 - A^{-4}x^3 - A^{-4}x - A^{-8}x^5 + 2A^{-8}x^3 - 2A^{-8}x + A^{-12}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - A^8x^5 + 2A^8x^3 - 2A^8x + A^4x^5 - A^4x^3 - A^4x - x^3 + x$
$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - A^2x^6 + 3A^2x^4 - 3A^2x^2 + 2A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - A^{-2}x^6 + 3A^{-2}x^4 - 3A^{-2}x^2 + A^{-6}x^4 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{10}) = A^{-4}x^3 - 2A^{-8}x^3 + A^{-8}x + 2A^{-12}x^3 - A^{-12}x - A^{-16}x^3 + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - A^{16}x^3 + 2A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x + A^4x^3$
$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - A^{-2}x^4 - A^{-2}x^2 + A^{-2} + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - A^{10}x^4 + 2A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 - A^2x^2 + A^2 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + A^{-4}x^5 - A^{-4}x - A^{-8}x^5 + 2A^{-12}x^3 - A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - A^8x^5 + A^4x^5 - A^4x - x^3$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + A^2x^4 - A^2 - 2A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + A^6x^4 - 2A^2x^4 + 2A^2x^2 - A^2 + A^{-2}x^4 - A^{-2} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + x^5 - x^3 + x - A^{-4}x^5 + A^{-4}x^3 - A^{-4}x + 2A^{-8}x^3 - 2A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - A^4x^5 + A^4x^3 - A^4x + x^5 - x^3 + x - A^{-4}x^3$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - A^2x^6 + 2A^2x^4 - 3A^2x^2 + A^2 + 2A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - A^{-2}x^6 + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - A^4x^5 + A^4x^3 - A^4x + x^5 - 2A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + x^5 - A^{-4}x^5 + A^{-4}x^3 - A^{-4}x + A^{-8}x^3$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - A^{10}x^4 + A^{10}x^2 - A^{10} + A^6x^4 + A^6x^2 - A^6 - A^2x^4 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - A^{-2}x^4 + A^{-6}x^4 + A^{-6}x^2 - A^{-6} - A^{-10}x^4 + A^{-10}x^2 - A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{18}) = -A^{-6}x^4 + A^{-6}x^2 + 2A^{-10}x^4 - 3A^{-10}x^2 + A^{-10} - A^{-14}x^4 + A^{-14} + A^{-18}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - A^{14}x^4 + A^{14} + 2A^{10}x^4 - 3A^{10}x^2 + A^{10} - A^6x^4 + A^6x^2$
$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + x^5 - x^3 - A^{-4}x^5 + 2A^{-4}x^3 - 2A^{-4}x + A^{-8}x^3 - A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - A^4x^5 + 2A^4x^3 - 2A^4x + x^5 - x^3 - A^{-4}x^3$
$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - A^{-2}x^6 + 3A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} + 2A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - A^2x^6 + 3A^2x^4 - 4A^2x^2 + A^2 + A^{-2}x^4 - A^{-2}x^2$
$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - A^8x^5 + A^8x^3 + A^4x^5 - A^4x^3 - x^3 + x$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + A^{-4}x^5 - A^{-4}x^3 - A^{-8}x^5 + A^{-8}x^3 + A^{-12}x^3$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + A^2x^4 - A^2x^2 - A^{-2}x^4 + A^{-2} + A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - A^2x^4 + A^2 + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{23}) = -A^{12}x + A^8x^3 - A^8x - A^4x^3 + A^4x + 2x$
$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{24}) = -A^{-6}x^4 + 3A^{-6}x^2 + A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} - A^6x^4 + 3A^6x^2$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-14}x^4 - A^{-14}x^2 + A^{-14} + A^{-18}x^4 + A^{-18}x^2 - 2A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + A^{18}x^4 + A^{18}x^2 - A^{14}x^4 - A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{26}) = -x + A^{-4}x^3 + A^{-4}x - 2A^{-8}x^3 + A^{-8}x + 2A^{-12}x^3 - A^{-16}x^3 - A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - A^{16}x^3 - A^{16}x + 2A^{12}x^3 - 2A^8x^3 + A^8x + A^4x^3 + A^4x - x$
$\mathcal{S}_{2,\infty}(5_{27}) = -x + A^{-4}x^3 + A^{-4}x - 2A^{-8}x^3 + A^{-8}x + 2A^{-12}x^3 - A^{-16}x^3 - A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - A^{16}x^3 - A^{16}x + 2A^{12}x^3 - 2A^8x^3 + A^8x + A^4x^3 + A^4x - x$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - A^2x^4 + 2A^{-2}x^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - A^{-2}x^4 + 3A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{29}) = -A^6x^2 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{29}}) = A^{14} - A^{10}x^2 - A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{30}) = A^8x - A^4x + 2x - A^{-4}x^3 + A^{-8}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + A^8x^3 - A^4x^3 + 2x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{31}) = A^8x - 2A^4x + x^3 + x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x^3 + A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + A^8x^3 + A^8x - 2A^4x^3 + A^4x + x^3 + x - 2A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{32}) = A^8x - 2A^4x + x^3 - A^{-4}x^3 - A^{-4}x + A^{-8}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + A^8x^3 - A^4x^3 - A^4x + x^3 - 2A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{33}) = -A^6x^2 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-2} + A^{-6}x^2 + A^{-6} - A^{-10}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{33}}) = A^{14} - A^{10}x^2 + A^6x^2 + A^6 - 2A^2x^2 + A^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - A^{-2}x^4 + 3A^{-6}x^2 - A^{-6} - A^{-10}x^2 - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - A^2x^4 + 3A^{-2}x^2 - A^{-2} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - A^{-14}x^4 - A^{-14} + A^{-18}x^4 + A^{-18}x^2 - A^{-18} - 2A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + A^{18}x^4 + A^{18}x^2 - A^{18} - A^{14}x^4 - A^{14} + 2A^{10}x^2 - A^{10} - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{36}) = -x + A^{-4}x^3 - A^{-8}x^3 - A^{-8}x + 2A^{-12}x^3 - A^{-12}x - A^{-16}x^3 - A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - A^{16}x^3 - A^{16}x + 2A^{12}x^3 - A^{12}x - A^8x^3 - A^8x + A^4x^3 - x$
$\mathcal{S}_{2,\infty}(5_{37}) = A^{14}x^2 - 2A^{10}x^2 + 2A^6x^2 - A^6 - 2A^2x^2 + A^{-2}x^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{37}}) = -A^6 + A^2x^2 - 2A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - A^{-10}x^4 + A^{-10}x^2 - A^{-10} + A^{-14}x^4 - A^{-14}x^2 - A^{-18}x^2 + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + A^{14}x^4 - A^{14}x^2 - A^{10}x^4 + A^{10}x^2 - A^{10} + A^6x^2 - A^6 - A^2$
$\mathcal{S}_{2,\infty}(5_{39}) = -A^{12}x^3 + A^{12}x + 2A^8x^3 - 2A^8x - A^4x^3 + x^3 - x - A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^4x + x^3 - x - A^{-4}x^3 + 2A^{-8}x^3 - 2A^{-8}x - A^{-12}x^3 + A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x^3 - A^{-8}x - A^{-12}x^3 + A^{-12}x + A^{-16}x^3 - A^{-16}x - A^{-20}x^3 + A^{-20}x + A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x^3 + A^{20}x + A^{16}x^3 - A^{16}x - A^{12}x^3 + A^{12}x + A^8x^3 - A^8x$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^4 + 2A^6x^2 - A^6 + A^2x^2 - A^2$
$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-10}x^2 - A^{-10} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{42}) = A^{-4}x^3 - A^{-4}x - A^{-8}x^3 + A^{-12}x^3 - A^{-16}x^3 + A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x^3 + A^{16}x + A^{12}x^3 - A^8x^3 + A^4x^3 - A^4x$

$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - A^2x^4 + A^2x^2 + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-10}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^4 + 2A^6x^2 - A^6 + A^2x^4 - A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + A^{-4}x^5 - A^{-4}x^3 - A^{-8}x^5 + A^{-8}x^3 - A^{-8}x + 2A^{-12}x^3 - 2A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - A^8x^5 + A^8x^3 - A^8x + A^4x^5 - A^4x^3 - x^3 + x$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + A^6x^4 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^4 - A^{-2}x^2 + A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + A^2x^4 - A^2x^2 + A^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + A^{-6}x^4 - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + A^{-4}x^5 - A^{-4}x^3 + A^{-4}x - A^{-8}x^5 + A^{-8}x + 2A^{-12}x^3 - A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - A^8x^5 + A^8x + A^4x^5 - A^4x^3 + A^4x - x^3 + x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - A^2x^6 + 2A^2x^4 - 2A^2x^2 + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - A^{-2}x^6 + 2A^{-2}x^4 - 2A^{-2}x^2 + 2A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - x^5 + 2x^3 - x + A^{-4}x^5 - A^{-4}x^3 - 2A^{-8}x^3 + 2A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + A^4x^5 - A^4x^3 - x^5 + 2x^3 - x + A^{-4}x^3 - A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - A^{10}x^4 + A^{10}x^2 + A^6x^4 - A^2x^4 + A^2x^2 - A^2 + A^{-2}x^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-10}x^4 + A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{50}) = -A^{-6}x^2 + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} + 2A^{-18}x^2 - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{50}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - 2A^{14}x^2 + A^{14} + A^{10}x^2 - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + A^{-12}x^3 + A^{-12}x - 2A^{-16}x^3 + A^{-16}x + A^{-20}x^3 - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{51}}) = -A^{24}x + A^{20}x^3 - 2A^{16}x^3 + A^{16}x + A^{12}x^3 + A^{12}x - A^8x + A^4x$
$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^8x + A^4x^3 - x^3 + x + A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^4x - x^3 + x + A^{-4}x^3 - A^{-8}x + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - A^{-6}x^4 + A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - A^6x^4 + A^6x^2 + A^2x^2$
$\mathcal{S}_{2,\infty}(5_{54}) = A^{20}x - A^{16}x^3 + A^{12}x^3 + A^{12}x - A^8x^3 + A^4x^3 - x$
$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + A^{-4}x^3 - A^{-8}x^3 + A^{-12}x^3 + A^{-12}x - A^{-16}x^3 + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + A^{14}x^4 - A^{10}x^4 - A^{10}x^2 + A^{10} + 2A^6x^2 - A^2x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - A^{-10}x^4 - A^{-10}x^2 + A^{-10} + A^{-14}x^4 - A^{-18}x^2$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - A^2x^4 + A^2x^2 - A^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-2} + 2A^{-6}x^2 - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{57}) = -A^{-2}x^2 + A^{-6}x^2 - A^{-10}x^2 + A^{-14}x^2 - A^{-18}x^2 + A^{-18} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{57}}) = A^{22} - A^{18}x^2 + A^{18} + A^{14}x^2 - A^{10}x^2 + A^6x^2 - A^2x^2$
$\mathcal{S}_{2,\infty}(5_{58}) = -A^4x + x^3 - A^{-4}x^3 + A^{-8}x^3 - A^{-12}x^3 + A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x^3 + A^{12}x + A^8x^3 - A^4x^3 + x^3 - A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - A^{-2}x^4 + A^{-6}x^4 + A^{-6}x^2 - A^{-10}x^4 + 2A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^4 + A^6x^4 + A^6x^2 - A^2x^4 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(5_{60}) = A^8x - A^4x^3 + 2x^3 - 2A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + A^8x^3 - 2A^4x^3 + A^4x + 2x^3 - A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - A^{-2}x^4 - A^{-2}x^2 + 2A^{-6}x^4 - A^{-6}x^2 - A^{-10}x^4 - A^{-10}x^2 + 2A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^4 - A^{10}x^2 + 2A^6x^4 - A^6x^2 - A^2x^4 - A^2x^2 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + x^5 + x^3 - x - A^{-4}x^5 - A^{-4}x + 2A^{-8}x^3 - A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - A^4x^5 - A^4x + x^5 + x^3 - x - 2A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + A^2x^4 - 2A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^4 + A^{-6}x^2 - 2A^{-10}x^2 + A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + A^6x^4 + A^6x^2 - 2A^2x^4 + A^2x^2 + A^{-2}x^4 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + x^5 + x - A^{-4}x^5 + 2A^{-8}x^3 - A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - A^4x^5 + x^5 + x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - A^{-2}x^6 + A^{-2}x^4 - 2A^{-2}x^2 + 3A^{-6}x^4 - 2A^{-6}x^2 - 3A^{-10}x^2 + A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - A^2x^6 + A^2x^4 - 2A^2x^2 + 2A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - A^6x^4 + A^6x^2 + A^2x^4 - A^{-2}x^4 + 2A^{-6}x^2 - A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - A^2x^4 + A^{-2}x^4 - A^{-6}x^4 + A^{-6}x^2 + A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - A^{-10}x^2 - A^{-14}x^4 + 2A^{-14}x^2 - A^{-14} + A^{-18}x^2 - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^4 + 2A^{14}x^2 - A^{14} + A^{10}x^4 - A^{10}x^2 - A^6x^4 + 2A^6x^2$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + A^4x^5 - 2A^4x^3 - x^5 + 3x^3 - 2x + A^{-4}x^3 - 2A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^5 + 3x^3 - 2x + A^{-4}x^5 - 2A^{-4}x^3 - A^{-8}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^6 + 3A^{-6}x^4 - 2A^{-6}x^2 + 2A^{-10}x^4 - 4A^{-10}x^2 + A^{-10} - A^{-14}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - A^6x^6 + 3A^6x^4 - 2A^6x^2 + A^2x^4 - 2A^2x^2$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - A^8x^5 + 2A^8x^3 - A^8x + A^4x^5 - 2A^4x^3 + A^4x - x^3 + 2x$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + A^{-4}x^5 - 2A^{-4}x^3 + A^{-4}x - A^{-8}x^5 + 2A^{-8}x^3 - A^{-8}x + A^{-12}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{71}) = A^{10}x^4 - 3A^{10}x^2 + A^{10} - A^6x^4 + 2A^6x^2 - A^2 + A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 3A^{-10}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^5 - 3A^{-8}x^3 + 2A^{-8}x - A^{-12}x^5 + 2A^{-12}x^3 + A^{-16}x^3 - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - A^{12}x^5 + 2A^{12}x^3 + A^8x^5 - 3A^8x^3 + 2A^8x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^6 + 4A^2x^4 - 5A^2x^2 + A^2 + A^{-2}x^4 - 2A^{-2}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^6 + 4A^{-2}x^4 - 5A^{-2}x^2 + A^{-2} + A^{-6}x^4 - 2A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{74}) = -A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 - A^{10}x^4 + 3A^{10}x^2 - A^{10}$
$\mathcal{S}_{2,\infty}(5_{75}) = A^{10}x^4 - 3A^{10}x^2 + A^{10} - A^6x^4 + 2A^6x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 3A^{-10}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{76}) = A^4x^3 - 2A^4x - x^3 + x + A^{-4}x^3 - 2A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = A^4x^3 - 2A^4x - x^3 + x + A^{-4}x^3 - 2A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-6}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^4 + 2A^2x^2 - A^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{78}) = A^4x^3 - 2A^4x - x^3 + 2x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{78}}) = A^8x - x^3 + 2x + A^{-4}x^3 - 2A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - A^{-14}x^4 + A^{-14}x^2 + A^{-18}x^4 - A^{-18}x^2 + A^{-18} - A^{-22}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + A^{18}x^4 - A^{18}x^2 + A^{18} - A^{14}x^4 + A^{14}x^2 + A^{10}x^2 - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^4 + 2A^2x^2 - A^2 + A^{-2}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-6}x^2 - A^{-10}x^2 + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} - A^{-14} + 2A^{-18}x^2 - A^{-18} - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} - A^{10}x^4 + 3A^{10}x^2 - A^{10}$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-10}x^4 - A^{-10} - A^{-14}x^4 + A^{-14}x^2 + A^{-18}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - A^{14}x^4 + A^{14}x^2 + A^{10}x^4 - A^{10} - A^6x^4 + 2A^6x^2 - A^6$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + x^5 - 2x^3 + 2x - A^{-4}x^5 + 2A^{-4}x^3 - A^{-4}x + A^{-8}x^3 - A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - A^4x^5 + 2A^4x^3 - A^4x + x^5 - 2x^3 + 2x - A^{-4}x^3 + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - A^{-6}x^6 + 3A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-14}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - A^6x^6 + 3A^6x^4 - 3A^6x^2 + A^6 + A^2x^4 - 2A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_{85}) = -A^{-10}x^6 + 5A^{-10}x^4 - 6A^{-10}x^2 + A^{-10} + A^{-14}x^4 - 3A^{-14}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^{10}x^6 + 5A^{10}x^4 - 6A^{10}x^2 + A^{10}$
$\mathcal{S}_{2,\infty}(5_{86}) = A^{14} - A^{10}x^2 + 2A^6x^2 - A^6 - 2A^2x^2 + A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{86}}) = -A^6 + A^2x^2 - A^2 - 2A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(5_{87}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-18} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{87}}) = A^{26} + A^{18} - A^{14}x^2 + A^{10}x^2 - A^{10} - A^6x^2 + A^6$

$\mathcal{S}_{2,\infty}(5_{88}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} + A^{-18}x^2 - A^{-22} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{88}}) = A^{26} - A^{22} + A^{18}x^2 - 2A^{14}x^2 + A^{14} + A^{10}x^2 - A^{6}x^2 + A^{6}$
$\mathcal{S}_{2,\infty}(5_{89}) = -A^{-6}x^2 + 2A^{-10}x^2 - 2A^{-10} - 2A^{-14}x^2 + A^{-18}x^2 - A^{-22} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{89}}) = A^{26} - A^{22} + A^{18}x^2 - 2A^{14}x^2 + 2A^{10}x^2 - 2A^{10} - A^{6}x^2$
$\mathcal{S}_{2,\infty}(5_{90}) = -A^{-2} + A^{-6} - A^{-10}x^2 + A^{-14}x^2 - A^{-14} - A^{-18}x^2 + A^{-22}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{90}}) = A^{22}x^2 - A^{22} - A^{18}x^2 + A^{14}x^2 - A^{14} - A^{10}x^2 + A^{6} - A^{2}$
$\mathcal{S}_{2,\infty}(5_{91}) = -A^{-2} - A^{-10} - A^{-18}x^2 + A^{-18} + A^{-22}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{91}}) = A^{22}x^2 - A^{22} - A^{18}x^2 + A^{18} - A^{10} - A^{2}$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + A^{-12}x^3 + A^{-12}x - A^{-16}x^3 + 2A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{92}}) = -A^{24}x + 2A^{20}x - A^{16}x^3 + A^{12}x^3 + A^{12}x - 2A^{8}x + A^{4}x$
$\mathcal{S}_{2,\infty}(5_{93}) = -A^{-10}x^2 + A^{-10} + A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_{93}}) = A^{30} - A^{10}x^2 + A^{10}$
$\mathcal{S}_{2,\infty}(5_{94}) = -A^{-6} - A^{-14}x^2 + A^{-18}x^2 - A^{-22}x^2 + A^{-26}x^2 - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{94}}) = A^{26}x^2 - A^{26} - A^{22}x^2 + A^{18}x^2 - A^{14}x^2 - A^{6}$
$\mathcal{S}_{2,\infty}(5_{95}) = -A^{-6} - A^{-14}x^2 - A^{-14} + 2A^{-18}x^2 - A^{-18} - 2A^{-22}x^2 + A^{-22} + A^{-26}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{95}}) = A^{26}x^2 - 2A^{22}x^2 + A^{22} + 2A^{18}x^2 - A^{18} - A^{14}x^2 - A^{14} - A^{6}$

## Knots in $L(2,1)$

$\mathcal{S}_{2,\infty}(\overline{1_1}) = A^{6} - 1 - A^{-2} - A^{-4}$
$\mathcal{S}_{2,\infty}(2_1) = -A^{-2} - A^{-4} - A^{-6} + A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{2_1}) = -A^{10} + A^{4} - A^{-2} - A^{-4}$
$\mathcal{S}_{2,\infty}(2_2) = -A^{4}x + x + A^{-8}x$
$\mathcal{S}_{2,\infty}(2_3) = A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^{8}x + x + A^{-4}x$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^{8}x + A^{-4}x$
$\mathcal{S}_{2,\infty}(3_2) = A^{6} + A^{4} - A^{-2} - A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = 1 - A^{-14} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(3_3) = -A^{-2} - A^{-4} + A^{-10} - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{3_3}) = A^{14} - A^{8} - A^{-2} - A^{-4}$
$\mathcal{S}_{2,\infty}(3_4) = -A^{4}x + 2x + A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^{8}x - A^{4}x + x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(3_5) = A^{2} + 1 - A^{-4} - A^{-6} + A^{-12} - A^{-14} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + A^{6} + 2A^{4} + A^{2} - A^{-2} - A^{-4} - A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{8} + A^{6} + A^{4} - A^{-2} - A^{-4} - A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(3_8) = -A^{-2} - A^{-10} - A^{-12} - A^{-14} + A^{-18} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{12} + A^{6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(4_2) = -A^{8}x + A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -A^{-8}x + A^{-20}x + A^{-24}x$
$\mathcal{S}_{2,\infty}(4_3) = 2A^{-4}x - A^{-8}x + A^{-12}x - 2A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + 2A^{12}x - A^{8}x + A^{4}x - x + A^{-4}x$
$\mathcal{S}_{2,\infty}(4_4) = A^{2} + 1 - 2A^{-4} - A^{-6} - A^{-8} - A^{-14} - A^{-16} + A^{-20} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10} - A^8 + A^6 + A^4 + A^2 - 1 - A^{-2} - A^{-4} + A^{-6} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(4_5) = -2A^4x + 2x - A^{-4}x + A^{-8}x - A^{-12}x + A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x + x - 2A^{-4}x + A^{-8}x + A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(4_6) = -A^{-6} - A^{-8} - A^{-10} + A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{4_6}) = -A^{22} + A^{18} + A^{16} - A^{10} - A^2 - 1$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + 2A^{-12}x - A^{-16}x + A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - 2A^{16}x + A^{12}x - A^8x + 2A^4x$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^8x + A^4x + A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x + 2A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(4_9) = A^{14} + A^{12} - A^8 - A^6 - A^4 - A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{4_9}) = A^2 - A^{-2} - A^{-4} - A^{-12} - A^{-14} - A^{-16} + A^{-18} + A^{-20}$
$\mathcal{S}_{2,\infty}(4_{10}) = -A^{-2} - A^{-4} - A^{-14} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{4_{10}}) = -A^{18} + A^{12} - A^{-2} - A^{-4}$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^4x + 2x - A^{-4}x + A^{-8}x - A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^8x - A^4x + 2x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2 + 1 - A^{-2} - 2A^{-4} - A^{-6} + A^{-10} - A^{-14} - 2A^{-16} + A^{-20} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - A^{10} - 2A^8 + A^4 + A^2 - A^{-2} - A^{-4} + A^{-6} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(4_{13}) = A^8x - 2A^4x + 2x - A^{-4}x + 2A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 3A^4x + 2x - 2A^{-4}x + 2A^{-8}x - A^{-12}x + A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6 - A^4 + 1 - A^{-4} - A^{-6} + A^{-10} + 2A^{-12} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + A^6 + 2A^4 - 1 - 2A^{-2} - A^{-4} - A^{-10} - A^{-12} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x - A^8x + A^4x + A^{-4}x$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6 + A^4 - 1 - A^{-2} - A^{-4} - A^{-8} - A^{-10} - A^{-12} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^4 - A^{-4} + A^{-10} + A^{-12} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(4_{18}) = -x + A^{-4}x - A^{-8}x + A^{-12}x - A^{-16}x + A^{-20}x + A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x - x + A^{-4}x + A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10} - A^8 + A^4 - A^{-2} + A^{-6} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = 1 - A^{-2} - A^{-4} - A^{-6} - A^{-14} - A^{-16} + A^{-20} + A^{-22}$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6} - A^{-8} - A^{-10} + A^{-12} - A^{-18} - A^{-20} + A^{-24} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14} - A^{12} - A^{10} + A^8 + A^6 + A^4 - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x + x - A^{-4}x + A^{-16}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x + A^{-12}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^8x - A^4x + A^{-4}x + A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{26}}) = -A^{10} + A^6 - 1 + A^{-6} - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x + A^{-12}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = x$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - A^{-10} - 2A^{-12} - A^{-14} + A^{-18} - A^{-22} - A^{-24} + A^{-28} + A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x + A^{-8}x + A^{-16}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(5_4) = A^{12} + A^{10} + A^8 - A^4 - A^2 - A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{-12}x$
$\mathcal{S}_{2,\infty}(5_6) = A^8 + A^6 + A^{-6} - A^{-16} - A^{-18} - A^{-20} - A^{-22} - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = -A^{-8} + A^{-12} + A^{-14} + A^{-16} + A^{-18} - A^{-28} - A^{-30} - A^{-32} - A^{-34} - A^{-36}$
$\mathcal{S}_{2,\infty}(5_7) = -A^{-6} - 2A^{-8} - A^{-10} + A^{-12} - A^{-18} + A^{-22} + 2A^{-24} + A^{-26} - A^{-28} - A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{16} - A^{14} - A^{12} - A^{10} + A^8 + A^6 - A^2 - 1 + A^{-4} + A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_8) = -x + A^{-4}x - A^{-8}x + 2A^{-12}x + A^{-20}x - A^{-28}x$

$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x - 2A^8x + A^4x + A^{-4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(5_9) = A^6 + 2A^4 - 1 - A^{-2} + A^{-6} - A^{-12} + A^{-16} + A^{-18} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^4 + 1 - A^{-4} + A^{-8} + 2A^{-10} + 2A^{-12} + A^{-14} - A^{-16} - A^{-18} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(5_{10}) = 2A^{-4}x - 2A^{-8}x + 2A^{-12}x - 2A^{-16}x + 2A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - 2A^{16}x + 2A^{12}x - 2A^8x + 2A^4x - x + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{11}) = A^2 + 1 - A^{-2} - 3A^{-4} - A^{-6} + A^{-10} - A^{-14} - A^{-16} + A^{-18} + 2A^{-20} + A^{-22} - A^{-24} - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14} + A^{12} - A^{10} - 2A^8 + A^4 + A^2 - 1 - 2A^{-2} - 2A^{-4} + A^{-6} + 2A^{-8} + A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{12}) = -2x + 3A^{-4}x - 2A^{-8}x + 3A^{-12}x - 2A^{-16}x + 2A^{-20}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 3A^{12}x - 3A^8x + 2A^4x - 2x + 2A^{-4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^6 - A^4 + 2 - A^{-4} - A^{-6} + A^{-10} + A^{-12} - A^{-14} - 2A^{-16} - A^{-18} + A^{-20} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10} - A^8 + A^6 + 2A^4 - 1 - 2A^{-2} + A^{-6} + A^{-8} - A^{-10} - 2A^{-12} - A^{-14} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{14}) = -2A^4x + 3x - 2A^{-4}x + 2A^{-8}x - A^{-12}x + 2A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x - 2A^4x + 3x - 2A^{-4}x + A^{-8}x - A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10} - A^8 + 2A^6 + 3A^4 + A^2 - 2 - 2A^{-2} + 2A^{-6} + 2A^{-8} - 2A^{-12} - A^{-14} + A^{-16} + A^{-18} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6 - A^4 + A^2 + 3 + A^{-2} - 2A^{-4} - A^{-6} + 2A^{-10} + 2A^{-12} - 2A^{-16} - A^{-18} + A^{-20} + A^{-22} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(5_{16}) = 2A^8x - 3A^4x + 3x - 2A^{-4}x + 2A^{-8}x - 2A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 3A^4x + 3x - 2A^{-4}x + 3A^{-8}x - A^{-12}x + A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14} + A^{12} - A^{10} - A^8 + 2A^4 + A^2 - 2A^{-2} - 2A^{-4} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2 + 1 - A^{-2} - 2A^{-4} - A^{-6} + A^{-8} + A^{-10} + A^{-12} - A^{-14} - A^{-16} + A^{-20} - A^{-24} - A^{-26}$
$\mathcal{S}_{2,\infty}(5_{18}) = -A^{-6} - 2A^{-8} + A^{-12} + A^{-14} - A^{-16} - A^{-18} - A^{-20} + A^{-22} + 2A^{-24} + A^{-26} - A^{-28} - A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18} + A^{16} - 2A^{12} - A^{10} + A^6 - A^2 - 1 + A^{-4} + A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{19}) = -2A^4x + 2x - A^{-4}x + 2A^{-8}x + A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x - 2A^4x + 2x - A^{-4}x + A^{-8}x - A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{20}) = A^2 + 2 - 2A^{-4} - A^{-6} + A^{-10} + A^{-12} - A^{-16} + A^{-20} + A^{-22} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10} - A^8 + A^6 + 2A^4 + A^2 - 1 - A^{-2} + 2A^{-6} + 2A^{-8} + A^{-10} - A^{-12} - A^{-14} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(5_{21}) = 2A^{12}x - 2A^8x + A^4x - x + A^{-4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x + 2A^{-4}x - A^{-8}x + 2A^{-12}x - A^{-16}x + A^{-20}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^6 - A^4 + 1 + A^{-2} - A^{-4} - A^{-8} + A^{-10} + A^{-12} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = 2A^6 + A^4 - 2 - 2A^{-2} - A^{-4} - A^{-10} - A^{-12} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x - A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{12} + A^{10} + A^8 + A^6 - A^{-2} - A^{-4} - A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6} - A^{-8} + A^{-10} + A^{-12} - 2A^{-16} - A^{-18} + A^{-22} + A^{-24} - A^{-26} - 2A^{-28} + A^{-32} + A^{-34}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22} - 2A^{20} + A^{18} + 2A^{16} + 2A^{14} - A^{10} - A^8 + A^6 + A^4 - A^2 - 2 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{26}) = -x + 3A^{-4}x - 2A^{-8}x + 3A^{-12}x - 3A^{-16}x + 2A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - 3A^{16}x + 3A^{12}x - 2A^8x + 3A^4x - 2x + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{27}) = -x + 3A^{-4}x - 2A^{-8}x + 3A^{-12}x - 3A^{-16}x + 2A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - 3A^{16}x + 3A^{12}x - 2A^8x + 3A^4x - 2x + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14} + A^{12} - A^{10} - A^8 + A^4 + A^2 - A^{-2} - A^{-4} + A^{-8} - A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2 + 2 - A^{-4} - A^{-6} - 2A^{-14} - 2A^{-16} + A^{-20}$
$\mathcal{S}_{2,\infty}(5_{29}) = -A^6 - A^4 + 1 - A^{-6} + A^{-12} - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{29}}) = A^{14} - 2A^{10} - A^8 + A^4 - A^{-2} + A^{-6} + A^{-8} - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(5_{30}) = A^8x - A^4x + 2x - 2A^{-4}x + A^{-8}x - A^{-12}x + A^{-16}x$

$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + 2A^{8}x - A^{4}x + 2x - 2A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{31}) = A^{8}x - 2A^{4}x + 3x - 2A^{-4}x + 2A^{-8}x - 2A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + 3A^{8}x - 2A^{4}x + 2x - 3A^{-4}x + 2A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{32}) = A^{8}x - 2A^{4}x + 2x - 2A^{-4}x + 2A^{-8}x - A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + 2A^{8}x - 2A^{4}x + 2x - 2A^{-4}x + 2A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{33}) = -A^{6} - A^{4} + 1 + A^{-2} - A^{-8} - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{33}}) = A^{14} - A^{10} - A^{8} + A^{6} - 1 - A^{-2} + A^{-6} + A^{-8} - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^{6} - A^{4} + A^{2} + 2 + A^{-2} - A^{-6} + A^{-12} - A^{-14} - 2A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - 2A^{10} - A^{8} + A^{6} + 2A^{4} + A^{2} - A^{-2} + A^{-6} + A^{-8} - 2A^{-10} - 2A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6} - A^{-8} + A^{-12} - A^{-14} - A^{-16} - A^{-18} + A^{-20} + A^{-22} + A^{-24} - A^{-26} - 2A^{-28} + A^{-32} + A^{-34}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22} - 2A^{20} + 2A^{16} + A^{14} + A^{12} - A^{10} + A^{6} + A^{4} - A^{2} - 2 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{36}) = -x + 2A^{-4}x - 2A^{-8}x + 3A^{-12}x - 2A^{-16}x + 2A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - 3A^{16}x + 2A^{12}x - 2A^{8}x + 3A^{4}x - x + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{37}) = A^{14} + A^{12} - A^{10} - A^{8} - A^{6} - A^{-2} - A^{-4} + A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{37}}) = -A^{6} + A^{2} + 1 - A^{-2} - A^{-4} - A^{-6} - A^{-14} - A^{-16} + A^{-18} + A^{-20}$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-8} - A^{-10} - A^{-12} - A^{-14} + A^{-18} - A^{-22} - A^{-24} + A^{-28} + A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^{4}x + x - A^{-4}x + 2A^{-8}x + A^{-16}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x + A^{-16}x - A^{-20}x + A^{-24}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x - A^{12}x + A^{8}x + x$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{8} - A^{2} - A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x + A^{12}x - 2A^{8}x + A^{4}x + A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{43}) = A^{6} + A^{4} - 1 - A^{-2} - A^{-10} + A^{-16} - A^{-20} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{8} + 1 - A^{-4} - A^{-6} - A^{-8} + A^{-12} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{44}) = -x + 2A^{-4}x - 2A^{-8}x + 2A^{-12}x - A^{-16}x + 2A^{-20}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x - 2A^{8}x + 2A^{4}x - x + A^{-4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10} - A^{8} + A^{6} + 2A^{4} + A^{2} - 1 - A^{-2} - A^{-4} + A^{-6} - A^{-10} - 2A^{-12} - A^{-14} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^{4} + A^{2} + 1 - 2A^{-4} - A^{-6} + A^{-10} + A^{-12} - A^{-14} - 2A^{-16} - A^{-18} + A^{-20} + A^{-22}$
$\mathcal{S}_{2,\infty}(5_{46}) = -x + 3A^{-4}x - 2A^{-8}x + 2A^{-12}x - 2A^{-16}x + 2A^{-20}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 3A^{12}x - 2A^{8}x + 2A^{4}x - 2x + A^{-4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10} - A^{8} + A^{6} + 3A^{4} + A^{2} - 1 - A^{-2} + 2A^{-6} + A^{-8} - 2A^{-12} - A^{-14} + A^{-16} + A^{-18} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^{4} + A^{2} + 2 - 2A^{-4} - A^{-6} + A^{-8} + 2A^{-10} + 2A^{-12} - 2A^{-16} - A^{-18} + A^{-20} + A^{-22} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(5_{48}) = A^{4}x - x + 2A^{-4}x - A^{-8}x + A^{-12}x - 2A^{-16}x + A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^{8}x + A^{4}x - x + 2A^{-4}x + A^{-12}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14} + A^{12} - A^{8} + A^{4} - 2A^{-2} - A^{-4} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = 1 - A^{-2} - A^{-4} + A^{-8} + A^{-10} - A^{-14} - A^{-16} + A^{-20} - A^{-24} - A^{-26}$
$\mathcal{S}_{2,\infty}(5_{50}) = -A^{-6} - A^{-8} - A^{-16} + A^{-22} + A^{-24} - A^{-28}$
$\mathcal{S}_{2,\infty}(\overline{5_{50}}) = A^{26} - A^{22} - A^{20} + A^{18} + A^{16} + A^{14} - A^{10} - A^{8} - A^{2} - 1$
$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + 3A^{-12}x - 2A^{-16}x + A^{-20}x - 2A^{-24}x + A^{-28}x$
$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^{8}x + 2A^{4}x + A^{-4}x - A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^{4}x - x + A^{-4}x - A^{-8}x + 2A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2} + A^{-4} - A^{-8} - A^{-10} - A^{-18} - A^{-20} - A^{-22} + A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18} + A^{16} + A^{8} + A^{6} - A^{2} - 1 - A^{-6} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + 2A^{-4}x - A^{-8}x + 3A^{-12}x - 2A^{-16}x + A^{-20}x - A^{-24}x$

$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18} - A^{16} + A^{14} + 2A^{12} + A^{10} - A^{8} - A^{6} - A^{4} + A^{2} + 1 - 2A^{-4} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^{2} - A^{-2} - A^{-4} + A^{-6} + A^{-8} - 2A^{-12} - 2A^{-14} - A^{-16} + A^{-18} + A^{-20} - A^{-22} - A^{-24} + A^{-28} + A^{-30}$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14} + A^{12} - A^{8} - A^{-2} + A^{-8} - A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^{6} + A^{2} + 2 - A^{-12} - 2A^{-14} - 2A^{-16} + A^{-20}$
$\mathcal{S}_{2,\infty}(5_{57}) = -A^{-2} - A^{-4} + A^{-18} - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_{57}}) = A^{22} - A^{16} - A^{-2} - A^{-4}$
$\mathcal{S}_{2,\infty}(5_{58}) = -A^{4}x + 2x - A^{-4}x + 2A^{-8}x - A^{-12}x + A^{-16}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x + A^{8}x - 2A^{4}x + 2x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{59}) = A^{2} + 1 - A^{-2} - 2A^{-4} + A^{-8} + A^{-10} - 2A^{-14} - A^{-16} + 2A^{-20} - A^{-24} - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + A^{14} + 2A^{12} - A^{8} + A^{4} + A^{2} - 2A^{-2} - 2A^{-4} + A^{-8} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{60}) = A^{8}x - 2A^{4}x + 3x - 2A^{-4}x + 2A^{-8}x - 2A^{-12}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + 2A^{8}x - 2A^{4}x + 3x - 2A^{-4}x + 2A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{61}) = A^{2} + 1 - 2A^{-2} - 3A^{-4} - A^{-6} + A^{-8} + A^{-10} - 2A^{-14} - A^{-16} + A^{-18} + 3A^{-20} + A^{-22} - A^{-24} - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + A^{14} + 2A^{12} - A^{10} - 2A^{8} - A^{6} + A^{4} + A^{2} - 2A^{-2} - 2A^{-4} + A^{-6} + 2A^{-8} + A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{62}) = A^{8}x - 4A^{4}x + 4x - 3A^{-4}x + 4A^{-8}x - 2A^{-12}x + 2A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 3A^{8}x - 4A^{4}x + 4x - 3A^{-4}x + 3A^{-8}x - 2A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^{6} - A^{4} + A^{2} + 2 - 2A^{-4} - A^{-6} + 2A^{-10} + A^{-12} - A^{-14} - 3A^{-16} - A^{-18} + A^{-20} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - A^{10} - 2A^{8} + A^{6} + 2A^{4} + A^{2} - 1 - 2A^{-2} - A^{-4} + A^{-6} + A^{-8} - A^{-10} - 2A^{-12} - A^{-14} + A^{-16} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{64}) = A^{8}x - 3A^{4}x + 4x - 3A^{-4}x + 3A^{-8}x - 2A^{-12}x + 2A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 3A^{8}x - 3A^{4}x + 4x - 3A^{-4}x + 2A^{-8}x - 2A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^{6} - A^{4} + 2A^{2} + 4 - 3A^{-4} - 2A^{-6} + A^{-8} + 3A^{-10} + 2A^{-12} - A^{-14} - 4A^{-16} - A^{-18} + 2A^{-20} + 2A^{-22} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 2A^{10} - 3A^{8} + A^{6} + 4A^{4} + 2A^{2} - 1 - 2A^{-2} + 3A^{-6} + 3A^{-8} - 3A^{-12} - 2A^{-14} + A^{-16} + A^{-18} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10} + A^{8} - A^{4} + 1 - A^{-4} - 2A^{-6} - A^{-8} + 2A^{-12} - A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + A^{6} + 2A^{4} - 1 - A^{-2} + A^{-6} - A^{-10} - A^{-12} + A^{-16} - A^{-20} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{16} + A^{8} + A^{6} - A^{2} - 1 - A^{-2} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^{8}x + A^{-4}x + A^{-8}x + A^{-12}x - A^{-20}x$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^{-4}x - A^{-16}x + A^{-24}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-4} - A^{-8} + A^{-12} + A^{-14} - A^{-20} + A^{-24} + A^{-26} - A^{-28} - A^{-30} - A^{-32} - A^{-34} - A^{-36}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{12} + A^{8} + A^{6} + 1 + A^{-2} + A^{-4} + A^{-6} - A^{-8} - A^{-10} - A^{-16} - A^{-18} - A^{-20} - A^{-22} - A^{-24}$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x - A^{8}x + A^{4}x - A^{-8}x + A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = A^{-4}x - A^{-8}x + A^{-12}x + A^{-20}x - A^{-28}x$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^{6} + 1 + A^{-4} - A^{-8} - A^{-10} - A^{-12} - A^{-14} - A^{-16} + A^{-24} + A^{-26}$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x + A^{-24}x - A^{-32}x$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x - A^{4}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^{6} + A^{4} - 1 - A^{-2} + A^{-6} + A^{-8} + A^{-10} - A^{-20} - A^{-22} - A^{-24} - A^{-26} - A^{-28}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = 1 - A^{-4} + A^{-10} + A^{-12} + A^{-14} - A^{-24} - A^{-26} - A^{-28} - A^{-30} - A^{-32}$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^{2} - A^{-8} - A^{-10} - A^{-12} - A^{-14} - A^{-16} + A^{-24} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^{6} + A^{4} + A^{-4} + A^{-6} - 2A^{-10} - 2A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-8} - A^{-16} - A^{-26} - A^{-28} + A^{-32} + A^{-34}$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22} - A^{20} + A^{18} + A^{16} + A^{14} + A^{6} - A^{2} - 2 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{12} - A^{-10} - A^{-12} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^{2} + 1 - A^{-10} - A^{-12} - 2A^{-14} - A^{-16} + A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22} - A^{20} + A^{16} + A^{14} + 2A^{12} - A^{2} - 1 - A^{-2} - A^{-4} - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6} - A^{-8} + 2A^{-12} + A^{-14} - A^{-18} - A^{-20} + A^{-24} - A^{-28} - A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18} + A^{16} - A^{12} - A^{10} + A^{8} + A^{6} + A^{4} - A^{2} - 1 - A^{-2} - A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^{4}x + 2x - A^{-4}x + A^{-8}x + A^{-16}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^{8}x - A^{4}x + 2x - A^{-4}x - A^{-12}x + A^{-20}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}+A^{-4}-2A^{-8}-A^{-10}+A^{-12}+A^{-14}+A^{-16}-A^{-20}+A^{-24}+A^{-26}-A^{-28}-A^{-30}-A^{-32}-A^{-34}-A^{-36}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14} - A^{12} + 2A^{8} + 2A^{6} + A^{-2} + A^{-4} + A^{-6} - A^{-8} - A^{-10} - A^{-16} - A^{-18} - A^{-20} - A^{-22} - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{6} + A^{4} + A^{2} + 1 + A^{-2} - A^{-8} - A^{-10} - A^{-12} - A^{-14} - A^{-16} - A^{-18} - A^{-20}$
$\mathcal{S}_{2,\infty}(5_{86}) = A^{14} - A^{10} - A^{8} + A^{4} - 2A^{-2} - A^{-4} + A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{86}}) = -A^{6} + 1 - A^{-2} - A^{-4} - A^{-6} + A^{-10} + A^{-12} - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{88}}) = A^{26} - A^{22} + A^{18} + A^{16} - A^{12} - A^{10} - A^{8} + A^{6} - A^{2} - 1$
$\mathcal{S}_{2,\infty}(\overline{5_{89}}) = A^{26} - A^{22} + A^{18} + A^{16} - A^{14} - A^{12} - 2A^{10} + A^{6} + A^{4} - A^{2} - 1$
$\mathcal{S}_{2,\infty}(5_{90}) = -A^{-2} + A^{-6} - A^{-10} - A^{-12} - A^{-14} - A^{-22} + A^{-26} + A^{-28}$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + 3A^{-12}x - A^{-16}x + 2A^{-20}x - 2A^{-24}x$
$\mathcal{S}_{2,\infty}(5_{94}) = -A^{-6} - A^{-14} - A^{-16} - A^{-26} + A^{-30} + A^{-32}$
$\mathcal{S}_{2,\infty}(5_{95}) = -A^{-6} - 2A^{-14} - A^{-16} + A^{-20} + A^{-22} - A^{-26} - A^{-28} + A^{-30} + A^{-32}$

## Knots in $L(3,1)$

$\mathcal{S}_{2,\infty}(1_1) = -A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{1_1}) = A^{6} - Ax - A^{-2}$
$\mathcal{S}_{2,\infty}(2_1) = -A^{-2} - A^{-3}x - A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{2_1}) = -A^{10} + A^{5}x - Ax - A^{-2}$
$\mathcal{S}_{2,\infty}(2_2) = -A^{4}x + x + A^{-3} + A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^{8}x + A + x + A^{-3}$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x + A^{-7} - A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^{8}x - A^{5} + x + A^{-3}$
$\mathcal{S}_{2,\infty}(3_2) = A^{6} + A^{5}x - Ax - A^{-2} - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = Ax - A^{-3}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(3_3) = -A^{-2} - A^{-3}x + A^{-7}x + A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{3_3}) = A^{14} - A^{9}x + A^{5}x - Ax - A^{-2}$
$\mathcal{S}_{2,\infty}(3_4) = -A^{4}x + 2x + A^{-3} - A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^{8}x - A^{4}x - A + x + A^{-7}$
$\mathcal{S}_{2,\infty}(3_5) = A^{2} + Ax - 2A^{-3}x - A^{-6} + A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + A^{6} + 2A^{5}x + A^{2} - 2Ax - A^{-2} - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{9}x + A^{6} - Ax - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{3_7}) = A^{18} - A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(3_8) = -A^{-2} - A^{-10} - A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-7}x - A^{-15}x + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{13}x + A^{9}x + A^{6} - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(4_2) = -A^{8}x - A^{5} + x + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(4_3) = 2A^{-4}x + A^{-7} - A^{-8}x - A^{-11} - A^{-15} + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + 2A^{12}x + A^{9} - A^{8}x - A^{5} - A + x + A^{-3}$

$\mathcal{S}_{2,\infty}(4_4) = A^2 + Ax - 3A^{-3}x - A^{-6} + A^{-7}x - A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10} - A^9x + A^6 + 2A^5x + A^2 - 2Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(4_5) = -2A^4x - A + 2x + A^{-3} - A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x - 2A + x + A^{-7} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(4_6) = -A^{-6} - A^{-7}x - A^{-10} + A^{-11}x - A^{-15}x + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{4_6}) = -A^{22} + A^{18} + A^{17}x - A^{13}x - A^{10} + A^9x - A^5x - A^2$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + 2A^{-12}x + A^{-15} - A^{-16}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - 2A^{16}x - A^{13} + A^{12}x + A^5 + A^4x$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^8x + A^4x + A + A^{-3}$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x + A^{-7} + A^{-11} + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_9}) = A^2 - A^{-2} - A^{-3}x + A^{-7}x - 2A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(4_{10}) = -A^{-2} - A^{-3}x + A^{-7}x - A^{-11}x - A^{-14} + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_{10}}) = -A^{18} + A^{13}x - A^9x + A^5x - Ax - A^{-2}$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^4x + 2x + A^{-3} - A^{-4}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^8x + A^5 - A^4x + x + A^{-7}$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2 + Ax - A^{-2} - 3A^{-3}x - A^{-6} + 2A^{-7}x + A^{-10} - 2A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - A^{10} - 2A^9x + 3A^5x + A^2 - 2Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(4_{13}) = A^8x - 2A^4x - A + 2x + A^{-3} + A^{-7} - A^{-11}$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 3A^4x - 2A + 2x + A^{-3} + A^{-7} - A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6 - A^5x + 2Ax - 2A^{-3}x - A^{-6} + 2A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + A^6 + 2A^5x - 3Ax - 2A^{-2} + A^{-3}x - A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(4_{16}) = A^{-4}x + A^{-7} - A^{-8}x + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x + A^9 - A^8x + x + A^{-3}$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6 + A^5x - 2Ax - A^{-2} - A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^5x + Ax - A^{-3}x + A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(4_{18}) = -x - A^{-3} + A^{-4}x + A^{-7} - A^{-15} + A^{-16}x + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x - 2A^5 + x + A^{-3} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10} - A^9x + 2A^5x - Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = Ax - A^{-2} - 2A^{-3}x - A^{-6} + A^{-7}x - A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14} - A^{13}x - A^{10} + 2A^9x + A^6 - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x - A + x + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{4_{22}}) = -A^{16}x + A^5 + A^4x + A$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x + A^{-11} + A^{-12}x - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^9 - A^8x - A^5 + x + A^{-3} + A^{-4}x + A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{4_{25}}) = -A^{10} - A^2 + A^{-3}x + A^{-6} - A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{26}}) = -A^{10} + A^6 - Ax + A^{-3}x + A^{-6} - A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(4_{27}) = A^8x - 2A^4x + x + A^{-3} - A^{-4}x + A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x + A^{-11} + A^{-12}x - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = -A^9 + A^4x + A$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - A^{-10} - 2A^{-11}x - A^{-14} + A^{-15}x + A^{-18} - A^{-19}x + A^{-23}x$
$\mathcal{S}_{2,\infty}(5_3) = -A^{12}x - A^9 + A^8x + A^5 + A^4x + A - A^{-3}$
$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x - A^{-7} + A^{-8}x + A^{-11} + A^{-12}x + A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(5_4) = A^{13}x + A^{10} - A^5x - A^2 - A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{13} - A^8x + A^{-4}x + A^{-7}$
$\mathcal{S}_{2,\infty}(5_6) = A^9x + A^6 - A^{-2} - A^{-10} - A^{-11}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_7) = -A^{-6} - 2A^{-7}x - A^{-10} + 2A^{-11}x - A^{-15}x + 2A^{-19}x - A^{-23}x$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{17}x - A^{14} - 2A^{13}x - A^{10} + 2A^{9}x + A^{6} - A^{5}x + Ax - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_8) = -x - A^{-3} + A^{-4}x + 2A^{-7} + A^{-11} + A^{-16}x + A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x + A^{9} - 2A^{8}x - A^{5} + A + x + A^{-3} - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_9) = A^{6} + 2A^{5}x - 2Ax - A^{-2} + A^{-3}x - A^{-7}x + A^{-11}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^{5}x + 2Ax - A^{-3}x + 2A^{-7}x - A^{-11}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(5_{10}) = 2A^{-4}x + A^{-7} - 2A^{-8}x - A^{-11} + A^{-12}x + A^{-19} - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - 2A^{16}x - A^{13} + 2A^{12}x + A^{9} - A^{8}x - A + x + A^{-3}$
$\mathcal{S}_{2,\infty}(5_{11}) = A^{2} + Ax - A^{-2} - 4A^{-3}x - A^{-6} + 3A^{-7}x + A^{-10} - 2A^{-11}x + 2A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14} + A^{13}x - A^{10} - 3A^{9}x + 3A^{5}x + A^{2} - 3Ax - A^{-2} + 2A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{12}) = -2x - A^{-3} + 3A^{-4}x + 3A^{-7} - A^{-8}x - A^{-15} + A^{-16}x + 2A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 3A^{12}x + 2A^{9} - 3A^{8}x - 2A^{5} + x + 2A^{-3} - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^{6} - A^{5}x + 3Ax - 3A^{-3}x - A^{-6} + 2A^{-7}x - 2A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10} - A^{9}x + A^{6} + 3A^{5}x - 3Ax - 2A^{-2} + 2A^{-3}x - 2A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{14}) = -2A^{4}x - A + 3x + 2A^{-3} - A^{-4}x + A^{-12}x + 2A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^{8}x + 2A^{5} - 2A^{4}x - A + x + A^{-7} - A^{-8}x - A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10} - A^{9}x + 2A^{6} + 4A^{5}x + A^{2} - 4Ax - 2A^{-2} + 3A^{-3}x - 2A^{-7}x + A^{-11}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^{6} - A^{5}x + A^{2} + 4Ax + A^{-2} - 4A^{-3}x - A^{-6} + 3A^{-7}x - 2A^{-11}x + A^{-15}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(5_{16}) = 2A^{8}x + A^{5} - 3A^{4}x - 2A + 2x + A^{-3} + A^{-7} - A^{-8}x - 2A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^{8}x - 3A^{4}x - 2A + 3x + 2A^{-3} + A^{-7} - A^{-11} + A^{-12}x + A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14} + A^{13}x - A^{10} - 2A^{9}x + 3A^{5}x + A^{2} - 3Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^{2} + Ax - A^{-2} - 3A^{-3}x - A^{-6} + 3A^{-7}x + A^{-10} - 2A^{-11}x + A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(5_{18}) = -A^{-6} - 2A^{-7}x + 2A^{-11}x + A^{-14} - 2A^{-15}x + 2A^{-19}x - A^{-23}x$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18} + A^{17}x - 3A^{13}x - A^{10} + 2A^{9}x + A^{6} - A^{5}x + Ax - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{19}) = -2A^{4}x - A + 2x + 2A^{-3} + A^{-7} + A^{-12}x + A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^{8}x + A^{5} - 2A^{4}x - A + x + A^{-3} + A^{-7} - A^{-8}x - A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{20}) = A^{2} + 2Ax - 3A^{-3}x - A^{-6} + 2A^{-7}x - A^{-11}x + A^{-15}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10} - A^{9}x + A^{6} + 3A^{5}x + A^{2} - 2Ax - A^{-2} + 2A^{-3}x - A^{-7}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{21}) = 2A^{12}x + A^{9} - 2A^{8}x - 2A^{5} + x + A^{-3} - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x - A^{-3} + 2A^{-4}x + 2A^{-7} - A^{-15} + A^{-16}x + A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^{6} - A^{5}x + 2Ax + A^{-2} - 2A^{-3}x + A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = 2A^{6} + A^{5}x - 3Ax - 2A^{-2} + A^{-3}x - A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{23}) = -A^{12}x + A^{8}x + A^{5} + x - A^{-3}$
$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x - A^{-7} + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{13}x + A^{10} + A^{6} - Ax - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6} - A^{-7}x + A^{-10} + 2A^{-11}x - 4A^{-15}x - A^{-18} + 3A^{-19}x + A^{-22} - 2A^{-23}x + A^{-27}x$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22} - 2A^{21}x + A^{18} + 4A^{17}x + 2A^{14} - 3A^{13}x - A^{10} + 2A^{9}x - 2A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(5_{26}) = -x + 3A^{-4}x + A^{-7} - 2A^{-8}x - A^{-11} + 2A^{-12}x - A^{-16}x + A^{-19} - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - 3A^{16}x - A^{13} + 3A^{12}x + A^{9} - A^{8}x + A^{4}x - A + A^{-3}$
$\mathcal{S}_{2,\infty}(5_{27}) = -x + 3A^{-4}x + A^{-7} - 2A^{-8}x - A^{-11} + 2A^{-12}x - A^{-16}x + A^{-19} - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - 3A^{16}x - A^{13} + 3A^{12}x + A^{9} - A^{8}x + A^{4}x - A + A^{-3}$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14} + A^{13}x - A^{10} - 2A^{9}x + 3A^{5}x + A^{2} - 3Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^{6} + 2A^{2} + 2Ax - 3A^{-3}x - A^{-6} + 2A^{-7}x - 3A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{29}}) = A^{14} - 2A^{10} - A^{9}x + 2A^{5}x - 2Ax - A^{-2} + 2A^{-3}x + A^{-6} - A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{30}) = A^{8}x - A^{4}x + 2x - 2A^{-4}x - A^{-7} + A^{-8}x + A^{-15}$

$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + 2A^8x + A^5 - A^4x + x - A^{-3} - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{31}) = A^8x - 2A^4x + 3x + A^{-3} - 2A^{-4}x - A^{-7} + A^{-8}x - A^{-11} + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + 3A^8x + A^5 - 2A^4x - A + x - A^{-3} - A^{-4}x + A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{32}) = A^8x - 2A^4x + 2x + A^{-3} - 2A^{-4}x + A^{-8}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + 2A^8x + A^5 - 2A^4x + x - A^{-4}x + A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{33}}) = A^{14} - A^{10} - A^9x + A^6 + A^5x - 2Ax - A^{-2} + 2A^{-3}x + A^{-6} - A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^6 - A^5x + A^2 + 3Ax + A^{-2} - 3A^{-3}x - A^{-6} + 2A^{-7}x - 2A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - 2A^{10} - A^9x + A^6 + 3A^5x + A^2 - 3Ax - A^{-2} + 2A^{-3}x + A^{-6} - 2A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6} - A^{-7}x + 2A^{-11}x - A^{-14} - 3A^{-15}x - A^{-18} + 3A^{-19}x + A^{-22} - 2A^{-23}x + A^{-27}x$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22} - 2A^{21}x + 4A^{17}x + A^{14} - 2A^{13}x - A^{10} + 2A^9x - 2A^5x - A^2$
$\mathcal{S}_{2,\infty}(5_{36}) = -x + 2A^{-4}x + A^{-7} - 2A^{-8}x + 2A^{-12}x + A^{-15} - A^{-16}x + A^{-19} - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - 3A^{16}x - A^{13} + 2A^{12}x + A^9 - A^8x + A^5 + A^4x + A^{-3}$
$\mathcal{S}_{2,\infty}(5_{37}) = A^{14} + A^{13}x - A^{10} - 2A^9x - A^6 + 2A^5x - 2Ax - A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{37}}) = -A^6 + A^2 + Ax - A^{-2} - 2A^{-3}x - A^{-6} + 2A^{-7}x - 2A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-7}x - A^{-10} - 2A^{-11}x - A^{-14} + A^{-15}x + A^{-18} - A^{-19}x + A^{-23}x$
$\mathcal{S}_{2,\infty}(5_{39}) = -A^{12}x - A^9 + A^8x + A^5 + A + A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^4x + x + A^{-3} - A^{-4}x + A^{-8}x + A^{-11} + A^{-12}x + A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x + A^{-11} - A^{-27}$
$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x - A^{17} + A^4x + A$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^9x - A^5x - A^2 - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{42}) = A^{-4}x + A^{-7} - A^{-8}x + A^{-12}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x - A^{13} + A^{12}x - A^8x + x + A^{-3}$
$\mathcal{S}_{2,\infty}(5_{43}) = A^6 + A^5x - 2Ax - A^{-2} + A^{-3}x - A^{-7}x + A^{-11}x - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^9x - A^5x + Ax - 2A^{-3}x + A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{44}) = -x - A^{-3} + 2A^{-4}x + 2A^{-7} - A^{-8}x + A^{-16}x + 2A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x + 2A^9 - 2A^8x - A^5 + x + A^{-3} - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10} - A^9x + A^6 + 3A^5x + A^2 - 3Ax - A^{-2} + A^{-3}x - 2A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^5x + A^2 + 2Ax - 3A^{-3}x - A^{-6} + 2A^{-7}x - 2A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(5_{46}) = -x - A^{-3} + 3A^{-4}x + 2A^{-7} - A^{-8}x - A^{-11} - A^{-15} + A^{-16}x + 2A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 3A^{12}x + 2A^9 - 2A^8x - 2A^5 - A + x + A^{-3} - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10} - A^9x + A^6 + 4A^5x + A^2 - 3Ax - A^{-2} + 2A^{-3}x - 2A^{-7}x + A^{-11}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^5x + A^2 + 3Ax - 3A^{-3}x - A^{-6} + 3A^{-7}x - 2A^{-11}x + A^{-15}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x + A - x - A^{-3} + A^{-4}x + A^{-7} - A^{-12}x - 2A^{-15} + A^{-16}x + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x - 2A^5 + A^4x + A + x + A^{-3} + A^{-8}x + A^{-11} - A^{-12}x - A^{-15}$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14} + A^{13}x - 2A^9x + 2A^5x - 2Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = Ax - A^{-2} - 2A^{-3}x + 2A^{-7}x + A^{-10} - 2A^{-11}x + A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(5_{50}) = -A^{-6} - A^{-7}x + A^{-11}x - 2A^{-15}x + 2A^{-19}x + A^{-22} - A^{-23}x$
$\mathcal{S}_{2,\infty}(\overline{5_{50}}) = A^{26} - A^{22} - A^{21}x + A^{18} + 2A^{17}x + A^{14} - 2A^{13}x - A^{10} + A^9x - A^5x - A^2$
$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + 3A^{-12}x + A^{-15} - 2A^{-16}x - A^{-19} - A^{-23} + A^{-27}$
$\mathcal{S}_{2,\infty}(\overline{5_{51}}) = -A^{24}x + 2A^{20}x + A^{17} - 2A^{16}x - A^{13} + A^{12}x - A^9 + A^5 + A^4x$
$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^8x + 2A^4x + A - A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^4x - x - A^{-3} + A^{-4}x + A^{-11} + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2} + A^{-3}x - 2A^{-7}x - A^{-10} + A^{-11}x - 2A^{-15}x + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18} + A^{17}x - A^{13}x + 2A^9x + A^6 - 2A^5x - A^2 - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{54}) = A^{20}x - 2A^{16}x - A^{13} + 2A^{12}x - A^8x + A^4x + A^{-3}$

$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + 2A^{-4}x + A^{-7} - A^{-8}x + 2A^{-12}x - A^{-16}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^{2} - A^{-2} - A^{-3}x + A^{-6} + 2A^{-7}x - 4A^{-11}x - 2A^{-14} + 2A^{-15}x + A^{-18} - A^{-19}x + A^{-23}x$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14} + A^{13}x - 2A^{9}x + 2A^{5}x - 2Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^{6} + A^{2} + 2Ax - 2A^{-3}x + A^{-7}x - 3A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{57}) = -A^{-2} - A^{-3}x + A^{-7}x - A^{-11}x + A^{-15}x + A^{-18} - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{57}}) = A^{22} - A^{17}x + A^{13}x - A^{9}x + A^{5}x - Ax - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{58}) = -A^{4}x + 2x + A^{-3} - A^{-4}x + A^{-8}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x - A^{9} + A^{8}x - A^{4}x + x + A^{-7}$
$\mathcal{S}_{2,\infty}(5_{59}) = A^{2} + Ax - A^{-2} - 3A^{-3}x + 3A^{-7}x + A^{-10} - 3A^{-11}x - A^{-14} + 2A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + A^{14} + 2A^{13}x - 3A^{9}x + 3A^{5}x + A^{2} - 3Ax - A^{-2} + A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{60}) = A^{8}x - 2A^{4}x - A + 3x + A^{-3} - A^{-4}x - A^{-11} + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + 2A^{8}x + A^{5} - 2A^{4}x - A + 2x + A^{-7} - A^{-11}$
$\mathcal{S}_{2,\infty}(5_{61}) = A^{2} + Ax - 2A^{-2} - 4A^{-3}x - A^{-6} + 4A^{-7}x + A^{-10} - 3A^{-11}x - A^{-14} + 3A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + A^{14} + 2A^{13}x - A^{10} - 4A^{9}x - A^{6} + 4A^{5}x + A^{2} - 3Ax - A^{-2} + 2A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{62}) = A^{8}x - 4A^{4}x - 2A + 4x + 3A^{-3} - A^{-4}x + A^{-7} - A^{-11} + A^{-12}x + 2A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 3A^{8}x + 2A^{5} - 4A^{4}x - 2A + 2x + A^{-3} + 2A^{-7} - A^{-8}x - 2A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^{6} - A^{5}x + A^{2} + 3Ax - 4A^{-3}x - A^{-6} + 3A^{-7}x + A^{-10} - 3A^{-11}x + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - A^{10} - 2A^{9}x + A^{6} + 4A^{5}x + A^{2} - 4Ax - 2A^{-2} + 2A^{-3}x - 2A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{64}) = A^{8}x - 3A^{4}x - 2A + 4x + 2A^{-3} - A^{-4}x - A^{-11} + A^{-12}x + 2A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 3A^{8}x + 2A^{5} - 3A^{4}x - 2A + 2x + A^{-7} - A^{-8}x - 2A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^{6} - A^{5}x + 2A^{2} + 5Ax - 6A^{-3}x - 2A^{-6} + 5A^{-7}x + A^{-10} - 4A^{-11}x + 2A^{-15}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 2A^{10} - 3A^{9}x + A^{6} + 7A^{5}x + 2A^{2} - 5Ax - 2A^{-2} + 4A^{-3}x - 3A^{-7}x + A^{-11}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10} + A^{9}x - 2A^{5}x + 2Ax - 3A^{-3}x - A^{-6} + 2A^{-7}x - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + A^{6} + 2A^{5}x - 3Ax - A^{-2} + 2A^{-3}x + A^{-6} - 2A^{-7}x + A^{-11}x - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{17}x - A^{13}x + A^{9}x + A^{6} - A^{5}x - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^{8}x - A^{5} + A + x + A^{-3} + A^{-7} + A^{-8}x + A^{-11} - A^{-12}x - A^{-15}$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-3}x - A^{-7}x + A^{-11}x - A^{-15}x + A^{-19}x - A^{-22} - A^{-23}x - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{13}x + 2A^{9}x + A^{6} + Ax - A^{-2} - A^{-3}x - A^{-10} - A^{-11}x - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x + A^{9} - A^{8}x - A^{5} + x - A^{-4}x - A^{-7} + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -A^{-3} + A^{-4}x + A^{-7} + A^{-16}x + A^{-19} - A^{-20}x - A^{-23}$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^{6} + Ax - A^{-7}x - A^{-10} - A^{-11}x - A^{-14} + A^{-18} + A^{-19}x$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x + A^{-11} - A^{-15} + A^{-20}x + A^{-23} - A^{-24}x - A^{-27}$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x + A^{13} - A^{9} - A^{8}x - A^{5} + A^{-4}x + A^{-7}$
$\mathcal{S}_{2,\infty}(5_{73}) = A^{6} + A^{5}x - Ax - A^{-2} + A^{-3}x - A^{-14} - A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = Ax - A^{-3}x + A^{-7}x - A^{-18} - A^{-19}x - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18} + A^{17}x + A^{14} - A^{5}x - A^{2} - Ax$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^{2} - A^{-7}x - A^{-10} - A^{-11}x - A^{-14} + A^{-18} + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = A - A^{-4}x + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^{6} + A^{5}x - Ax + A^{-3}x + A^{-6} - 2A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{78}}) = A^{8}x - A^{-3} - A^{-4}x + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-7}x + A^{-11}x - 2A^{-15}x + A^{-19}x - A^{-23}x + A^{-27}x$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22} - A^{21}x + A^{18} + 2A^{17}x + A^{14} - A^{13}x + A^{9}x - 2A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{13}x - A^{9}x + A^{5}x - Ax - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^{2} + Ax - A^{-3}x - A^{-10} - 2A^{-11}x - A^{-14} + A^{-15}x + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22} - A^{21}x + 2A^{17}x + A^{14} - A^{5}x - A^{2} - Ax$

$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18} + A^{17}x - 2A^{13}x - A^{10} + 2A^{9}x + A^{6} - A^{5}x - A^{-2} - A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^{4}x - A + 2x + A^{-3} + A^{-12}x + A^{-15} - A^{-16}x - A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^{8}x + A^{5} - A^{4}x - A + x - A^{-8}x - A^{-11} + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2} + A^{-3}x - 2A^{-7}x - A^{-10} + 2A^{-11}x - A^{-15}x + A^{-19}x - A^{-22} - A^{-23}x - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14} - A^{13}x + 3A^{9}x + 2A^{6} - A^{5}x + Ax - A^{-2} - A^{-3}x - A^{-10} - A^{-11}x - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{9}x + A^{6} + A^{5}x - A^{-2} - A^{-3}x - A^{-6} - A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{86}) = A^{14} - A^{10} - A^{9}x + 2A^{5}x - 2Ax - 2A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{86}}) = -A^{6} + Ax - A^{-2} - 2A^{-3}x - A^{-6} + 2A^{-7}x + A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{87}}) = A^{26} + A^{18} - A^{14} - A^{13}x - A^{10} + A^{9}x + A^{6} - A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(\overline{5_{88}}) = A^{26} - A^{22} + A^{18} + A^{17}x - 2A^{13}x - A^{10} + A^{9}x + A^{6} - A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(5_{89}) = -A^{-6} - A^{-7}x - A^{-10} + 2A^{-11}x - 2A^{-15}x - A^{-18} + A^{-19}x + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{89}}) = A^{26} - A^{22} + A^{18} + A^{17}x - A^{14} - 2A^{13}x - 2A^{10} + 2A^{9}x + A^{6} - A^{5}x - A^{2}$
$\mathcal{S}_{2,\infty}(5_{90}) = -A^{-2} + A^{-6} - A^{-10} - A^{-11}x - A^{-14} + A^{-15}x - A^{-19}x - A^{-22} + A^{-23}x + A^{-26}$
$\mathcal{S}_{2,\infty}(5_{91}) = -A^{-2} - A^{-10} - A^{-19}x - A^{-22} + A^{-23}x + A^{-26}$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + 3A^{-12}x + A^{-15} - A^{-16}x + A^{-20}x - A^{-23} - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{92}}) = -A^{24}x + 2A^{20}x - 2A^{16}x - A^{13} + 2A^{12}x - A^{8}x + A^{5} + A^{4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{93}}) = A^{30} - A^{9}x - A^{6}$
$\mathcal{S}_{2,\infty}(5_{94}) = -A^{-6} - A^{-14} - A^{-15}x + A^{-19}x - A^{-23}x - A^{-26} + A^{-27}x + A^{-30}$
$\mathcal{S}_{2,\infty}(5_{95}) = -A^{-6} - 2A^{-14} - A^{-15}x + 2A^{-19}x + A^{-22} - 2A^{-23}x - A^{-26} + A^{-27}x + A^{-30}$
$\mathcal{S}_{2,\infty}(\overline{5_{95}}) = A^{26} + A^{25}x - 2A^{21}x - A^{18} + 2A^{17}x - A^{13}x - A^{10} - A^{6}$

## Knots in $L(4,1)$

$\mathcal{S}_{2,\infty}(2_2) = -A^{4}x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(2_3) = A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^{8}x + A^{2}x + x$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x + A^{-6}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^{8}x - A^{6}x + A^{2}x + x$
$\mathcal{S}_{2,\infty}(3_2) = A^{6}x^{2} - A^{2}x^{2} - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^{2}x^{2} - A^{2} - A^{-2}x^{2} - A^{-10}$
$\mathcal{S}_{2,\infty}(3_4) = -A^{4}x + 2x + A^{-2}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^{8}x - A^{4}x - A^{2}x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(3_5) = A^{2}x^{2} - 2A^{-2}x^{2} + A^{-2} + A^{-6}x^{2} - A^{-6} - 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^{6}x^{2} - A^{6} - 2A^{2}x^{2} + A^{2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^{2} - A^{10} - A^{2}x^{2} - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^{2} + A^{14} + A^{10}x^{2} + A^{6} - A^{2}x^{2} + A^{2} - A^{-2}$
$\mathcal{S}_{2,\infty}(4_2) = -A^{8}x - A^{6}x + A^{2}x + x + A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -A^{-2}x + A^{-6}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(4_3) = 2A^{-4}x + A^{-6}x - A^{-8}x - 2A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + 2A^{12}x + A^{10}x - A^{8}x - 2A^{6}x + A^{2}x + x$
$\mathcal{S}_{2,\infty}(4_4) = A^{2}x^{2} - 3A^{-2}x^{2} + 2A^{-2} + A^{-6}x^{2} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^{2} + 2A^{6}x^{2} - 2A^{2}x^{2} + 2A^{2} + A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(4_5) = -2A^{4}x - A^{2}x + 2x + 2A^{-2}x - A^{-6}x + A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x - 2A^2x + x + 2A^{-2}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + 2A^{-12}x + A^{-14}x - A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - 2A^{16}x - A^{14}x + A^{12}x + A^{10}x + A^4x$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^8x + A^4x + A^2x$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x + A^{-6}x + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^4x + 2x + A^{-2}x - A^{-4}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^8x + A^6x - A^4x - A^2x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 - 2A^2x^2 + A^2 + A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(4_{13}) = A^8x - 2A^4x - A^2x + 2x + 2A^{-2}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 3A^4x - 2A^2x + 2x + 3A^{-2}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-2} + A^{-6}x^2 - 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 + A^{-2}x^2 - A^{-2} - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(4_{16}) = A^{-4}x + A^{-6}x - A^{-8}x - A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x + A^{10}x - A^8x - A^6x + A^2x + x$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - 2A^2x^2 + A^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^2 - A^{-2}x^2 + A^{-2} + A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(4_{18}) = -x - A^{-2}x + A^{-4}x + 2A^{-6}x - A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x - 2A^6x + 2A^2x + x + A^{-6}x$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^6x^2 - A^6 - A^2x^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-6}x^2 - A^{-6} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 + A^{-10}x^2 - 2A^{-10} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} - A^2x^2 + A^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x - A^2x + x + A^{-2}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{4_{22}}) = -A^{16}x + A^6x + A^4x$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x + A^{-10}x + A^{-12}x - A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{10}x - A^8x + A^2x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(4_{27}) = A^8x - 2A^4x + x + A^{-2}x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x + A^{-10}x + A^{-12}x - A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = -A^{10}x + A^6x + A^4x$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2 - A^{-14} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^2 + A^{10} - A^6x^2 + 2A^6 - A^2$
$\mathcal{S}_{2,\infty}(5_3) = -A^{12}x - A^{10}x + A^8x + 2A^6x + A^4x - A^2x$
$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x - A^{-6}x + A^{-8}x + 2A^{-10}x + A^{-12}x - A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 - A^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-10}x^2 + A^{-14}x^2 - 2A^{-14}$
$\mathcal{S}_{2,\infty}(5_5) = A^{-10}x + A^{-18}x - A^{-22}x$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{14}x - A^{10}x - A^8x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} + A^6 - A^2x^2 + A^2 - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} + 2A^{-10}x^2 - 2A^{-10} - A^{-14} + A^{-18} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 + 2A^{10}x^2 - 2A^{10} - A^2x^2 + 2A^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_8) = -x - A^{-2}x + A^{-4}x + 3A^{-6}x - A^{-10}x + A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x + A^{10}x - 2A^8x - 2A^6x + 2A^2x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 - 2A^2x^2 + A^2 - A^{-10}x^2 + 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 + A^{-2} + A^{-6} - A^{-10} - A^{-14}x^2 + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{10}) = 2A^{-4}x + A^{-6}x - 2A^{-8}x - 2A^{-10}x + A^{-12}x + 2A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - 2A^{16}x - A^{14}x + 2A^{12}x + 2A^{10}x - A^{8}x - 2A^{6}x + A^{2}x + x$
$\mathcal{S}_{2,\infty}(5_{11}) = A^{2}x^2 - 4A^{-2}x^2 + 2A^{-2} + 3A^{-6}x^2 - A^{-6} - A^{-10}x^2 + 2A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} + 3A^{6}x^2 - A^{6} - 2A^{2}x^2 + A^{2} + 2A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{12}) = -2x - A^{-2}x + 3A^{-4}x + 4A^{-6}x - A^{-8}x - 3A^{-10}x + 2A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 3A^{12}x + 2A^{10}x - 3A^{8}x - 4A^{6}x + 3A^{2}x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^{6}x^2 + 3A^{2}x^2 - 2A^{2} - 3A^{-2}x^2 + A^{-2} + A^{-6}x^2 - 2A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^{6}x^2 - A^{6} - 3A^{2}x^2 + A^{2} + A^{-2}x^2 - A^{-2} - 2A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{14}) = -2A^{4}x - A^{2}x + 3x + 3A^{-2}x - A^{-4}x - 2A^{-6}x + 2A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^{8}x + 2A^{6}x - 2A^{4}x - 3A^{2}x + x + 2A^{-2}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^{6}x^2 - A^{6} - 4A^{2}x^2 + 3A^{2} + A^{-2}x^2 - 2A^{-6} - A^{-10}x^2 + 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^{6}x^2 + 4A^{2}x^2 - 2A^{2} - 4A^{-2}x^2 + 3A^{-2} + A^{-6}x^2 + A^{-6} - 2A^{-10} - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(5_{16}) = 2A^{8}x + A^{6}x - 3A^{4}x - 3A^{2}x + 2x + 3A^{-2}x - 2A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^{8}x - 3A^{4}x - 2A^{2}x + 3x + 4A^{-2}x - 2A^{-6}x + A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 + 3A^{6}x^2 - 2A^{6} - 2A^{2}x^2 + A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^{2}x^2 - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - 2A^{-6} - A^{-10}x^2 - A^{-10} + A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} + 2A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-14} + 2A^{-18} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} + 2A^{10}x^2 - A^{10} - A^{2}x^2 + 2A^{2} - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{19}) = -2A^{4}x - A^{2}x + 2x + 3A^{-2}x - A^{-6}x + A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^{8}x + A^{6}x - 2A^{4}x - 2A^{2}x + x + 2A^{-2}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{20}) = 2A^{2}x^2 - A^{2} - 3A^{-2}x^2 + 2A^{-2} + A^{-6}x^2 - A^{-10} - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^{6}x^2 - A^{6} - 2A^{2}x^2 + 2A^{2} + A^{-2} - A^{-6} - A^{-10}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{21}) = 2A^{12}x + A^{10}x - 2A^{8}x - 3A^{6}x + 2A^{2}x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x - A^{-2}x + 2A^{-4}x + 3A^{-6}x - 2A^{-10}x + A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^{6}x^2 + 2A^{2}x^2 - A^{2} - 2A^{-2}x^2 + 2A^{-2} + 2A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^{6}x^2 + A^{6} - 3A^{2}x^2 + 2A^{2} + A^{-2}x^2 - A^{-2} - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{23}) = -A^{12}x + A^{8}x + A^{6}x - A^{2}x + x$
$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} - A^{-10}x^2 + A^{-14}x^2 - 2A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^{6} - A^{2}x^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} + 3A^{-18}x^2 - A^{-18} - A^{-22}x^2 - A^{-22} + 2A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} - 3A^{14}x^2 + 2A^{14} + A^{10}x^2 + A^{10} - A^{6}x^2 - A^{6}$
$\mathcal{S}_{2,\infty}(5_{26}) = -x + 3A^{-4}x + A^{-6}x - 2A^{-8}x - 2A^{-10}x + 2A^{-12}x + 2A^{-14}x - A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - 3A^{16}x - A^{14}x + 3A^{12}x + 2A^{10}x - A^{8}x - 2A^{6}x + A^{4}x + A^{2}x$
$\mathcal{S}_{2,\infty}(5_{27}) = -x + 3A^{-4}x + A^{-6}x - 2A^{-8}x - 2A^{-10}x + 2A^{-12}x + 2A^{-14}x - A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - 3A^{16}x - A^{14}x + 3A^{12}x + 2A^{10}x - A^{8}x - 2A^{6}x + A^{4}x + A^{2}x$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^{6}x^2 - A^{6} - 3A^{2}x^2 + A^{2} + A^{-2}x^2 - 2A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^{6} + 2A^{2}x^2 - 3A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 - A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{30}) = A^{8}x - A^{4}x + 2x - 2A^{-4}x - A^{-6}x + A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + 2A^{8}x + A^{6}x - A^{4}x - A^{2}x + x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{31}) = A^{8}x - 2A^{4}x + 3x + A^{-2}x - 2A^{-4}x - 2A^{-6}x + A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + 3A^{8}x + A^{6}x - 2A^{4}x - 2A^{2}x + x + A^{-2}x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{32}) = A^{8}x - 2A^{4}x + 2x + A^{-2}x - 2A^{-4}x - A^{-6}x + A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + 2A^{8}x + A^{6}x - 2A^{4}x - A^{2}x + x + A^{-2}x - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^{6}x^2 + 3A^{2}x^2 - A^{2} - 3A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 - 2A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}x^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 + 3A^{-18}x^2 - 2A^{-18} - A^{-22}x^2 - A^{-22} + 2A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} - 2A^{14}x^2 + A^{10}x^2 - A^6x^2 - A^6$
$\mathcal{S}_{2,\infty}(5_{36}) = -x + 2A^{-4}x + A^{-6}x - 2A^{-8}x - A^{-10}x + 2A^{-12}x + 2A^{-14}x - A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - 3A^{16}x - A^{14}x + 2A^{12}x + 2A^{10}x - A^8x - A^6x + A^4x + A^2x$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-14}x^2 - A^{-14} + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^2 - 2A^2$
$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^4x + x + A^{-2}x - A^{-4}x - A^{-6}x + A^{-8}x + 2A^{-10}x + A^{-12}x - A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x + A^{-10}x - A^{-14}x + A^{-18}x - A^{-22}x$
$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x - A^{18}x + A^{14}x - A^{10}x + A^6x + A^4x$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 - A^2 - A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 - A^{-10} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{42}) = A^{-4}x + A^{-6}x - A^{-8}x - A^{-10}x + A^{-12}x + A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x - A^{14}x + A^{12}x + A^{10}x - A^8x - A^6x + A^2x + x$
$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 + A^{-2}x^2 - A^{-2} - A^{-6} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 + A^2x^2 - A^2 - A^{-2}x^2 + A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{44}) = -x - A^{-2}x + 2A^{-4}x + 3A^{-6}x - A^{-8}x - 2A^{-10}x + 2A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x + 2A^{10}x - 2A^8x - 3A^6x + 2A^2x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + 2A^2 + A^{-2} - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 - 3A^{-2}x^2 + 2A^{-2} + A^{-6}x^2 - 2A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(5_{46}) = -x - A^{-2}x + 3A^{-4}x + 3A^{-6}x - A^{-8}x - 3A^{-10}x + 2A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 3A^{12}x + 2A^{10}x - 2A^8x - 4A^6x + 2A^2x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 - 3A^2x^2 + 2A^2 + A^{-2} - A^{-6} - A^{-10}x^2 + 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + 2A^{-2} + A^{-6}x^2 - 2A^{-10} - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x + A^2x - x - 2A^{-2}x + A^{-4}x + 2A^{-6}x - 2A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x - 2A^6x + A^4x + 3A^2x + x - A^{-2}x + A^{-6}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - A^6 - A^2x^2 - A^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 + A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + 3A^{-12}x + A^{-14}x - 2A^{-16}x - 2A^{-18}x + A^{-22}x$
$\mathcal{S}_{2,\infty}(\overline{5_{51}}) = -A^{24}x + 2A^{20}x + A^{18}x - 2A^{16}x - 2A^{14}x + A^{12}x + A^{10}x + A^4x$
$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^8x + 2A^4x + A^2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^4x - x - A^{-2}x + A^{-4}x + A^{-6}x + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} + A^{-10}x^2 - A^{-10} - A^{-14}x^2 - A^{-14} + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{54}) = A^{20}x - 2A^{16}x - A^{14}x + 2A^{12}x + A^{10}x - A^8x - A^6x + A^4x + A^2x$
$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + 2A^{-4}x + A^{-6}x - A^{-8}x - A^{-10}x + 2A^{-12}x + A^{-14}x - A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} - 3A^{10}x^2 + 2A^{10} + A^6x^2 + A^6 - A^2x^2 - A^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} + 2A^{-14}x^2 - A^{-14} - A^{-18} + A^{-22}$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 + A^{-2}x^2 - A^{-2} - 2A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-6}x^2 - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{58}) = -A^4x + 2x + A^{-2}x - A^{-4}x - A^{-6}x + A^{-8}x + A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x - A^{10}x + A^8x + A^6x - A^4x - A^2x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-14}x^2 - 2A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 - 2A^2x^2 + A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{60}) = A^8x - 2A^4x - A^2x + 3x + 2A^{-2}x - A^{-4}x - 2A^{-6}x + A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + 2A^{8}x + A^{6}x - 2A^{4}x - 2A^{2}x + 2x + 2A^{-2}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{61}) = A^{2}x^{2} - 4A^{-2}x^{2} + A^{-2} + 4A^{-6}x^{2} - 2A^{-6} - 2A^{-10}x^{2} + A^{-14}x^{2} + A^{-14} - 2A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^{2} - A^{14} - 4A^{10}x^{2} + A^{10} + 4A^{6}x^{2} - 2A^{6} - 2A^{2}x^{2} + 2A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{62}) = A^{8}x - 4A^{4}x - 2A^{2}x + 4x + 5A^{-2}x - A^{-4}x - 3A^{-6}x + 2A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 3A^{8}x + 2A^{6}x - 4A^{4}x - 4A^{2}x + 2x + 4A^{-2}x - 2A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^{6}x^{2} + 3A^{2}x^{2} - A^{2} - 4A^{-2}x^{2} + 2A^{-2} + 2A^{-6}x^{2} - A^{-10}x^{2} - A^{-10} + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^{2} + A^{10} + 4A^{6}x^{2} - A^{6} - 4A^{2}x^{2} + 2A^{2} + A^{-2}x^{2} - 2A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{64}) = A^{8}x - 3A^{4}x - 2A^{2}x + 4x + 4A^{-2}x - A^{-4}x - 3A^{-6}x + 2A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 3A^{8}x + 2A^{6}x - 3A^{4}x - 4A^{2}x + 2x + 3A^{-2}x - 2A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^{6}x^{2} + 5A^{2}x^{2} - 2A^{2} - 6A^{-2}x^{2} + 3A^{-2} + 3A^{-6}x^{2} - A^{-6} - A^{-10}x^{2} - 2A^{-10} - A^{-14}x^{2} + 4A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^{2} + A^{10} + 7A^{6}x^{2} - 3A^{6} - 5A^{2}x^{2} + 3A^{2} + A^{-2}x^{2} + A^{-2} - 3A^{-6} - A^{-10}x^{2} + 2A^{-10}$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^{2} - 2A^{6}x^{2} + A^{6} + 2A^{2}x^{2} - A^{2} - 2A^{-2}x^{2} + A^{-6}x^{2} - 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^{6}x^{2} - A^{6} - 3A^{2}x^{2} + A^{2} + 2A^{-2}x^{2} - A^{-2} - A^{-6}x^{2} + A^{-10} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^{2} - A^{18} - A^{14}x^{2} + A^{10}x^{2} - A^{10} - A^{2}x^{2} + A^{2} - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^{8}x - A^{6}x + 2A^{2}x + x + A^{-6}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^{2}x - A^{-2}x + A^{-6}x - A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^{2} - A^{-2} - A^{-6}x^{2} + A^{-6} - A^{-18}x^{2} + 2A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^{2} + A^{14} + 2A^{10}x^{2} - A^{10} + A^{6} - A^{2}x^{2} + A^{2} - A^{-2} - A^{-6}x^{2} + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x + A^{10}x - A^{8}x - 2A^{6}x + A^{2}x + x - A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -A^{-2}x + A^{-4}x + 2A^{-6}x - A^{-10}x + A^{-14}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^{6} + A^{2}x^{2} - A^{2} - A^{-2} - A^{-6}x^{2} + A^{-6} - A^{-10}x^{2} + A^{-14}x^{2} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x + A^{14}x - 2A^{10}x - A^{8}x + x + A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^{6}x^{2} - A^{2}x^{2} + A^{2} - A^{-10}x^{2} + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^{2}x^{2} - A^{2} - A^{-2}x^{2} + A^{-2} + A^{-6} - A^{-14}x^{2} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^{2} - A^{6}x^{2} - A^{2}$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^{2} - A^{-6}x^{2} + A^{-6} - A^{-10}x^{2} + A^{-14}x^{2} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = A^{2}x - A^{-2}x - A^{-4}x + A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^{6}x^{2} - A^{2}x^{2} + A^{-2}x^{2} - A^{-2} - A^{-6}x^{2}$
$\mathcal{S}_{2,\infty}(\overline{5_{78}}) = A^{8}x - A^{-2}x - A^{-4}x + A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^{2} + A^{-6} + A^{-10}x^{2} - 2A^{-14}x^{2} + A^{-14} + A^{-18}x^{2} - A^{-22} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^{2} + 2A^{18}x^{2} - A^{14}x^{2} + A^{14} + A^{10} - A^{6}x^{2}$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^{2} - A^{14} - A^{10}x^{2} + A^{6}x^{2} - A^{2}x^{2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^{2}x^{2} - A^{-2}x^{2} - A^{-10}x^{2} - A^{-10} + A^{-14}x^{2} - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-14}x^{2} - A^{-14} + 2A^{-18}x^{2} - 2A^{-18} - A^{-22}x^{2} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^{2} + 2A^{18}x^{2} - A^{18} - A^{14} - A^{6}x^{2} - A^{2}$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^{2} + 2A^{-10}x^{2} - 2A^{-10} - A^{-14}x^{2} + A^{-18} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^{2} - 2A^{14}x^{2} + A^{14} + 2A^{10}x^{2} - 2A^{10} - A^{6} - A^{2}x^{2} + A^{2} - A^{-2}$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^{4}x - A^{2}x + 2x + 2A^{-2}x - A^{-6}x + A^{-10}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^{8}x + A^{6}x - A^{4}x - 2A^{2}x + x + A^{-2}x - A^{-6}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^{2} - 2A^{-6}x^{2} + 2A^{-6} + A^{-10}x^{2} - A^{-10} - A^{-14} - A^{-18}x^{2} + 2A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^{2} + 3A^{10}x^{2} - 2A^{10} - A^{6}x^{2} + 2A^{6} - A^{2}x^{2} + 2A^{2} - A^{-2} - A^{-6}x^{2} + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{10}x^{2} + A^{6} - A^{2}x^{2} + A^{2} - A^{-2}x^{2}$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + 3A^{-12}x + A^{-14}x - A^{-16}x - A^{-18}x + A^{-20}x - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{92}}) = -A^{24}x + 2A^{20}x - 2A^{16}x - A^{14}x + 2A^{12}x + A^{10}x - A^{8}x + A^{4}x$

## Knots in $L(5,1)$

$\mathcal{S}_{2,\infty}(2_2) = -A^4x + x + A^{-1}x^2 - A^{-1} - A^{-5}$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^8x + A^3x^2 - A^3 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-9}x^2 + A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^8x - A^7x^2 + A^7 + A^3x^2 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^2 - A^{-1}x$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^2 - A^{-5}x + A^{-6}$
$\mathcal{S}_{2,\infty}(3_4) = -A^4x + 2x + A^{-1}x^2 - A^{-1} - A^{-5}x^2 + A^{-9}$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^8x - A^4x - A^3x^2 + A^3 + x + A^{-1}x^2 - A^{-5}$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - 2A^{-2}x^2 + A^{-2} - A^{-5}x + A^{-6}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + A^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(3_6) = -A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^2 + A^{-6} - A^{-9}x + A^{-10} + A^{-13}x$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^2 + A^7x - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(4_2) = -A^8x - A^7x^2 + A^7 + A^3x^2 + x + A^{-5}$
$\mathcal{S}_{2,\infty}(4_3) = 2A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-8}x - 2A^{-9}x^2 + A^{-9} + A^{-13}x^2 + A^{-13} - A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + 2A^{12}x + A^{11}x^2 - A^{11} - A^8x - 2A^7x^2 + A^7 + A^3x^2 + A^3 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(4_4) = A^2x^2 - 3A^{-2}x^2 + 2A^{-2} - A^{-5}x + A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + 2A^6x^2 + A^3x - 2A^2x^2 + A^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(4_5) = -2A^4x - A^3x^2 + A^3 + 2x + 2A^{-1}x^2 - A^{-1} - A^{-5}x^2 + 2A^{-9}$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x - 2A^3x^2 + 2A^3 + x + 2A^{-1}x^2 - A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + 2A^{-12}x + A^{-13}x^2 - A^{-13} - A^{-16}x - A^{-17}x^2 + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - 2A^{16}x - A^{15}x^2 + A^{15} + A^{12}x + A^{11}x^2 - A^7 + A^4x$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^8x + A^4x + A^3x^2 - A^3 - A^{-1}$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-9} + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^4x + 2x + A^{-1}x^2 - A^{-1} - A^{-4}x - A^{-5}x^2 + A^{-9}x^2 - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^8x + A^7x^2 - A^7 - A^4x - A^3x^2 + x + A^{-1}x^2 - A^{-5}$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-5}x + 2A^{-6}x^2 + A^{-9}x - A^{-10}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 + A^3x - 2A^2x^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(4_{13}) = A^8x - 2A^4x - A^3x^2 + A^3 + 2x + 2A^{-1}x^2 - A^{-1} - A^{-5}x^2 - A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 3A^4x - 2A^3x^2 + 2A^3 + 2x + 3A^{-1}x^2 - A^{-1} - A^{-5}x^2 - A^{-5} + 2A^{-9}$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + 2A^2x^2 - A^2 + A^{-1}x - 2A^{-2}x^2 - A^{-5}x + A^{-6}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - A^{-1}x + A^{-2}x^2 + A^{-5}x - A^{-6}$
$\mathcal{S}_{2,\infty}(4_{16}) = A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-8}x - A^{-9}x^2 + A^{-13}x^2 - A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x + A^{11}x^2 - A^{11} - A^8x - A^7x^2 + A^3x^2 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - 2A^2x^2 + A^2 - A^{-1}x + A^{-2} + A^{-5}x$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^2 + A^{-1}x - A^{-2}x^2 - A^{-5}x + A^{-6}$
$\mathcal{S}_{2,\infty}(4_{18}) = -x - A^{-1}x^2 + A^{-1} + A^{-4}x + 2A^{-5}x^2 - A^{-5} - A^{-9}x^2 + 2A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x - 2A^7x^2 + 2A^7 + 2A^3x^2 + x - A^{-1} + A^{-5}$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^6x^2 - A^6 + A^3x - A^2x^2 - A^2 - A^{-1}x$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-5}x + A^{-6}x^2 + A^{-9}x - A^{-10}$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 - A^{-9}x + A^{-10}x^2 - A^{-10} + A^{-13}x - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} + A^7x - A^6 - A^3x - A^2x^2 + A^2$

$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x - A^3x^2 + A^3 + x + A^{-1}x^2 + A^{-9}$
$\mathcal{S}_{2,\infty}(\overline{4_{22}}) = -A^{16}x + A^7x^2 - A^7 + A^4x - A^3$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x + A^{-9}x^2 - A^{-9} + A^{-12}x - A^{-13}x^2 - A^{-16}x + A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{11}x^2 + A^{11} - A^8x + A^7 + A^3x^2 + x$
$\mathcal{S}_{2,\infty}(4_{27}) = A^8x - 2A^4x + x + A^{-1}x^2 - A^{-1} - A^{-4}x - A^{-5} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x + A^{-9}x^2 - A^{-9} + A^{-12}x - A^{-13}x^2 - A^{-16}x + A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = -A^{11}x^2 + A^{11} + A^7x^2 + A^4x - A^3$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} - A^{-13}x + A^{-14}x^2 + A^{-17}x$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{11}x - A^{10}x^2 - A^7x - A^6x^2 + 2A^6$
$\mathcal{S}_{2,\infty}(5_3) = -A^{12}x - A^{11}x^2 + A^{11} + A^8x + 2A^7x^2 - A^7 + A^4x - A^3x^2 - A^3 + A^{-1}$
$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x + 2A^{-9}x^2 - A^{-9} + A^{-12}x - A^{-13}x^2 - A^{-13} - A^{-16}x + A^{-17}$
$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 - A^3x$
$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-9}x - A^{-10}x^2 + A^{-10} + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{15}x^2 - A^{15} - A^{11}x^2 - A^8x + A^3x^2 - A^3 + x$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} + A^7x - A^3x - A^2x^2 + A^2 - A^{-5}x$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-5}x - A^{-9}x - A^{-17}x$
$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} - A^{-9}x + 2A^{-10}x^2 - A^{-10} + 2A^{-13}x - 2A^{-14} - A^{-17}x$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 - A^{11}x + 2A^{10}x^2 - A^{10} + 2A^7x - A^6 - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_8) = -x - A^{-1}x^2 + A^{-1} + A^{-4}x + 3A^{-5}x^2 - 2A^{-5} - A^{-9}x^2 - A^{-9} + A^{-13} - A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x + A^{11}x^2 - A^{11} - 2A^8x - 2A^7x^2 + A^7 + 2A^3x^2 - A^3 + x - 2A^{-1} + A^{-5}$
$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 + A^3x - 2A^2x^2 - 2A^{-1}x + A^{-2} + A^{-5}x - A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 + 2A^{-1}x - A^{-2}x^2 - A^{-2} - 2A^{-5}x + A^{-6} - A^{-13}x$
$\mathcal{S}_{2,\infty}(5_{10}) = 2A^{-4}x + A^{-5}x^2 - A^{-5} - 2A^{-8}x - 2A^{-9}x^2 + A^{-9} + A^{-12}x + 2A^{-13}x^2 - A^{-17}x^2 - A^{-17} + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - 2A^{16}x - A^{15}x^2 + A^{15} + 2A^{12}x + 2A^{11}x^2 - A^{11} - A^8x - 2A^7x^2 + A^3x^2 + A^3 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - 4A^{-2}x^2 + 2A^{-2} - A^{-5}x + 3A^{-6}x^2 + 2A^{-9}x - A^{-10}x^2 - A^{-10} - A^{-13}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} - A^7x + 3A^6x^2 + 2A^3x - 2A^2x^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(5_{12}) = -2x - A^{-1}x^2 + A^{-1} + 3A^{-4}x + 4A^{-5}x^2 - 3A^{-5} - A^{-8}x - 3A^{-9}x^2 + A^{-13}x^2 + 2A^{-13} - 2A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 3A^{12}x + 2A^{11}x^2 - 2A^{11} - 3A^8x - 4A^7x^2 + 2A^7 + 3A^3x^2 + x - 3A^{-1} + A^{-5}$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + 3A^2x^2 - 2A^2 + A^{-1}x - 3A^{-2}x^2 - 2A^{-5}x + A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^3x - 3A^2x^2 - 2A^{-1}x + A^{-2}x^2 + A^{-5}x - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{14}) = -2A^4x - A^3x^2 + A^3 + 3x + 3A^{-1}x^2 - 2A^{-1} - A^{-4}x - 2A^{-5}x^2 + A^{-9}x^2 + A^{-9} - 2A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x + 2A^7x^2 - 2A^7 - 2A^4x - 3A^3x^2 + A^3 + x + 2A^{-1}x^2 - 2A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^6x^2 - A^6 + 2A^3x - 4A^2x^2 + A^2 - 3A^{-1}x + A^{-2}x^2 + A^{-2} + A^{-5}x - A^{-6} - A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 4A^2x^2 - 2A^2 + 2A^{-1}x - 4A^{-2}x^2 + A^{-2} - 3A^{-5}x + A^{-6}x^2 + 2A^{-6} + A^{-9}x - A^{-10} - A^{-13}x$
$\mathcal{S}_{2,\infty}(5_{16}) = 2A^8x + A^7x^2 - A^7 - 3A^4x - 3A^3x^2 + 2A^3 + 2x + 3A^{-1}x^2 - A^{-1} - A^{-5}x^2 - 2A^{-5} + 2A^{-9}$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 3A^4x - 2A^3x^2 + 2A^3 + 3x + 4A^{-1}x^2 - 2A^{-1} - 2A^{-5}x^2 - A^{-5} + 2A^{-9} - A^{-13}$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 - A^7x + 3A^6x^2 - A^6 + A^3x - 2A^2x^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-5}x + 3A^{-6}x^2 - A^{-6} + A^{-9}x - A^{-10}x^2 - A^{-10} - A^{-13}x + A^{-14}$
$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} - A^{-9}x + 2A^{-10}x^2 + 2A^{-13}x - A^{-14}x^2 - A^{-17}x + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} - A^{11}x + 2A^{10}x^2 + 2A^7x - A^6 - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_{19}) = -2A^4x - A^3x^2 + A^3 + 2x + 3A^{-1}x^2 - 2A^{-1} - A^{-5}x^2 - A^{-5} + A^{-9} - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x + A^7x^2 - A^7 - 2A^4x - 2A^3x^2 + A^3 + x + 2A^{-1}x^2 - A^{-1} - 2A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(5_{20}) = 2A^2x^2 - A^2 + A^{-1}x - 3A^{-2}x^2 + A^{-2} - 2A^{-5}x + A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10} - A^{-13}x$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + 2A^3x - 2A^2x^2 - 2A^{-1}x + A^{-2} - A^{-9}x$
$\mathcal{S}_{2,\infty}(5_{21}) = 2A^{12}x + A^{11}x^2 - A^{11} - 2A^8x - 3A^7x^2 + 2A^7 + 2A^3x^2 + x - 2A^{-1} + A^{-5}$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x - A^{-1}x^2 + A^{-1} + 2A^{-4}x + 3A^{-5}x^2 - 2A^{-5} - 2A^{-9}x^2 + 2A^{-13} - A^{-17}$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + 2A^2x^2 - A^2 + A^{-1}x - 2A^{-2}x^2 + A^{-2} - A^{-5}x + 2A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - 3A^2x^2 + 2A^2 - A^{-1}x + A^{-2}x^2 + A^{-5}x - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{23}) = -A^{12}x + A^8x + A^7x^2 - A^7 - A^3x^2 + x + A^{-1}$
$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-9}x^2 - A^{-13}$
$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} - A^{-9}x - A^{-10}x^2 + A^{-10} + A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^6 - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} - A^{-17}x + 3A^{-18}x^2 + A^{-21}x - A^{-22}x^2 - A^{-22} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} + A^{15}x - 3A^{14}x^2 + A^{14} - A^{11}x + A^{10}x^2 + A^{10} - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{26}) = -x + 3A^{-4}x + A^{-5}x^2 - A^{-5} - 2A^{-8}x - 2A^{-9}x^2 + A^{-9} + 2A^{-12}x + 2A^{-13}x^2 - A^{-16}x - A^{-17}x^2 - A^{-17} + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - 3A^{16}x - A^{15}x^2 + A^{15} + 3A^{12}x + 2A^{11}x^2 - A^{11} - A^8x - 2A^7x^2 + A^4x + A^3x^2 + A^3 - A^{-1}$
$\mathcal{S}_{2,\infty}(5_{27}) = -x + 3A^{-4}x + A^{-5}x^2 - A^{-5} - 2A^{-8}x - 2A^{-9}x^2 + A^{-9} + 2A^{-12}x + 2A^{-13}x^2 - A^{-16}x - A^{-17}x^2 - A^{-17} + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - 3A^{16}x - A^{15}x^2 + A^{15} + 3A^{12}x + 2A^{11}x^2 - A^{11} - A^8x - 2A^7x^2 + A^4x + A^3x^2 + A^3 - A^{-1}$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - A^{-1}x + A^{-2}x^2 + A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-5}x + 2A^{-6}x^2 - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{30}) = A^8x - A^4x + 2x - 2A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x + A^{-9}x^2 - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + 2A^8x + A^7x^2 - A^7 - A^4x - A^3x^2 + x + A^{-1} - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{31}) = A^8x - 2A^4x + 3x + A^{-1}x^2 - A^{-1} - 2A^{-4}x - 2A^{-5}x^2 + A^{-5} + A^{-8}x + A^{-9}x^2 + A^{-9} - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + 3A^8x + A^7x^2 - A^7 - 2A^4x - 2A^3x^2 + A^3 + x + A^{-1}x^2 + A^{-1} - A^{-4}x - A^{-5} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{32}) = A^8x - 2A^4x + 2x + A^{-1}x^2 - A^{-1} - 2A^{-4}x - A^{-5}x^2 + A^{-8}x + A^{-9}x^2 - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + 2A^8x + A^7x^2 - A^7 - 2A^4x - A^3x^2 + x + A^{-1}x^2 - A^{-4}x - A^{-5} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} - A^{-5}x + 2A^{-6}x^2 - A^{-10}x^2 - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - A^{-1}x + 2A^{-2}x^2 - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 - A^{-17}x + 3A^{-18}x^2 - A^{-18} + A^{-21}x - A^{-22}x^2 - A^{-22} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} + A^{15}x - 2A^{14}x^2 - A^{14} - A^{11}x + A^{10}x^2 - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{36}) = -x + 2A^{-4}x + A^{-5}x^2 - A^{-5} - 2A^{-8}x - A^{-9}x^2 + 2A^{-12}x + 2A^{-13}x^2 - A^{-13} - A^{-16}x - A^{-17}x^2 - A^{-17} + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - 3A^{16}x - A^{15}x^2 + A^{15} + 2A^{12}x + 2A^{11}x^2 - A^{11} - A^8x - A^7x^2 - A^7 + A^4x + A^3x^2 - A^{-1}$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 - A^{-13}x + A^{-14}x^2 + A^{-17}x$
$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{11}x - A^{10}x^2 - A^{10} - A^7x - A^2$
$\mathcal{S}_{2,\infty}(5_{39}) = -A^{12}x - A^{11}x^2 + A^{11} + A^8x + 2A^7x^2 - A^7 - A^3x^2 - A^3 + A^{-1}x^2 - A^{-5}$
$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^4x + x + A^{-1}x^2 - A^{-1} - A^{-4}x - A^{-5}x^2 + A^{-8}x + 2A^{-9}x^2 - A^{-9} + A^{-12}x - A^{-13}x^2 - A^{-13} - A^{-16}x + A^{-17}$
$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x + A^{-9}x^2 - A^{-9} - A^{-13}x^2 + A^{-17}x^2 - A^{-21}x^2 + A^{-25}$
$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x - A^{19}x^2 + A^{19} + A^{15}x^2 - A^{11}x^2 + A^7x^2 + A^4x - A^3$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 - A^3x$
$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 - A^{-9}x + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{42}) = A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-8}x - A^{-9}x^2 + A^{-12}x + A^{-13}x^2 - A^{-17}x^2 + A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x - A^{15}x^2 + A^{15} + A^{12}x + A^{11}x^2 - A^8x - A^7x^2 + A^3x^2 + x - A^{-1}$
$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 - A^{-1}x + A^{-2}x^2 + A^{-5}x - A^{-6} - A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 - A^3x + A^2x^2 + A^{-1}x - A^{-2}x^2 - A^{-5}x + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{44}) = -x - A^{-1}x^2 + A^{-1} + 2A^{-4}x + 3A^{-5}x^2 - 2A^{-5} - A^{-8}x - 2A^{-9}x^2 + A^{-13}x^2 + A^{-13} - 2A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x + 2A^{11}x^2 - 2A^{11} - 2A^8x - 3A^7x^2 + A^7 + 2A^3x^2 + x - 2A^{-1} + A^{-5}$

$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^3x - 3A^2x^2 + A^2 - 2A^{-1}x + 2A^{-2} + A^{-5}x$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 + A^{-1}x - 3A^{-2}x^2 + A^{-2} - 2A^{-5}x + A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10}$
$\mathcal{S}_{2,\infty}(5_{46}) = -x - A^{-1}x^2 + A^{-1} + 3A^{-4}x + 3A^{-5}x^2 - 2A^{-5} - A^{-8}x - 3A^{-9}x^2 + A^{-9} + A^{-13}x^2 + 2A^{-13} - 2A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 3A^{12}x + 2A^{11}x^2 - 2A^{11} - 2A^8x - 4A^7x^2 + 2A^7 + 2A^3x^2 + A^3 + x - 2A^{-1} + A^{-5}$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 + 2A^3x - 3A^2x^2 - 3A^{-1}x + 2A^{-2} + A^{-5}x - A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 + 2A^{-1}x - 3A^{-2}x^2 - 3A^{-5}x + A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10} - A^{-13}x$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x + A^3x^2 - A^3 - x - 2A^{-1}x^2 + A^{-1} + A^{-4}x + 2A^{-5}x^2 - A^{-5} - A^{-9}x^2 - A^{-9} + 2A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x - 2A^7x^2 + 2A^7 + A^4x + 3A^3x^2 - A^3 + x - A^{-1}x^2 - A^{-1} + A^{-5} - A^{-9}$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} - A^7x + 2A^6x^2 + A^3x - A^2x^2 - A^2 - A^{-1}x$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-5}x + 2A^{-6}x^2 + A^{-9}x - A^{-10}x^2 - A^{-13}x + A^{-14}$
$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + 3A^{-12}x + A^{-13}x^2 - A^{-13} - 2A^{-16}x - 2A^{-17}x^2 + A^{-17} + A^{-21}x^2 + A^{-21} - A^{-25}$
$\mathcal{S}_{2,\infty}(\overline{5_{51}}) = -A^{24}x + 2A^{20}x + A^{19}x^2 - A^{19} - 2A^{16}x - 2A^{15}x^2 + A^{15} + A^{12}x + A^{11}x^2 + A^{11} - A^7 + A^4x$
$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^8x + 2A^4x + A^3x^2 - A^3 - A^{-1}x^2 + A^{-5}$
$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^4x - x - A^{-1}x^2 + A^{-1} + A^{-4}x + A^{-5}x^2 - A^{-9} + A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} - A^{-9}x + A^{-10}x^2 - A^{-14}x^2 + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 - A^3x + A^2$
$\mathcal{S}_{2,\infty}(5_{54}) = A^{20}x - 2A^{16}x - A^{15}x^2 + A^{15} + 2A^{12}x + A^{11}x^2 - A^8x - A^7x^2 + A^4x + A^3x^2 - A^{-1}$
$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + 2A^{-4}x + A^{-5}x^2 - A^{-5} - A^{-8}x - A^{-9}x^2 + 2A^{-12}x + A^{-13}x^2 - A^{-16}x - A^{-17}x^2 + A^{-21}$
$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} + A^{11}x - 3A^{10}x^2 + A^{10} - A^7x + A^6x^2 + A^6 - A^2x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} - A^{-13}x + 2A^{-14}x^2 + A^{-17}x - A^{-18}$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 - A^{-1}x + A^{-2}x^2 - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-5}x + A^{-6}x^2 + A^{-6} - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{58}) = -A^4x + 2x + A^{-1}x^2 - A^{-1} - A^{-4}x - A^{-5}x^2 + A^{-8}x + A^{-9}x^2 - A^{-13}x^2 + A^{-17}$
$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x - A^{11}x^2 + A^{11} + A^8x + A^7x^2 - A^4x - A^3x^2 + x + A^{-1}x^2 - A^{-5}$
$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-5}x + 3A^{-6}x^2 + A^{-9}x - 2A^{-10}x^2 - A^{-13}x + A^{-14}x^2 - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} - A^7x + 3A^6x^2 + A^3x - 2A^2x^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(5_{60}) = A^8x - 2A^4x - A^3x^2 + A^3 + 3x + 2A^{-1}x^2 - A^{-1} - A^{-4}x - 2A^{-5}x^2 + A^{-9}x^2 + A^{-9} - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + 2A^8x + A^7x^2 - A^7 - 2A^4x - 2A^3x^2 + A^3 + 2x + 2A^{-1}x^2 - A^{-5}x^2 - A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - 4A^{-2}x^2 + A^{-2} - A^{-5}x + 4A^{-6}x^2 - A^{-6} + 2A^{-9}x - 2A^{-10}x^2 - A^{-10} - A^{-13}x + A^{-14}x^2 - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 4A^{10}x^2 + A^{10} - A^7x + 4A^6x^2 - A^6 + 2A^3x - 2A^2x^2 - A^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 4A^4x - 2A^3x^2 + 2A^3 + 4x + 5A^{-1}x^2 - 3A^{-1} - A^{-4}x - 3A^{-5}x^2 - A^{-5} + A^{-9}x^2 + 2A^{-9} - 2A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 3A^8x + 2A^7x^2 - 2A^7 - 4A^4x - 4A^3x^2 + 2A^3 + 2x + 4A^{-1}x^2 - A^{-1} - A^{-5}x^2 - 3A^{-5} + 2A^{-9}$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + 3A^2x^2 - A^2 + A^{-1}x - 4A^{-2}x^2 + A^{-2} - 2A^{-5}x + 2A^{-6}x^2 + A^{-6} + A^{-9}x - A^{-10}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 + A^3x - 4A^2x^2 + A^2 - 2A^{-1}x + A^{-2}x^2 + A^{-2} + A^{-5}x - A^{-6}$
$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 3A^4x - 2A^3x^2 + 2A^3 + 4x + 4A^{-1}x^2 - 2A^{-1} - A^{-4}x - 3A^{-5}x^2 + A^{-9}x^2 + 2A^{-9} - 2A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 3A^8x + 2A^7x^2 - 2A^7 - 3A^4x - 4A^3x^2 + 2A^3 + 2x + 3A^{-1}x^2 - A^{-5}x^2 - 2A^{-5} + 2A^{-9}$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 5A^2x^2 - 2A^2 + 2A^{-1}x - 6A^{-2}x^2 + A^{-2} - 4A^{-5}x + 3A^{-6}x^2 + A^{-6} + 2A^{-9}x - A^{-10}x^2 - A^{-10} - A^{-13}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 7A^6x^2 - 3A^6 + 3A^3x - 5A^2x^2 - 4A^{-1}x + A^{-2}x^2 + 2A^{-2} + A^{-5}x - A^{-6} - A^{-9}x$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - 2A^6x^2 + A^6 - A^3x + 2A^2x^2 + A^{-1}x - 2A^{-2}x^2 - A^{-5}x + A^{-6}x^2 - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - A^{-1}x + 2A^{-2}x^2 + A^{-5}x - A^{-6}x^2 - A^{-9}x + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^2 + A^{-6} - A^{-9}x + A^{-10}x^2 + A^{-13}x - A^{-14} - A^{-17}x$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^2 - A^{11}x + A^{10}x^2 + A^7x - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x - A^7x^2 + A^7 + 2A^3x^2 - A^3 + x - A^{-1} - A^{-9}$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^2 - A^{-2} + A^{-5}x - A^{-6}x^2 - 2A^{-9}x + A^{-10} + A^{-13}x - A^{-17}x$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10} + 2A^7x - A^6 - 2A^3x - A^2x^2 + A^2 - A^{-5}x$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x + A^{11}x^2 - A^{11} - A^8x - 2A^7x^2 + A^7 + A^3x^2 + x - A^{-1} + A^{-5}$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -A^{-1}x^2 + A^{-1} + A^{-4}x + 2A^{-5}x^2 - A^{-5} - A^{-9}x^2 + A^{-13} - A^{-17}$
$\mathcal{S}_{2,\infty}(5_{71}) = A^7x - A^3x - A^2x^2 + A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-9}x - A^{-10}x^2 + A^{-10} + A^{-13}x + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x + A^{-9}x^2 - A^{-9} - A^{-13}x^2 + A^{-13} + A^{-17} - A^{-21}$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x + A^{15}x^2 - A^{15} - 2A^{11}x^2 + A^{11} - A^8x + A^7 + A^3x^2 - A^3 + x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^2 + A^3x - A^2x^2 - A^{-1}x - A^{-9}x$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^2 - A^2 + A^{-1}x - A^{-2}x^2 - A^{-5}x + A^{-6} - A^{-13}x$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 - A^7x - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^2 + A^{-6} - A^{-9}x - A^{-10}x^2 + A^{-10} + A^{-13}x + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = A^3x^2 - A^3 - A^{-1}x^2 - A^{-4}x + A^{-5}x^2 + A^{-8}x - A^{-9}$
$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 - A^{-5}x + A^{-6} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^2 - A^{-1}x + A^{-2}x^2 - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{78}}) = A^8x - A^{-1}x^2 + A^{-1} - A^{-4}x + A^{-5}x^2 + A^{-8}x - A^{-9}$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} - A^{-17}x + A^{-18}x^2 + A^{-18} + A^{-21}x - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + 2A^{18}x^2 + A^{15}x - A^{14}x^2 - A^{11}x + A^{10} - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2 - A^{-1}x + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^2 - A^{-5}x + A^{-6} - A^{-10}x^2 + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-13}x - A^{-14}x^2 + 2A^{-18}x^2 - A^{-18} - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} - A^7x - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^2 - A^{-9}x + 2A^{-10}x^2 - A^{-10} + A^{-13}x - A^{-14}x^2 - A^{-17}x + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - 2A^{14}x^2 + A^{14} - A^{11}x + 2A^{10}x^2 - A^{10} + A^7x - A^6 - A^3x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x - A^3x^2 + A^3 + 2x + 2A^{-1}x^2 - A^{-1} - A^{-5}x^2 + A^{-9} - A^{-13}$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x + A^7x^2 - A^7 - A^4x - 2A^3x^2 + A^3 + x + A^{-1}x^2 - A^{-5} + A^{-9}$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^2 + A^{-5}x - 2A^{-6}x^2 + A^{-6} - 2A^{-9}x + A^{-10}x^2 + A^{-13}x - A^{-14} - A^{-17}x$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 3A^{10}x^2 - 2A^{10} + 2A^7x - A^6x^2 - 2A^3x - A^2x^2 + 2A^2 - A^{-5}x$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{11}x + A^{10}x^2 - A^{10} - A^3x - A^2x^2 + A^2 - A^{-1}x$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + 3A^{-12}x + A^{-13}x^2 - A^{-13} - A^{-16}x - A^{-17}x^2 + A^{-20}x + A^{-21} - A^{-24}x$
$\mathcal{S}_{2,\infty}(\overline{5_{92}}) = -A^{24}x + 2A^{20}x - 2A^{16}x - A^{15}x^2 + A^{15} + 2A^{12}x + A^{11}x^2 - A^8x - A^7 + A^4x$

## Knots in $L(5,2)$

$\mathcal{S}_{2,\infty}(2_2) = -A^4x + x - A^{-4}x^2 + A^{-4} + A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{2_3}) = -A^8x - x^2 + x + 1 + A^{-4}$
$\mathcal{S}_{2,\infty}(3_1) = A^{-4}x - A^{-8}x^2 + A^{-8} + A^{-12}x^2 - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{3_1}) = A^{12}x - A^8x + A^4x^2 - A^4 - x^2 + x + A^{-4}$
$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^2 + A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6}x + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^2 + A^{-6}x^2 - A^{-10}x^2 + A^{-10}x + A^{-14}$
$\mathcal{S}_{2,\infty}(3_4) = -A^4x + 2x - A^{-4}x^2 + A^{-4} + A^{-8}x^2 - A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{3_4}) = A^8x - A^4x + x^2 + x - 1 - A^{-4}x^2 + A^{-8}$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - 2A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6} - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + A^2 + A^{-2}x^2 - A^{-6}x^2 + A^{-6}x + A^{-10}$

$\mathcal{S}_{2,\infty}(3_6) = A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-14}x + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} - A^{-2}x^2 + A^{-2}x + A^{-6}$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14}x + A^{-14} + A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^2 - A^{6}x^2 + A^{6} + A^{2}x^2 - A^{2}x - A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}$
$\mathcal{S}_{2,\infty}(4_2) = -A^{8}x + A^{4}x^2 - A^{4} - x^2 + x + A^{-4}x^2 - A^{-4}x - A^{-8}x^2 + A^{-8}x - A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = A^{-4}x^2 - A^{-4} - A^{-8}x^2 + A^{-12}x^2 - A^{-12}x - A^{-16}x^2 + A^{-16}x - A^{-16}$
$\mathcal{S}_{2,\infty}(4_3) = 2A^{-4}x - A^{-8}x^2 - A^{-8}x + A^{-8} + 2A^{-12}x^2 - A^{-12} - A^{-16}x^2 - A^{-16} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{4_3}) = -A^{16}x + 2A^{12}x - A^{8}x^2 - A^{8}x + A^{8} + 2A^{4}x^2 - A^{4} - x^2 + x - 1 + A^{-4}$
$\mathcal{S}_{2,\infty}(4_4) = A^{2}x^2 - 3A^{-2}x^2 + 2A^{-2} + 2A^{-6}x^2 - 2A^{-10}x^2 + A^{-10}x + A^{-14}x^2 - A^{-14}x + A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + 2A^{6}x^2 - 3A^{2}x^2 + 2A^{2} + 2A^{-2}x^2 - A^{-2}x - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(4_5) = -2A^{4}x + x^2 + 2x - 1 - 2A^{-4}x^2 + A^{-4} + 2A^{-8}x^2 - A^{-8}x - A^{-12}x^2 + A^{-12}x - 2A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^{8}x - 2A^{4}x + 2x^2 + x - 2 - 2A^{-4}x^2 + A^{-8}x^2 - A^{-8}x + A^{-8} - A^{-12}x^2 + A^{-12}x - A^{-12}$
$\mathcal{S}_{2,\infty}(4_7) = A^{-4}x - A^{-8}x + 2A^{-12}x - A^{-16}x^2 - A^{-16}x + A^{-16} + A^{-20}x^2 - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{4_7}) = A^{20}x - 2A^{16}x + A^{12}x^2 + A^{12}x - A^{12} - A^{8}x^2 + A^{4}x + A^{4}$
$\mathcal{S}_{2,\infty}(4_8) = -A^{16}x + A^{12}x - A^{8}x + A^{4}x - x^2 + 1 + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{4_8}) = -x + A^{-4}x - A^{-8}x^2 + A^{-8} + A^{-12}x + A^{-12} - A^{-16}x$
$\mathcal{S}_{2,\infty}(4_{11}) = -A^{4}x + 2x - A^{-4}x^2 - A^{-4}x + A^{-4} + A^{-8}x^2 - A^{-12}x^2 + A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{4_{11}}) = -A^{12}x + A^{8}x - A^{4}x^2 - A^{4}x + A^{4} + x^2 + x - A^{-4}x^2 + A^{-8}$
$\mathcal{S}_{2,\infty}(4_{12}) = A^{2}x^2 - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - A^{-6} - 3A^{-10}x^2 + A^{-10}x + A^{-10} + A^{-14}x^2 - A^{-14}x + 2A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^{6}x^2 - A^{6} - 3A^{2}x^2 + A^{2} + 2A^{-2}x^2 - A^{-2}x - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(4_{13}) = A^{8}x - 2A^{4}x + x^2 + 2x - 1 - 2A^{-4}x^2 + A^{-4} + A^{-8}x^2 + A^{-8} - A^{-12}$
$\mathcal{S}_{2,\infty}(4_{14}) = A^{8}x - 3A^{4}x + 2x^2 + 2x - 2 - 3A^{-4}x^2 + A^{-4} + 2A^{-8}x^2 - A^{-8}x + A^{-8} - A^{-12}x^2 + A^{-12}x - 2A^{-12}$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^{6}x^2 + 2A^{2}x^2 - A^{2} - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - A^{-6}x - A^{-6} - A^{-10}x^2 + A^{-10}x - 2A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^{6}x^2 - A^{6} - 3A^{2}x^2 + A^{2} + 2A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6}x + A^{-10}x^2 - A^{-10}x + A^{-10} - A^{-14}$
$\mathcal{S}_{2,\infty}(4_{16}) = A^{-4}x - A^{-8}x^2 - A^{-8}x + A^{-8} + A^{-12}x^2 - A^{-16}x^2 + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{4_{16}}) = -A^{16}x + A^{12}x - A^{8}x^2 - A^{8}x + A^{8} + A^{4}x^2 - x^2 + x + A^{-4}$
$\mathcal{S}_{2,\infty}(4_{17}) = A^{6}x^2 - 2A^{2}x^2 + A^{2} + A^{-2}x^2 - 2A^{-6}x^2 + A^{-6}x + A^{-6} + A^{-10}x^2 - A^{-10}x + A^{-10} - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^{6}x^2 + A^{6} + A^{2}x^2 - 2A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6}x - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(4_{18}) = -x + A^{-4}x^2 + A^{-4}x - A^{-4} - 2A^{-8}x^2 + A^{-8} + 2A^{-12}x^2 - A^{-12}x - A^{-16}x^2 + A^{-16}x - 2A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^{8}x + 2A^{4}x^2 - 2A^{4} - 2x^2 + x + A^{-4}x^2 - A^{-4}x + A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^{6}x^2 - A^{6} - 2A^{2}x^2 + 2A^{-2}x^2 - A^{-2}x - A^{-2} - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^{2}x^2 - A^{2} - 2A^{-2}x^2 + 2A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-10}x + A^{-14}x^2 - A^{-14}x + A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 + 2A^{-10}x^2 - 2A^{-10} - 2A^{-14}x^2 + A^{-14}x + A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} - A^{6}x^2 + A^{2}x^2 - A^{2}x - A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^{4}x + x^2 + x - 1 - A^{-4}x^2 + A^{-8}x^2 - A^{-8}x - A^{-12}x^2 + A^{-12}x - A^{-12}$
$\mathcal{S}_{2,\infty}(4_{22}) = -A^{-12}x^2 + A^{-12}x + A^{-12} - A^{-16}x + A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{4_{22}}) = -A^{16}x - A^{4}x^2 + A^{4}x + A^{4} + 1$
$\mathcal{S}_{2,\infty}(\overline{4_{23}}) = A^{-8}x - A^{-12}x^2 + A^{-12}x + A^{-12} + A^{-16}x^2 - A^{-16}x - A^{-20}$
$\mathcal{S}_{2,\infty}(4_{24}) = A^{-16}x^2 - A^{-16}x - A^{-16} - A^{-20}x^2 + A^{-20}x - A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = A^{8}x^2 - A^{8}x - A^{8} - A^{4} - A^{-4}x^2 + A^{-4}x$
$\mathcal{S}_{2,\infty}(4_{27}) = A^{8}x - 2A^{4}x + x - A^{-4}x^2 - A^{-4}x + A^{-4} + A^{-8}x + A^{-8}$
$\mathcal{S}_{2,\infty}(5_1) = A^{-8}x - A^{-12}x^2 + A^{-12}x + A^{-12} + A^{-16}x^2 - A^{-16}x - A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_1}) = A^{8}x^2 - A^{8} - A^{4}x^2 + A^{4}x + 1$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} + 2A^{-14}x^2 - A^{-14} - 2A^{-18}x^2 + A^{-18}x + A^{-18} + A^{-22}x^2 - A^{-22}x + A^{-22} - A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - 2A^{10}x^2 + A^{10} + A^{6}x^2 - A^{6}x + A^{6} - A^{2}x^2 + A^{2}x - A^{2} + A^{-2}$

$\mathcal{S}_{2,\infty}(5_3) = -A^{12}x + A^8x^2 + A^8x - A^8 - 2A^4x^2 + A^4x + A^4 + x^2 + 1 - A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{5_3}) = x - A^{-4}x + A^{-8}x^2 + A^{-8}x - A^{-8} - 2A^{-12}x^2 + A^{-12}x + A^{-12} + A^{-16}x^2 - A^{-16}x + A^{-16} - A^{-20}$

$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 + A^2x^2 - A^2 - A^{-2}x^2 + A^{-2}x + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} + A^{-14}x - A^{-14} + A^{-18}$

$\mathcal{S}_{2,\infty}(5_5) = -A^{-12}x^2 + A^{-12} + A^{-16}x^2 - A^{-16}x - 2A^{-20}x^2 + 2A^{-20}x + A^{-24}x^2 - A^{-24}x + A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_5}) = -A^{12}x^2 + A^{12} + A^8x^2 - A^8x - A^4x^2 + A^4x + x^2 - x + 1 - A^{-4}x^2 + A^{-4}x$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} - A^6x^2 + A^6 + A^2x^2 - A^2x - A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}x^2 - A^{-6}x + A^{-6} - A^{-10}x^2 + A^{-10}x + A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = -A^{-6}x^2 + A^{-6} + 2A^{-10}x^2 - A^{-10}x - A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}x^2 + A^{-22}x + A^{-26}x$

$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} + 3A^{-10}x^2 - 2A^{-10} - 3A^{-14}x^2 + A^{-14}x + 3A^{-18}x^2 - 2A^{-18}x - A^{-22}x^2 + A^{-22}x - 2A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 + 3A^{10}x^2 - 2A^{10} - 3A^6x^2 + A^6x + A^6 + 2A^2x^2 - 2A^2x + A^2 - A^{-2}x^2 + A^{-2}x - 2A^{-2} + A^{-6}$

$\mathcal{S}_{2,\infty}(5_8) = -x + A^{-4}x^2 + A^{-4}x - A^{-4} - 3A^{-8}x^2 + 2A^{-8} + 2A^{-12}x^2 - A^{-12}x + A^{-12} - 2A^{-16}x^2 + 2A^{-16}x - A^{-16} + A^{-20}x^2 - A^{-20}x + A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x - A^8x^2 - 2A^8x + A^8 + 2A^4x^2 - A^4 - 3x^2 + 2x + 1 + 2A^{-4}x^2 - 2A^{-4}x + 2A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 - 3A^2x^2 + A^2 + 3A^{-2}x^2 - A^{-2}x - A^{-2} - 3A^{-6}x^2 + 2A^{-6}x + 2A^{-10}x^2 - 2A^{-10}x + 2A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} + 4A^{-6}x^2 - 2A^{-6}x - A^{-6} - 2A^{-10}x^2 + 2A^{-10}x - 2A^{-10} + A^{-14}x^2 - A^{-14}x + 2A^{-14} - A^{-18}x^2 + A^{-18}x + A^{-22}x$

$\mathcal{S}_{2,\infty}(5_{10}) = 2A^{-4}x - A^{-8}x^2 - 2A^{-8}x + A^{-8} + 2A^{-12}x^2 + A^{-12}x - A^{-12} - 2A^{-16}x^2 + A^{-20}x^2 + A^{-20} - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{10}}) = A^{20}x - 2A^{16}x + A^{12}x^2 + 2A^{12}x - A^{12} - 2A^8x^2 - A^8x + A^8 + 2A^4x^2 - x^2 + x - 1 + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - 4A^{-2}x^2 + 2A^{-2} + 4A^{-6}x^2 - A^{-6} - 4A^{-10}x^2 + A^{-10}x + A^{-10} + 3A^{-14}x^2 - 2A^{-14}x + A^{-14} - A^{-18}x^2 + A^{-18}x - 2A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 - 5A^2x^2 + A^2x + 2A^2 + 3A^{-2}x^2 - 2A^{-2}x + A^{-2} - A^{-6}x^2 + A^{-6}x - 2A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{12}) = -2x + A^{-4}x^2 + 3A^{-4}x - A^{-4} - 4A^{-8}x^2 - A^{-8}x + 3A^{-8} + 4A^{-12}x^2 - A^{-12}x - 3A^{-16}x^2 + 2A^{-16}x - 2A^{-16} + A^{-20}x^2 - A^{-20}x + 2A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 3A^{12}x - 2A^8x^2 - 3A^8x + 2A^8 + 4A^4x^2 - 2A^4 - 4x^2 + 2x + 2A^{-4}x^2 - 2A^{-4}x + 3A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + 3A^2x^2 - 2A^2 - 4A^{-2}x^2 + A^{-2} + 4A^{-6}x^2 - A^{-6}x - A^{-6} - 3A^{-10}x^2 + 2A^{-10}x - A^{-10} + A^{-14}x^2 - A^{-14}x + 2A^{-14} - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^6x^2 - A^6 - 4A^2x^2 + A^2 + 4A^{-2}x^2 - A^{-2}x - 2A^{-2} - 3A^{-6}x^2 + 2A^{-6}x - A^{-6} + A^{-10}x^2 - A^{-10}x + 2A^{-10} - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{14}) = -2A^4x + x^2 + 3x - 1 - 3A^{-4}x^2 - A^{-4}x + 2A^{-4} + 3A^{-8}x^2 - A^{-8}x - 3A^{-12}x^2 + 2A^{-12}x - A^{-12} + A^{-16}x^2 - A^{-16}x + 2A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x - 2A^4x^2 - 2A^4x + 2A^4 + 3x^2 + x - 1 - 3A^{-4}x^2 + A^{-4}x + 2A^{-8}x^2 - 2A^{-8}x + 2A^{-8} - A^{-12}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^6x^2 - A^6 - 6A^2x^2 + 3A^2 + 6A^{-2}x^2 - 2A^{-2}x - 2A^{-2} - 4A^{-6}x^2 + 3A^{-6}x - 2A^{-6} + 2A^{-10}x^2 - 2A^{-10}x + 3A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 4A^2x^2 - 2A^2 - 6A^{-2}x^2 + 3A^{-2} + 6A^{-6}x^2 - 2A^{-6}x - A^{-6} - 4A^{-10}x^2 + 3A^{-10}x - 2A^{-10} + 2A^{-14}x^2 - 2A^{-14}x + 3A^{-14} - A^{-18}x^2 + A^{-18}x - A^{-18} + A^{-22}x$

$\mathcal{S}_{2,\infty}(5_{16}) = 2A^8x - A^4x^2 - 3A^4x + A^4 + 3x^2 + 2x - 2 - 4A^{-4}x^2 + A^{-4}x + A^{-4} + 3A^{-8}x^2 - 2A^{-8}x + 2A^{-8} - A^{-12}x^2 + A^{-12}x - 2A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 3A^4x + 2x^2 + 3x - 2 - 4A^{-4}x^2 + 2A^{-4} + 3A^{-8}x^2 - A^{-8}x + A^{-8} - 2A^{-12}x^2 + 2A^{-12}x - 2A^{-12} + A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 + 4A^6x^2 - 2A^6 - 4A^2x^2 + A^2x + A^2 + 2A^{-2}x^2 - A^{-2}x + A^{-2} - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + 4A^{-6}x^2 - 2A^{-6} - 3A^{-10}x^2 + A^{-10}x + 2A^{-14}x^2 - A^{-14}x + A^{-14} - A^{-18}x^2 + A^{-18}x - A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} + 3A^{-10}x^2 - A^{-10} - 4A^{-14}x^2 + A^{-14}x + 2A^{-14} + 3A^{-18}x^2 - 2A^{-18}x + A^{-18} - A^{-22}x^2 + A^{-22}x - 2A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} + 3A^{10}x^2 - A^{10} - 3A^6x^2 + A^6x + A^6 + 2A^2x^2 - 2A^2x + A^2 - A^{-2}x^2 + A^{-2}x - 2A^{-2} + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{19}) = -2A^4x + x^2 + 2x - 1 - 3A^{-4}x^2 + 2A^{-4} + 2A^{-8}x^2 - A^{-8}x + A^{-8} - 2A^{-12}x^2 + 2A^{-12}x - A^{-12} + A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x - A^4x^2 - 2A^4x + A^4 + 2x^2 + x - 1 - 3A^{-4}x^2 + A^{-4}x + A^{-4} + 2A^{-8}x^2 - 2A^{-8}x + 2A^{-8} - A^{-12}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{20}) = 2A^2x^2 - A^2 - 4A^{-2}x^2 + 2A^{-2} + 4A^{-6}x^2 - A^{-6}x - A^{-6} - 3A^{-10}x^2 + 2A^{-10}x - A^{-10} + 2A^{-14}x^2 - 2A^{-14}x + 2A^{-14} - A^{-18}x^2 + A^{-18}x - A^{-18} + A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^6x^2 - A^6 - 4A^2x^2 + 2A^2 + 4A^{-2}x^2 - 2A^{-2}x - A^{-2} - 2A^{-6}x^2 + 2A^{-6}x - 2A^{-6} + A^{-10}x^2 - A^{-10}x + 2A^{-10} - A^{-14}x^2 + A^{-14}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{21}) = 2A^{12}x - A^8x^2 - 2A^8x + A^8 + 3A^4x^2 - 2A^4 - 3x^2 + 2x + 2A^{-4}x^2 - 2A^{-4}x + 2A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x + A^{-4}x^2 + 2A^{-4}x - A^{-4} - 3A^{-8}x^2 + 2A^{-8} + 3A^{-12}x^2 - A^{-12}x - 2A^{-16}x^2 + 2A^{-16}x - 2A^{-16} + A^{-20}x^2 - A^{-20}x + A^{-20}$

$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + 2A^2x^2 - A^2 - 3A^{-2}x^2 + 2A^{-2} + 2A^{-6}x^2 - A^{-6}x + A^{-6} - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - 3A^2x^2 + 2A^2 + 2A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6}x + A^{-10}x^2 - A^{-10}x + A^{-10} - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{23}) = -A^{12}x + A^8x - A^4x^2 + A^4 + x^2 + x - A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{5_{23}}) = 2x - A^{-4}x + A^{-8}x^2 - A^{-8} - A^{-12}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} + A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^6 - A^{-2}x^2 + A^{-2}x + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} + 4A^{-18}x^2 - A^{-18} - 3A^{-22}x^2 + A^{-22}x + A^{-26}x^2 - A^{-26}x + 2A^{-26} - A^{-30}$

$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} - 4A^{14}x^2 + 2A^{14} + 3A^{10}x^2 - A^{10}x - 2A^6x^2 + A^6x - A^6 + A^2$

$\mathcal{S}_{2,\infty}(5_{26}) = -x + 3A^{-4}x - A^{-8}x^2 - 2A^{-8}x + A^{-8} + 2A^{-12}x^2 + 2A^{-12}x - A^{-12} - 2A^{-16}x^2 - A^{-16}x + A^{-20}x^2 + A^{-20} - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{26}}) = A^{20}x - 3A^{16}x + A^{12}x^2 + 3A^{12}x - A^{12} - 2A^8x^2 - A^8x + A^8 + 2A^4x^2 + A^4x - x^2 - 1 + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{27}) = -x + 3A^{-4}x - A^{-8}x^2 - 2A^{-8}x + A^{-8} + 2A^{-12}x^2 + 2A^{-12}x - A^{-12} - 2A^{-16}x^2 - A^{-16}x + A^{-20}x^2 + A^{-20} - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{27}}) = A^{20}x - 3A^{16}x + A^{12}x^2 + 3A^{12}x - A^{12} - 2A^8x^2 - A^8x + A^8 + 2A^4x^2 + A^4x - x^2 - 1 + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 + 2A^{-2}x^2 - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - A^{-6} - 3A^{-10}x^2 + A^{-10}x + A^{-14}x^2 + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{30}) = A^8x - A^4x + 2x - 2A^{-4}x + A^{-8}x^2 + A^{-8}x - A^{-8} - A^{-12}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{30}}) = -A^{12}x + 2A^8x - A^4x^2 - A^4x + A^4 + x^2 + x - A^{-4}x - A^{-4} + A^{-8}x$

$\mathcal{S}_{2,\infty}(5_{31}) = A^8x - 2A^4x + 3x - A^{-4}x^2 - 2A^{-4}x + A^{-4} + 2A^{-8}x^2 + A^{-8}x - A^{-8} - A^{-12}x^2 - A^{-12} + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{31}}) = -A^{12}x + 3A^8x - A^4x^2 - 2A^4x + A^4 + 2x^2 + x - 1 - A^{-4}x^2 - A^{-4}x - A^{-4} + A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{32}) = A^8x - 2A^4x + 2x - A^{-4}x^2 - 2A^{-4}x + A^{-4} + A^{-8}x^2 + A^{-8}x - A^{-12}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{32}}) = -A^{12}x + 2A^8x - A^4x^2 - 2A^4x + A^4 + x^2 + x - A^{-4}x^2 - A^{-4}x + A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} + 3A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-10}x - A^{-10} + 2A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 + 3A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6}x + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 + 4A^{-18}x^2 - 2A^{-18} - 3A^{-22}x^2 + A^{-22}x + A^{-26}x^2 - A^{-26}x + 2A^{-26} - A^{-30}$

$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} - 3A^{14}x^2 + 3A^{10}x^2 - A^{10}x - A^{10} - 2A^6x^2 + A^6x - A^6 + A^2$

$\mathcal{S}_{2,\infty}(5_{36}) = -x + 2A^{-4}x - A^{-8}x^2 - 2A^{-8}x + A^{-8} + A^{-12}x^2 + 2A^{-12}x - 2A^{-16}x^2 - A^{-16}x + A^{-16} + A^{-20}x^2 + A^{-20} - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{36}}) = A^{20}x - 3A^{16}x + A^{12}x^2 + 2A^{12}x - A^{12} - 2A^8x^2 - A^8x + A^8 + A^4x^2 + A^4x + A^4 - x^2 + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + 2A^{-14}x^2 - A^{-14} - 2A^{-18}x^2 + A^{-18}x + A^{-18} + A^{-22}x^2 - A^{-22}x + A^{-22} - A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - 2A^{10}x^2 + 2A^6x^2 - A^6x - A^6 - A^2x^2 + A^2x - 2A^2 + A^{-2}$

$\mathcal{S}_{2,\infty}(5_{39}) = -A^{12}x + A^8x^2 + A^8x - A^8 - 2A^4x^2 + A^4 + x^2 + 1 - A^{-4}x^2 + A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{39}}) = -A^4x + x - A^{-4}x^2 - A^{-4}x + A^{-4} + A^{-8}x^2 + A^{-8}x - 2A^{-12}x^2 + A^{-12}x + A^{-12} + A^{-16}x^2 - A^{-16}x + A^{-16} - A^{-20}$

$\mathcal{S}_{2,\infty}(5_{40}) = A^{-8}x - A^{-12}x^2 + A^{-12} + A^{-16}x^2 - A^{-20}x^2 + A^{-24}x^2 - A^{-28}$

$\mathcal{S}_{2,\infty}(\overline{5_{40}}) = A^{24}x - A^{20}x + A^{16}x^2 - A^{16} - A^{12}x^2 + A^8x^2 - A^4x^2 + A^4x + 1$

$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 + A^2x^2 - A^2 - A^{-2}x^2 + A^{-2}x + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-10}x^2 - A^{-10} - A^{-14}x^2 + A^{-14}x + 2A^{-18}$

$\mathcal{S}_{2,\infty}(5_{42}) = A^{-4}x - A^{-8}x^2 - A^{-8}x + A^{-8} + A^{-12}x^2 + A^{-12}x - A^{-16}x^2 + A^{-20}x^2 - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{42}}) = A^{20}x - A^{16}x + A^{12}x^2 + A^{12}x - A^{12} - A^8x^2 - A^8x + A^4x^2 - x^2 + x + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 + 2A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6}x + 2A^{-10}x^2 - A^{-10}x - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 + 2A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2}x + A^{-2} + 2A^{-6}x^2 - A^{-6}x + A^{-6} - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{44}) = -x + A^{-4}x^2 + 2A^{-4}x - A^{-4} - 3A^{-8}x^2 - A^{-8}x + 2A^{-8} + 3A^{-12}x^2 - A^{-12}x - 3A^{-16}x^2 + 2A^{-16}x - A^{-16} + A^{-20}x^2 - A^{-20}x + 2A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x - 2A^8x^2 - 2A^8x + 2A^8 + 3A^4x^2 - A^4 - 3x^2 + 2x + 2A^{-4}x^2 - 2A^{-4}x + 2A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 - 4A^2x^2 + 2A^2 + 3A^{-2}x^2 - A^{-2}x - 3A^{-6}x^2 + 2A^{-6}x + A^{-10}x^2 - A^{-10}x + 2A^{-10} - A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 - 4A^{-2}x^2 + 2A^{-2} + 4A^{-6}x^2 - A^{-6}x - A^{-6} - 3A^{-10}x^2 + 2A^{-10}x - A^{-10} + A^{-14}x^2 - A^{-14}x + 2A^{-14} - A^{-18}$

$\mathcal{S}_{2,\infty}(5_{46}) = -x + A^{-4}x^2 + 3A^{-4}x - A^{-4} - 3A^{-8}x^2 - A^{-8}x + 2A^{-8} + 4A^{-12}x^2 - A^{-12}x - A^{-12} - 3A^{-16}x^2 + 2A^{-16}x - 2A^{-16} + A^{-20}x^2 - A^{-20}x + 2A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 3A^{12}x - 2A^8x^2 - 2A^8x + 2A^8 + 4A^4x^2 - 2A^4 - 3x^2 + 2x - 1 + 2A^{-4}x^2 - 2A^{-4}x + 2A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 - 5A^2x^2 + 2A^2 + 5A^{-2}x^2 - 2A^{-2}x - A^{-2} - 4A^{-6}x^2 + 3A^{-6}x - A^{-6} + 2A^{-10}x^2 - 2A^{-10}x + 3A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 - 5A^{-2}x^2 + 2A^{-2} + 6A^{-6}x^2 - 2A^{-6}x - 2A^{-6} - 4A^{-10}x^2 + 3A^{-10}x - 2A^{-10} + 2A^{-14}x^2 - 2A^{-14}x + 3A^{-14} - A^{-18}x^2 + A^{-18}x - A^{-18} + A^{-22}x$

$\mathcal{S}_{2,\infty}(5_{48}) = A^4x - x^2 - x + 1 + 2A^{-4}x^2 + A^{-4}x - A^{-4} - 3A^{-8}x^2 + A^{-8}x + A^{-8} + 3A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^2 + A^{-16}x - 2A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x + 2A^4x^2 + A^4x - 2A^4 - 3x^2 + x + 1 + 2A^{-4}x^2 - A^{-4}x + A^{-4} - 2A^{-8}x^2 + 2A^{-8}x - A^{-8} + A^{-12}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2x + 2A^{-2}x^2 - A^{-2}x - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + 3A^{-6}x^2 - A^{-6} - 3A^{-10}x^2 + A^{-10}x + A^{-10} + 2A^{-14}x^2 - A^{-14}x + A^{-14} -$

$A^{-18}x^2 + A^{-18}x - A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(5_{51}) = A^{-4}x - A^{-8}x + 3A^{-12}x - A^{-16}x^2 - 2A^{-16}x + A^{-16} + 2A^{-20}x^2 - A^{-20} - A^{-24}x^2 - A^{-24} + A^{-28}$

$\mathcal{S}_{2,\infty}(\overline{5_{51}}) = -A^{24}x + 2A^{20}x - A^{16}x^2 - 2A^{16}x + A^{16} + 2A^{12}x^2 + A^{12}x - A^{12} - A^8x^2 - A^8 + A^4x + A^4$

$\mathcal{S}_{2,\infty}(5_{52}) = -A^{16}x + A^{12}x - A^8x + 2A^4x - x^2 + 1 + A^{-4}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{52}}) = A^4x - x + A^{-4}x^2 + A^{-4}x - A^{-4} - A^{-8}x^2 + A^{-12}x + A^{-12} - A^{-16}x$

$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} + 2A^{-10}x^2 - A^{-10} - 2A^{-14}x^2 + A^{-14}x + A^{-18}x^2 + A^{-18} - A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 + A^2x^2 - A^{-2}x^2 + A^{-2}x + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{54}) = A^{20}x - 2A^{16}x + A^{12}x^2 + 2A^{12}x - A^{12} - A^8x^2 - A^8x + A^4x^2 + A^4x - x^2 + A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{5_{54}}) = -x + 2A^{-4}x - A^{-8}x^2 - A^{-8}x + A^{-8} + A^{-12}x^2 + 2A^{-12}x - A^{-16}x^2 - A^{-16}x + A^{-20}x^2 - A^{-24}$

$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} - 4A^{10}x^2 + 2A^{10} + 3A^6x^2 - A^6x - 2A^2x^2 + A^2x - A^2 + 2A^{-2}$

$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} + 3A^{-14}x^2 - A^{-14} - 2A^{-18}x^2 + A^{-18}x + A^{-22}x^2 - A^{-22}x + A^{-22} - A^{-26}$

$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 + 2A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + 2A^{-6}x^2 - 3A^{-10}x^2 + A^{-10}x + A^{-10} + A^{-14}x^2 + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{58}) = -A^4x + 2x - A^{-4}x^2 - A^{-4}x + A^{-4} + A^{-8}x^2 + A^{-8}x - A^{-12}x^2 + A^{-16}x^2 - A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{58}}) = A^{16}x - A^{12}x + A^8x^2 + A^8x - A^8 - A^4x^2 - A^4x + x^2 + x - A^{-4}x^2 + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + 4A^{-6}x^2 - A^{-6} - 4A^{-10}x^2 + A^{-10}x + A^{-10} + 3A^{-14}x^2 - A^{-14}x - A^{-18}x^2 + A^{-18}x - 2A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 - 4A^2x^2 + A^2x + A^2 + 2A^{-2}x^2 - A^{-2}x + A^{-2} - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{60}) = A^8x - 2A^4x + x^2 + 3x - 1 - 2A^{-4}x^2 - A^{-4}x + A^{-4} + 2A^{-8}x^2 - A^{-12}x^2 - A^{-12} + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{60}}) = -A^{12}x + 2A^8x - A^4x^2 - 2A^4x + A^4 + 2x^2 + 2x - 1 - 2A^{-4}x^2 + A^{-8}x^2 + A^{-8} - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - 4A^{-2}x^2 + A^{-2} + 5A^{-6}x^2 - 2A^{-6} - 5A^{-10}x^2 + A^{-10}x + A^{-10} + 4A^{-14}x^2 - 2A^{-14}x - A^{-18}x^2 + A^{-18}x - 3A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 4A^{10}x^2 + A^{10} + 5A^6x^2 - 2A^6 - 5A^2x^2 + A^2x + A^2 + 3A^{-2}x^2 - 2A^{-2}x + A^{-2} - A^{-6}x^2 + A^{-6}x - 2A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 4A^4x + 2x^2 + 4x - 2 - 5A^{-4}x^2 - A^{-4}x + 3A^{-4} + 4A^{-8}x^2 - A^{-8}x + A^{-8} - 3A^{-12}x^2 + 2A^{-12}x - 2A^{-12} + A^{-16}x^2 - A^{-16}x + 2A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 3A^8x - 2A^4x^2 - 4A^4x + 2A^4 + 4x^2 + 2x - 2 - 5A^{-4}x^2 + A^{-4}x + A^{-4} + 3A^{-8}x^2 - 2A^{-8}x + 3A^{-8} - A^{-12}x^2 + A^{-12}x - 2A^{-12}$

$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + 3A^2x^2 - A^2 - 5A^{-2}x^2 + 2A^{-2} + 5A^{-6}x^2 - A^{-6}x - A^{-6} - 4A^{-10}x^2 + 2A^{-10}x + A^{-14}x^2 - A^{-14}x + 3A^{-14} - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 - 5A^2x^2 + 2A^2 + 4A^{-2}x^2 - A^{-2}x - A^{-2} - 3A^{-6}x^2 + 2A^{-6}x - A^{-6} + A^{-10}x^2 - A^{-10}x + 2A^{-10} - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 3A^4x + 2x^2 + 4x - 2 - 4A^{-4}x^2 - A^{-4}x + 2A^{-4} + 4A^{-8}x^2 - A^{-8}x - 3A^{-12}x^2 + 2A^{-12}x - 2A^{-12} + A^{-16}x^2 - A^{-16}x + 2A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 3A^8x - 2A^4x^2 - 3A^4x + 2A^4 + 4x^2 + 2x - 2 - 4A^{-4}x^2 + A^{-4}x + 3A^{-8}x^2 - 2A^{-8}x + 2A^{-8} - A^{-12}x^2 + A^{-12}x - 2A^{-12}$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 5A^2x^2 - 2A^2 - 8A^{-2}x^2 + 3A^{-2} + 9A^{-6}x^2 - 2A^{-6}x - 3A^{-6} - 7A^{-10}x^2 + 4A^{-10}x - A^{-10} + 3A^{-14}x^2 - 3A^{-14}x + 5A^{-14} - A^{-18}x^2 + A^{-18}x - 2A^{-18} + A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 7A^6x^2 - 3A^6 - 8A^2x^2 + 3A^2 + 8A^{-2}x^2 - 3A^{-2}x - 2A^{-2} - 5A^{-6}x^2 + 4A^{-6}x - 3A^{-6} + 2A^{-10}x^2 - 2A^{-10}x + 4A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - 2A^6x^2 + A^6 + 3A^2x^2 - A^2 - 4A^{-2}x^2 + A^{-2}x + A^{-2} + 3A^{-6}x^2 - A^{-6}x - A^{-10}x^2 + A^{-10}x - 2A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 + 3A^{-2}x^2 - A^{-2} - 3A^{-6}x^2 + A^{-6}x + A^{-6} + 2A^{-10}x^2 - A^{-10}x +$

$A^{-10} - A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^2 + A^{-6} + 2A^{-10}x^2 - A^{-10} - 2A^{-14}x^2 + A^{-14}x + 2A^{-18}x^2 - A^{-18}x - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6x + A^6 + A^2x^2 - A^2x + A^2 - A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x + A^4x^2 - A^4 - 2x^2 + x + 1 + A^{-4}x^2 - A^{-4}x + A^{-4} - 2A^{-8}x^2 + 2A^{-8}x + A^{-12}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = -x^2 + 1 + A^{-4}x^2 - 2A^{-8}x^2 + A^{-8}x + A^{-8} + 2A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^2 + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6} + 3A^{-10}x^2 - A^{-10}x - A^{-10} - 3A^{-14}x^2 + 2A^{-14}x + 2A^{-18}x^2 - 2A^{-18}x + 2A^{-18} - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}x$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 + 3A^2x^2 - 2A^2x - A^2 - 2A^{-2}x^2 + 2A^{-2}x - 2A^{-2} + A^{-6}x^2 - A^{-6}x + 2A^{-6} - A^{-10}x^2 + A^{-10}x + A^{-14}x$

$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x - A^8x^2 - A^8x + A^8 + 2A^4x^2 - A^4 - 2x^2 + 2x + 2A^{-4}x^2 - 2A^{-4}x + A^{-4} - A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = A^{-4}x^2 + A^{-4}x - A^{-4} - 2A^{-8}x^2 + A^{-8} + 2A^{-12}x^2 - A^{-12}x - 2A^{-16}x^2 + 2A^{-16}x - A^{-16} + A^{-20}x^2 - A^{-20}x + A^{-20}$

$\mathcal{S}_{2,\infty}(5_{71}) = -A^6x^2 + A^6 + A^2x^2 - A^2x - A^2 + A^{-2}x - 2A^{-2}$

$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-14}x^2 + A^{-14}x + A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x - A^{-12}x^2 + A^{-12} + 2A^{-16}x^2 - A^{-16}x - A^{-16} - 2A^{-20}x^2 + 2A^{-20}x - A^{-20} + A^{-24}x^2 - A^{-24}x + A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x - A^{12}x^2 + A^{12} + 2A^8x^2 - A^8x - A^8 - A^4x^2 + A^4x - A^4 + x^2 - x + 1 - A^{-4}x^2 + A^{-4}x$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^2 - 2A^2x^2 + A^2 + 2A^{-2}x^2 - A^{-2}x - A^{-2} - A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}x^2 - A^{-10}x + A^{-10} - A^{-14}x^2 + A^{-14}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-2} + 2A^{-6}x^2 - A^{-6}x - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}x^2 - A^{-14}x + A^{-14} - A^{-18}x^2 + A^{-18}x + A^{-22}x$

$\mathcal{S}_{2,\infty}(5_{74}) = A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 - A^2x^2 + A^2x + A^{-2}$

$\mathcal{S}_{2,\infty}(5_{75}) = -A^6x^2 + A^6 + A^2x^2 - A^2x - A^{-2}x^2 + A^{-2}x + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^2 + A^{-6} - A^{-14}x^2 + A^{-14}x + A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{76}}) = -x^2 + 1 + A^{-4}x^2 - A^{-4}x - A^{-8}x^2 + A^{-8}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 + A^{-6}x^2 - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^2 + 2A^{-2}x^2 - A^{-2} - 2A^{-6}x^2 + A^{-6}x + A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{78}) = -x^2 + x + 1 + A^{-4}x^2 - A^{-4}x + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{78}}) = A^8x + A^{-4}x^2 - A^{-4}x - A^{-4} - A^{-8}x^2 + A^{-8}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} + 2A^{-18}x^2 - 2A^{-22}x^2 + A^{-22}x + A^{-26}x^2 - A^{-26}x + A^{-26} - A^{-30}$

$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + 2A^{18}x^2 - 2A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10}x - 2A^6x^2 + A^6x + A^2$

$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2 + A^{-2}x^2 - A^{-6}x^2 + A^{-6}x + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^2 + A^{-6}x^2 - 2A^{-10}x^2 + A^{-10}x + A^{-14}x^2$

$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-14} + A^{-18}x^2 + A^{-18}x - A^{-18} - A^{-22}x^2 + A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} - A^2x^2 + A^2x + A^{-2}$

$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^2 + 3A^{-10}x^2 - 2A^{-10} - 3A^{-14}x^2 + A^{-14}x + A^{-14} + 2A^{-18}x^2 - A^{-18}x + A^{-18} - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - 2A^{14}x^2 + A^{14} + 3A^{10}x^2 - 2A^{10} - 2A^6x^2 + A^6x + A^2x^2 - A^2x + A^2 - A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x + x^2 + 2x - 1 - 2A^{-4}x^2 + A^{-4} + 2A^{-8}x^2 - A^{-8}x - 2A^{-12}x^2 + 2A^{-12}x - A^{-12} + A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x - A^4x^2 - A^4x + A^4 + 2x^2 + x - 1 - 2A^{-4}x^2 + A^{-4}x + 2A^{-8}x^2 - 2A^{-8}x + A^{-8} - A^{-12}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^2 - 3A^{-6}x^2 + 2A^{-6} + 4A^{-10}x^2 - A^{-10}x - 2A^{-10} - 3A^{-14}x^2 + 2A^{-14}x - A^{-14} + 2A^{-18}x^2 - 2A^{-18}x + 2A^{-18} - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}x$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 3A^{10}x^2 - 2A^{10} - 3A^6x^2 + 2A^6 + 3A^2x^2 - 2A^2x - 2A^{-2}x^2 + 2A^{-2}x - 2A^{-2} + A^{-6}x^2 - A^{-6}x + 2A^{-6} - A^{-10}x^2 + A^{-10}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{85}) = A^{-22}x^2 - A^{-22}x - A^{-26}x^2 + A^{-26}x + A^{-30}x$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^6x^2 - A^6x - A^2 - A^{-6}x^2 + A^{-6}x + A^{-6} + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{92}) = A^{-4}x - 2A^{-8}x + 3A^{-12}x - A^{-16}x^2 - A^{-16}x + A^{-16} + A^{-20}x^2 + A^{-20}x - A^{-24}x - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_{92}}) = -A^{24}x + 2A^{20}x - 2A^{16}x + A^{12}x^2 + 2A^{12}x - A^{12} - A^8x^2 - A^8x + A^4x + A^4$

## Knots in $L(6,1)$

$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^2 - x^2 + 1 + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^2 - A^{-4}x^2 + A^{-4} + A^{-6} + A^{-8}$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - 2A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + A^2 - x^2 + 1 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} - A^4x^2 + A^4 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} + A^{-10} + A^{-12}x^2 - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^2 + A^8x^2 - A^8 - A^4x^2 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x + x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x$
$\mathcal{S}_{2,\infty}(4_4) = A^2x^2 - 3A^{-2}x^2 + 2A^{-2} - A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8}x^2 - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + 2A^6x^2 + A^4x^2 - A^4 - 2A^2x^2 + A^2 - x^2 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x - A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - A^{-4}x$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 2A^{-6}x^2 + A^{-8}x^2 - A^{-10}x^2 - A^{-12} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 + A^4x^2 - A^4 - 2A^2x^2 - x^2 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x - A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + 2A^2x^2 - A^2 + x^2 - 1 - 2A^{-2}x^2 - A^{-4}x^2 + A^{-6}x^2 + A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - x^2 + 1 + A^{-2}x^2 + A^{-4}x^2 - A^{-6} - A^{-8}$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - 2A^2x^2 + A^2 - x^2 + 1 + A^{-2} + A^{-4}x^2 - A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^2 + x^2 - 1 - A^{-2}x^2 - A^{-4}x^2 + A^{-6} + A^{-8}$
$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x - A^{-8}x^3 + A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^6x^2 - A^6 + A^4x^2 - A^4 - A^2x^2 - A^2 - x^2 + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-8}x^2 - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 - A^{-8}x^2 + A^{-8} + A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-14} - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} + A^8x^2 - A^8 - A^6 - A^4x^2 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3$
$\mathcal{S}_{2,\infty}(4_{24}) = A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^4x^3 - A^4x$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} - A^{-12}x^2 + A^{-12} + A^{-14}x^2 + A^{-16}x^2 - A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{12}x^2 - A^{12} - A^{10}x^2 - A^8x^2 - A^6x^2 + 2A^6 + A^4$
$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 - A^4x^2 + A^4 + 1$
$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} - A^{-10}x^2 + A^{-10} + A^{-12} + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x - A^8x + A^4x^3 - A^4x$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} + A^8x^2 - A^8 - A^4x^2 - A^2x^2 + A^2 - A^{-4}$
$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} + 2A^{-10}x^2 - A^{-10} + 2A^{-12}x^2 - A^{-12} - 2A^{-14} - A^{-16}x^2 - A^{-16} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 - A^{12}x^2 + A^{12} + 2A^{10}x^2 - A^{10} + 2A^8x^2 - A^8 - A^6 - A^4x^2 - A^4 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x - A^{-8}x^3$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x + 2A^4x^3 - 3A^4x$
$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 + A^4x^2 - A^4 - 2A^2x^2 - 2x^2 + 1 + A^{-2} + A^{-4}x^2 - 2A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 + 2x^2 - 2 - A^{-2}x^2 - A^{-2} - 2A^{-4}x^2 + A^{-6} + A^{-8} - A^{-12}$
$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - 4A^{-2}x^2 + 2A^{-2} - A^{-4}x^2 + A^{-4} + 3A^{-6}x^2 + 2A^{-8}x^2 - A^{-8} - A^{-10}x^2 - A^{-10} - A^{-12}x^2 - A^{-12} + A^{-14} + A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} - A^8x^2 + A^8 + 3A^6x^2 + 2A^4x^2 - A^4 - 2A^2x^2 - x^2 - 1 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x - 3A^{-8}x^3 + 2A^{-8}x + A^{-12}x^3 - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x + 3A^4x^3 - 3A^4x - x$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + 3A^2x^2 - 2A^2 + x^2 - 1 - 3A^{-2}x^2 - 2A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8}x^2 + A^{-8} - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^4x^2 - A^4 - 3A^2x^2 - 2x^2 + 1 + A^{-2}x^2 + A^{-4}x^2 + A^{-4} - A^{-6} - A^{-8}$
$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-8}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x + 2x^3 - x - A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^6x^2 - A^6 + 2A^4x^2 - 2A^4 - 4A^2x^2 + A^2 - 3x^2 + 1 + A^{-2}x^2 + A^{-2} + A^{-4}x^2 + A^{-4} - A^{-6} - 2A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 4A^2x^2 - 2A^2 + 2x^2 - 2 - 4A^{-2}x^2 + A^{-2} - 3A^{-4}x^2 + A^{-4} + A^{-6}x^2 + 2A^{-6} + A^{-8}x^2 + A^{-8} - A^{-10} - 2A^{-12}$
$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x + 3x^3 - 2x - A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 - A^8x^2 + A^8 + 3A^6x^2 - A^6 + A^4x^2 - 2A^2x^2 - x^2 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 3A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-10}x^2 - A^{-10} - A^{-12}x^2 + A^{-14} + A^{-16}$
$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} + 2A^{-10}x^2 + 2A^{-12}x^2 - A^{-12} - A^{-14}x^2 - A^{-16}x^2 - A^{-16} + A^{-18} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} - A^{12}x^2 + A^{12} + 2A^{10}x^2 + 2A^8x^2 - A^8 - A^6 - A^4x^2 - A^4 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x - A^{-4}x^3$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x + 2x^3 - 2x - A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{20}) = 2A^2x^2 - A^2 + x^2 - 1 - 3A^{-2}x^2 + A^{-2} - 2A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8}x^2 - A^{-10} - 2A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + 2A^4x^2 - 2A^4 - 2A^2x^2 - 2x^2 + A^{-2} + A^{-4} - A^{-8}$
$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x + 2A^4x^3 - 2A^4x$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + 2A^2x^2 - A^2 + x^2 - 1 - 2A^{-2}x^2 + A^{-2} - A^{-4}x^2 + 2A^{-6} + A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - 3A^2x^2 + 2A^2 - x^2 + 1 + A^{-2}x^2 + A^{-4}x^2 - A^{-6} - A^{-8}$
$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} - A^{-8}x^2 + A^{-8} - A^{-10}x^2 + A^{-10} + A^{-12} + A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^6 - A^4x^2 + A^4 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} - A^{-16}x^2 + A^{-16} + 3A^{-18}x^2 + A^{-20}x^2 - A^{-22}x^2 - A^{-22} - A^{-24} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} + A^{16}x^2 - A^{16} - 3A^{14}x^2 + A^{14} - A^{12}x^2 + A^{10}x^2 + A^{10} + A^8 - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - x^2 + 1 + A^{-2}x^2 + A^{-2} + A^{-4} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 2A^{-6}x^2 + A^{-8} - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 2A^{-6}x^2 + A^{-8} - A^{-10}x^2 - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - x^2 + 1 + 2A^{-2}x^2 + A^{-4} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 - A^{-16}x^2 + A^{-16} + 3A^{-18}x^2 - A^{-18} + A^{-20}x^2 - A^{-22}x^2 - A^{-22} - A^{-24} + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} + A^{16}x^2 - A^{16} - 2A^{14}x^2 - A^{14} - A^{12}x^2 + A^{10}x^2 + A^8 - A^6x^2$

$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 - A^{-12}x^2 + A^{-12} + A^{-14}x^2 + A^{-16}x^2 - A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{12}x^2 - A^{12} - A^{10}x^2 - A^{10} - A^8x^2 + A^4 - A^2$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 - A^4x^2 + A^4 + 1$
$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 - A^{-8}x^2 + A^{-8} + A^{-12} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 - x^2 + 1 + A^{-2}x^2 + A^{-4}x^2 - A^{-6} - A^{-8}x^2 + A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 - A^4x^2 + A^4 + A^2x^2 + x^2 - A^{-2}x^2 - A^{-4}x^2 + A^{-6} + A^{-8}$
$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 2A^{-8}x^3 + A^{-8}x + A^{-12}x^3 - A^{-12}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^4x^2 - A^4 - 3A^2x^2 + A^2 - 2x^2 + 1 + 2A^{-2} + A^{-4}x^2 + A^{-4} - A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 + x^2 - 1 - 3A^{-2}x^2 + A^{-2} - 2A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8}x^2 + A^{-8} - A^{-10} - A^{-12}$
$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x - 3A^{-8}x^3 + 3A^{-8}x + A^{-12}x^3 - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x + 2A^4x^3 - A^4x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 + 2A^4x^2 - 2A^4 - 3A^2x^2 - 3x^2 + 1 + 2A^{-2} + A^{-4}x^2 + A^{-4} - 2A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 + 2x^2 - 2 - 3A^{-2}x^2 - 3A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8}x^2 + A^{-8} - A^{-10} - 2A^{-12}$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x - A^{-8}x^3 + A^{-8}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - x^3 + x$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} - A^8x^2 + A^8 + 2A^6x^2 + A^4x^2 - A^2x^2 - A^2 - x^2 + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-4}x^2 + A^{-4} + 2A^{-6}x^2 + A^{-8}x^2 - A^{-10}x^2 - A^{-12}x^2 + A^{-14} + A^{-16}$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} + A^{-10}x^2 + A^{-12} - A^{-14}x^2 + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 - A^4x^2 + A^4 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} + A^{12}x^2 - A^{12} - 3A^{10}x^2 + A^{10} - A^8x^2 + A^6x^2 + A^6 + A^4 - A^2x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} - A^{-12}x^2 + A^{-12} + 2A^{-14}x^2 + A^{-16}x^2 - A^{-18} - A^{-20}$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 - x^2 + 1 + A^{-2}x^2 + A^{-4} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-4}x^2 + A^{-4} + A^{-6}x^2 + A^{-6} + A^{-8} - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 3A^{-6}x^2 + A^{-8}x^2 - 2A^{-10}x^2 - A^{-12}x^2 + A^{-14}x^2 + A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} - A^8x^2 + A^8 + 3A^6x^2 + A^4x^2 - 2A^2x^2 - x^2 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - 4A^{-2}x^2 + A^{-2} - A^{-4}x^2 + A^{-4} + 4A^{-6}x^2 - A^{-6} + 2A^{-8}x^2 - A^{-8} - 2A^{-10}x^2 - A^{-10} - A^{-12}x^2 - A^{-12} + A^{-14}x^2 + A^{-16} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 4A^{10}x^2 + A^{10} - A^8x^2 + A^8 + 4A^6x^2 - A^6 + 2A^4x^2 - A^4 - 2A^2x^2 - A^2 - x^2 - 1 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x - 3A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x + 4x^3 - 3x - A^{-4}x^3 - A^{-4}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + 3A^2x^2 - A^2 + x^2 - 1 - 4A^{-2}x^2 + A^{-2} - 2A^{-4}x^2 + A^{-4} + 2A^{-6}x^2 + A^{-6} + A^{-8}x^2 + A^{-8} - A^{-10}x^2 - A^{-12} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 + A^4x^2 - A^4 - 4A^2x^2 + A^2 - 2x^2 + 1 + A^{-2}x^2 + A^{-2} + A^{-4}x^2 + A^{-4} - A^{-6} - A^{-8}$
$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x - 3A^{-4}x^3 + 2A^{-4}x + A^{-8}x^3 - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x + 3x^3 - x - A^{-4}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 5A^2x^2 - 2A^2 + 2x^2 - 2 - 6A^{-2}x^2 + A^{-2} - 4A^{-4}x^2 + 2A^{-4} + 3A^{-6}x^2 + A^{-6} + 2A^{-8}x^2 + A^{-8} - A^{-10}x^2 - A^{-10} - 3A^{-12} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 7A^6x^2 - 3A^6 + 3A^4x^2 - 3A^4 - 5A^2x^2 - 4x^2 + 1 + A^{-2}x^2 + 2A^{-2} + A^{-4}x^2 + 2A^{-4} - A^{-6} - 2A^{-8}$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - 2A^6x^2 + A^6 - A^4x^2 + A^4 + 2A^2x^2 + x^2 - 2A^{-2}x^2 - A^{-4}x^2 + A^{-6}x^2 + A^{-8} - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - x^2 + 1 + 2A^{-2}x^2 + A^{-4}x^2 - A^{-6}x^2 - A^{-8}x^2 + A^{-10} + A^{-12}$
$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} + A^{-10}x^2 + A^{-12}x^2 - A^{-14} - A^{-16}x^2 + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^2 - A^{12}x^2 + A^{12} + A^{10}x^2 + A^8x^2 - A^4x^2 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x - A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x + A^{-4}x^3 - 2A^{-4}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6}x^2 - 2A^{-8}x^2 + A^{-8} + A^{-10} + A^{-12}x^2 - 2A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10} + 2A^8x^2 - 2A^8 - A^6 - 2A^4x^2 - A^2x^2 + A^2 + 1 - A^{-4}$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x + A^4x^3 - A^4x + x$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x - A^{-8}x^3 + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{71}) = A^8x^2 - A^8 - A^4x^2 - A^2x^2 + 1 + A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} - A^{-10}x^2 + A^{-10} + A^{-12}x^2 + A^{-14}x^2 - A^{-14} - A^{-16}$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x - A^{-12}x^3 + 2A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x + A^4x^3 - A^4x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^2 + A^4x^2 - A^4 - A^2x^2 - x^2 - A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^2 - A^2 + x^2 - 1 - A^{-2}x^2 - A^{-4}x^2 + A^{-6} - A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 - A^8x^2 + A^8 - A^6x^2 + A^6 + A^4$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^2 + A^{-6} - A^{-8}x^2 + A^{-8} - A^{-10}x^2 + A^{-10} + A^{-12}x^2 + A^{-14}x^2 - A^{-14} - A^{-16}$
$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 - A^{-4}x^2 + A^{-4} + A^{-6} + A^{-8} - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^2 - x^2 + 1 + A^{-2}x^2 + A^{-4} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} - A^{-16}x^2 + A^{-16} + A^{-18}x^2 + A^{-18} + A^{-20}x^2 - A^{-22} - A^{-24}$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + 2A^{18}x^2 + A^{16}x^2 - A^{16} - A^{14}x^2 - A^{12}x^2 + A^{10} + A^8 - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2 - x^2 + 1 + A^{-2} + A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^2 - A^{-4}x^2 + A^{-4} + A^{-6} + A^{-8} - A^{-10}x^2 + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-12}x^2 + A^{-12} - A^{-14}x^2 + A^{-16} + 2A^{-18}x^2 - A^{-18} - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} - A^8x^2 + A^8 - A^6x^2 + A^6 + A^4$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^2 - A^{-8}x^2 + A^{-8} + 2A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-14}x^2 - A^{-16}x^2 + A^{-18} + A^{-20}$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - 2A^{14}x^2 + A^{14} - A^{12}x^2 + A^{12} + 2A^{10}x^2 - A^{10} + A^8x^2 - A^6 - A^4x^2 - A^2x^2 + A^2 + 1$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x - A^{-4}x^3 + A^{-4}x$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x + x^3$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^2 + A^{-4}x^2 - A^{-4} - 2A^{-6}x^2 + A^{-6} - 2A^{-8}x^2 + A^{-8} + A^{-10}x^2 + A^{-12}x^2 - A^{-14} - 2A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 3A^{10}x^2 - 2A^{10} + 2A^8x^2 - 2A^8 - A^6x^2 - 2A^4x^2 - A^2x^2 + 2A^2 + 1 - A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{12}x^2 - A^{12} + A^{10}x^2 - A^{10} - A^8 - A^4x^2 - A^2x^2 + A^2$

## Knots in $L(7,1)$

$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^2 - Ax^3 + 2Ax + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - 2A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + A^{-6}x^2 + A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + A^2 - Ax^3 + 2Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} - A^5x^3 + 2A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x + A^{-10} + A^{-11}x^3 - A^{-11}x - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^2 + A^9x^3 - 2A^9x - A^5x^3 + A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x + Ax^2 - A - A^{-3}$

$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x + A^{-7}x^2 - A^{-7} - A^{-8}x - A^{-11}$
$\mathcal{S}_{2,\infty}(4_4) = A^2x^2 - 3A^{-2}x^2 + 2A^{-2} - A^{-3}x^3 + 2A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-7}x - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + 2A^6x^2 + A^5x^3 - 2A^5x - 2A^2x^2 + A^2 - Ax^3 + Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x + A^{-3}x^2 - A^{-3} - A^{-4}x^3 - A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x + A^{-3}x^2 - A^{-3} - 2A^{-4}x - A^{-7}$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 2A^{-6}x^2 + A^{-7}x^3 - A^{-7}x - A^{-10}x^2 - A^{-11}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 + A^5x^3 - 2A^5x - 2A^2x^2 - Ax^3 + Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x + A^{-3}x^2 - A^{-3} - A^{-4}x^3 - A^{-4}x - A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + 2A^2x^2 - A^2 + Ax^3 - 2Ax - 2A^{-2}x^2 - A^{-3}x^3 + A^{-3}x + A^{-6}x^2 + A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-3}x^3 - A^{-3}x - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - 2A^2x^2 + A^2 - Ax^3 + 2Ax + A^{-2} + A^{-3}x^3 - A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^2 + Ax^3 - 2Ax - A^{-2}x^2 - A^{-3}x^3 + A^{-3}x + A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x + A^{-7}x^2 - A^{-7} - A^{-8}x^3 - A^{-11} + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x + Ax^2 - A - x - A^{-3}$
$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^6x^2 - A^6 + A^5x^3 - 2A^5x - A^2x^2 - A^2 - Ax^3 + Ax + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + A^{-6}x^2 + A^{-7}x^3 - A^{-7}x - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 - A^{-7}x^3 + 2A^{-7}x + A^{-10}x^2 - A^{-10} + A^{-11}x^3 - A^{-11}x - A^{-14} - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} + A^9x^3 - 2A^9x - A^6 - A^5x^3 + A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 + A^{-3}x^2 - A^{-3} - A^{-4}x - A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^5x^2 - A^5 + A^4x^3 - 2A^4x - A$
$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} - A^{-11}x^3 + 2A^{-11}x + A^{-14}x^2 + A^{-15}x^3 - A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{13}x^3 - 2A^{13}x - A^{10}x^2 - A^9x^3 + A^9x - A^6x^2 + 2A^6 + A^5x$
$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 - A^5x^3 + 2A^5x + Ax$
$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x - A^{-10}x^2 + A^{-10} + A^{-11}x + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x + A^{-11}x^2 - A^{-11} - A^{-12}x - A^{-15}x^2 + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x - A^9x^2 + A^9 + A^5x^2 + A^4x^3 - 2A^4x - A$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} + A^9x^3 - 2A^9x - A^5x^3 + A^5x - A^2x^2 + A^2$
$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x + 2A^{-10}x^2 - A^{-10} + 2A^{-11}x^3 - 3A^{-11}x - 2A^{-14} - A^{-15}x^3 + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 - A^{13}x^3 + 2A^{13}x + 2A^{10}x^2 - A^{10} + 2A^9x^3 - 3A^9x - A^6 - A^5x^3 - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x + A^{-7}x^2 - A^{-7} - A^{-8}x^3 - A^{-8}x - A^{-11}x^2 + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x - A^5x^2 + A^5 + 2A^4x^3 - 2A^4x + Ax^2 - x - A^{-3}$
$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 + A^5x^3 - 2A^5x - 2A^2x^2 - 2Ax^3 + 3Ax + A^{-2} + A^{-3}x^3 - A^{-3}x - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 + 2Ax^3 - 4Ax - A^{-2}x^2 - A^{-2} - 2A^{-3}x^3 + 2A^{-3}x + A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - 4A^{-2}x^2 + 2A^{-2} - A^{-3}x^3 + 2A^{-3}x + 3A^{-6}x^2 + 2A^{-7}x^3 - 3A^{-7}x - A^{-10}x^2 - A^{-10} - A^{-11}x^3 + A^{-14} + A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} - A^9x^3 + 2A^9x + 3A^6x^2 + 2A^5x^3 - 3A^5x - 2A^2x^2 - Ax^3 + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x + A^{-7}x^2 - A^{-7} - 3A^{-8}x^3 + A^{-8}x - A^{-11}x^2 + A^{-12}x^3 + A^{-12}x + A^{-15} - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x - A^5x^2 + A^5 + 3A^4x^3 - 2A^4x + Ax^2 - 2x - A^{-3}$
$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + 3A^2x^2 - 2A^2 + Ax^3 - 2Ax - 3A^{-2}x^2 - 2A^{-3}x^3 + 3A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^5x^3 - 2A^5x - 3A^2x^2 - 2Ax^3 + 3Ax + A^{-2}x^2 + A^{-3}x^3 - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x + A^{-3}x^2 - A^{-3} - 2A^{-4}x^3 - A^{-7}x^2 + A^{-8}x^3 + A^{-11} - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x - Ax^2 + A + 2x^3 + A^{-3}x^2 - 2A^{-4}x - A^{-7}$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^6x^2 - A^6 + 2A^5x^3 - 4A^5x - 4A^2x^2 + A^2 - 3Ax^3 + 4Ax + A^{-2}x^2 + A^{-2} + A^{-3}x^3 - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 4A^2x^2 - 2A^2 + 2Ax^3 - 4Ax - 4A^{-2}x^2 + A^{-2} - 3A^{-3}x^3 + 4A^{-3}x + A^{-6}x^2 + 2A^{-6} + A^{-7}x^3 - A^{-10} - A^{-11}x$

$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x - Ax^2 + A + 3x^3 - x + A^{-3}x^2 - A^{-4}x^3 - A^{-4}x - A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x + A^{-3}x^2 - A^{-3} - 2A^{-4}x^3 - A^{-7}x^2 + 2A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 - A^9x^3 + 2A^9x + 3A^6x^2 - A^6 + A^5x^3 - A^5x - 2A^2x^2 - Ax^3 + Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 3A^{-6}x^2 - A^{-6} + A^{-7}x^3 - A^{-7}x - A^{-10}x^2 - A^{-10} - A^{-11}x^3 + A^{-11}x + A^{-14} + A^{-15}x$
$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x + 2A^{-10}x^2 + 2A^{-11}x^3 - 3A^{-11}x - A^{-14}x^2 - A^{-15}x^3 + A^{-18} + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} - A^{13}x^3 + 2A^{13}x + 2A^{10}x^2 + 2A^9x^3 - 3A^9x - A^6 - A^5x^3 - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x + A^{-3}x^2 - A^{-3} - A^{-4}x^3 - A^{-4}x - A^{-7}x^2 + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x - Ax^2 + A + 2x^3 - x + A^{-3}x^2 - 2A^{-4}x - A^{-7}$
$\mathcal{S}_{2,\infty}(5_{20}) = 2A^2x^2 - A^2 + Ax^3 - 2Ax - 3A^{-2}x^2 + A^{-2} - 2A^{-3}x^3 + 3A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-7}x - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^6x^2 - A^6 + 2A^5x^3 - 4A^5x - 2A^2x^2 - 2Ax^3 + 2Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x - A^5x^2 + A^5 + 2A^4x^3 - A^4x + Ax^2 - x - A^{-3}$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-7}x^2 - A^{-7} - 2A^{-8}x^3 + A^{-8}x - A^{-11}x^2 + 2A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + 2A^2x^2 - A^2 + Ax^3 - 2Ax - 2A^{-2}x^2 + A^{-2} - A^{-3}x^3 + A^{-3}x + 2A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - 3A^2x^2 + 2A^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-3}x^3 - A^{-3}x - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} - A^{-7}x^3 + 2A^{-7}x - A^{-10}x^2 + A^{-10} + A^{-11}x + A^{-14}x^2 - A^{-14} - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^6 - A^5x^3 + 2A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} - A^{-15}x^3 + 2A^{-15}x + 3A^{-18}x^2 + A^{-19}x^3 - A^{-19}x - A^{-22}x^2 - A^{-22} - A^{-23}x + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} + A^{17}x^3 - 2A^{17}x - 3A^{14}x^2 + A^{14} - A^{13}x^3 + A^{13}x + A^{10}x^2 + A^{10} + A^9x - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-2} + A^{-3}x - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 2A^{-6}x^2 + A^{-7}x - 2A^{-10}x^2 + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 2A^{-6}x^2 + A^{-7}x - A^{-10}x^2 - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 - Ax^3 + 2Ax + 2A^{-2}x^2 + A^{-3}x - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 - A^{-15}x^3 + 2A^{-15}x + 3A^{-18}x^2 - A^{-18} + A^{-19}x^3 - A^{-19}x - A^{-22}x^2 - A^{-22} - A^{-23}x + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} + A^{17}x^3 - 2A^{17}x - 2A^{14}x^2 - A^{14} - A^{13}x^3 + A^{13}x + A^{10}x^2 + A^9x - A^6x^2$
$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 - A^{-11}x^3 + 2A^{-11}x + A^{-14}x^2 + A^{-15}x^3 - A^{-15}x - A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} + A^{13}x^3 - 2A^{13}x - A^{10}x^2 - A^{10} - A^9x^3 + A^9x + A^5x - A^2$
$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 - A^5x^3 + 2A^5x + Ax$
$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 - A^{-7}x^3 + 2A^{-7}x + A^{-11}x + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-3}x^3 - A^{-3}x - A^{-6} - A^{-7}x^3 + A^{-7}x + A^{-11}x$
$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 - A^5x^3 + 2A^5x + A^2x^2 + Ax^3 - Ax - A^{-2}x^2 - A^{-3}x^3 + A^{-3}x + A^{-6} + A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-7}x^2 - A^{-7} - 2A^{-8}x^3 - A^{-11}x^2 + A^{-12}x^3 + A^{-15} - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x - A^5x^2 + A^5 + 2A^4x^3 - A^4x + Ax^2 - x - A^{-3}$
$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 + A^5x^3 - 2A^5x - 3A^2x^2 + A^2 - 2Ax^3 + 3Ax + 2A^{-2} + A^{-3}x^3 - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 + Ax^3 - 2Ax - 3A^{-2}x^2 + A^{-2} - 2A^{-3}x^3 + 3A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x + A^{-7}x^2 - A^{-7} - 3A^{-8}x^3 + 2A^{-8}x - A^{-11}x^2 + A^{-12}x^3 + A^{-12}x + A^{-15} - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x - A^5x^2 + A^5 + 2A^4x^3 + Ax^2 - x - A^{-3}$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 + 2A^5x^3 - 4A^5x - 3A^2x^2 - 3Ax^3 + 4Ax + 2A^{-2} + A^{-3}x^3 - A^{-7}x$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 + 2Ax^3 - 4Ax - 3A^{-2}x^2 - 3A^{-3}x^3 + 4A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-10} - A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x - A^{-3}x^2 + A^{-3} + 2A^{-4}x^3 - A^{-4}x + A^{-7}x^2 - A^{-8}x^3 - A^{-11} + A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x + Ax^2 - A - x^3 - A^{-3}x^2 + A^{-4}x + A^{-7}$
$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} - A^9x^3 + 2A^9x + 2A^6x^2 + A^5x^3 - A^5x - A^2x^2 - A^2 - Ax^3 + Ax + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + 2A^{-6}x^2 + A^{-7}x^3 - A^{-7}x - A^{-10}x^2 - A^{-11}x^3 + A^{-11}x + A^{-14} + A^{-15}x$
$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x + A^{-10}x^2 + A^{-11}x - A^{-14}x^2 + A^{-18}x^2 - A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 - A^5x^3 + 2A^5x + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} + A^{13}x^3 - 2A^{13}x - 3A^{10}x^2 + A^{10} - A^9x^3 + A^9x + A^6x^2 + A^6 + A^5x - A^2x^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} - A^{-11}x^3 + 2A^{-11}x + 2A^{-14}x^2 + A^{-15}x^3 - A^{-15}x - A^{-18} - A^{-19}x$
$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-3}x - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + A^{-6}x^2 + A^{-6} + A^{-7}x - 2A^{-10}x^2 + A^{-10} + A^{-14}x^2$
$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 3A^{-6}x^2 + A^{-7}x^3 - A^{-7}x - 2A^{-10}x^2 - A^{-11}x^3 + A^{-11}x + A^{-14}x^2 + A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} - A^9x^3 + 2A^9x + 3A^6x^2 + A^5x^3 - A^5x - 2A^2x^2 - Ax^3 + Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - 4A^{-2}x^2 + A^{-2} - A^{-3}x^3 + 2A^{-3}x + 4A^{-6}x^2 - A^{-6} + 2A^{-7}x^3 - 3A^{-7}x - 2A^{-10}x^2 - A^{-10} - A^{-11}x^3 + A^{-14}x^2 + A^{-15}x - A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 4A^{10}x^2 + A^{10} - A^9x^3 + 2A^9x + 4A^6x^2 - A^6 + 2A^5x^3 - 3A^5x - 2A^2x^2 - A^2 - Ax^3 + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x + A^{-3}x^2 - A^{-3} - 3A^{-4}x^3 - A^{-7}x^2 + A^{-8}x^3 + A^{-8}x + A^{-11} - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x - Ax^2 + A + 4x^3 - 2x + A^{-3}x^2 - A^{-4}x^3 - 2A^{-4}x - A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + 3A^2x^2 - A^2 + Ax^3 - 2Ax - 4A^{-2}x^2 + A^{-2} - 2A^{-3}x^3 + 3A^{-3}x + 2A^{-6}x^2 + A^{-6} + A^{-7}x^3 - A^{-10}x^2 - A^{-11}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 + A^5x^3 - 2A^5x - 4A^2x^2 + A^2 - 2Ax^3 + 3Ax + A^{-2}x^2 + A^{-2} + A^{-3}x^3 - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x + A^{-3}x^2 - A^{-3} - 3A^{-4}x^3 + A^{-4}x - A^{-7}x^2 + A^{-8}x^3 + A^{-8}x + A^{-11} - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x - Ax^2 + A + 3x^3 + A^{-3}x^2 - A^{-4}x^3 - A^{-4}x - A^{-7} + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 5A^2x^2 - 2A^2 + 2Ax^3 - 4Ax - 6A^{-2}x^2 + A^{-2} - 4A^{-3}x^3 + 6A^{-3}x + 3A^{-6}x^2 + A^{-6} + 2A^{-7}x^3 - A^{-7}x - A^{-10}x^2 - A^{-10} - 2A^{-11}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 7A^6x^2 - 3A^6 + 3A^5x^3 - 6A^5x - 5A^2x^2 - 4Ax^3 + 5Ax + A^{-2}x^2 + 2A^{-2} + A^{-3}x^3 + A^{-3}x - A^{-6} - A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - 2A^6x^2 + A^6 - A^5x^3 + 2A^5x + 2A^2x^2 + Ax^3 - Ax - 2A^{-2}x^2 - A^{-3}x^3 + A^{-3}x + A^{-6}x^2 + A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 - Ax^3 + 2Ax + 2A^{-2}x^2 + A^{-3}x^3 - A^{-3}x - A^{-6}x^2 - A^{-7}x^3 + A^{-7}x + A^{-10} + A^{-11}x$
$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x + A^{-10}x^2 + A^{-11}x^3 - A^{-11}x - A^{-14} - A^{-15}x^3 + A^{-15}x + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^2 - A^{13}x^3 + 2A^{13}x + A^{10}x^2 + A^9x^3 - A^9x - A^5x^3 + A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x + Ax^2 - A - x - A^{-3}x^2 + A^{-7}$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x - A^{-3}x^2 + A^{-3} + A^{-4}x^3 - A^{-4}x + A^{-7}x^2 - A^{-8}x - A^{-11}$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^2 - A^{-2} + A^{-3}x^3 - 2A^{-3}x - A^{-6}x^2 - 2A^{-7}x^3 + 3A^{-7}x + A^{-10} + A^{-11}x^3 - A^{-11}x - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10} + 2A^9x^3 - 4A^9x - A^6 - 2A^5x^3 + 2A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x - A^5x^2 + A^5 + A^4x^3 + Ax^2 - A^{-3}$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x + A^{-7}x^2 - A^{-7} - A^{-8}x^3 - A^{-11}x^2 + A^{-12}x + A^{-15}$
$\mathcal{S}_{2,\infty}(5_{71}) = A^9x^3 - 2A^9x - A^5x^3 + A^5x - A^2x^2 + Ax + A^{-2}x^2 - A^{-2} - A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x - A^{-10}x^2 + A^{-10} + A^{-11}x^3 - A^{-11}x + A^{-14}x^2 - A^{-14} - A^{-15}x$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x + A^{-11}x^2 - A^{-11} - A^{-12}x^3 + A^{-12}x - A^{-15}x^2 + A^{-16}x + A^{-19}$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x - A^9x^2 + A^9 + A^8x + A^5x^2 + A^4x^3 - 2A^4x - A$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^2 + A^5x^3 - 2A^5x - A^2x^2 - Ax^3 + Ax$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^2 - A^2 + Ax^3 - 2Ax - A^{-2}x^2 - A^{-3}x^3 + A^{-3}x + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 - A^9x^3 + 2A^9x - A^6x^2 + A^6 + A^5x$
$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^2 + A^{-6} - A^{-7}x^3 + 2A^{-7}x - A^{-10}x^2 + A^{-10} + A^{-11}x^3 - A^{-11}x + A^{-14}x^2 - A^{-14} - A^{-15}x$
$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + A^{-6} + A^{-7}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^2 - Ax^3 + 2Ax + A^{-2}x^2 + A^{-3}x - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} - A^{-15}x^3 + 2A^{-15}x + A^{-18}x^2 + A^{-18} + A^{-19}x^3 - A^{-19}x - A^{-22} - A^{-23}x$
$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + 2A^{18}x^2 + A^{17}x^3 - 2A^{17}x - A^{14}x^2 - A^{13}x^3 + A^{13}x + A^{10} + A^9x - A^6x^2 + A^6$
$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2 - Ax^3 + 2Ax + A^{-2} + A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^2 - A^{-3}x^3 + 2A^{-3}x + A^{-6} + A^{-7}x - A^{-10}x^2 + A^{-14}x^2 - A^{-14}$
$\mathcal{S}_{2,\infty}(5_{81}) = -A^{-11}x^3 + 2A^{-11}x - A^{-14}x^2 + A^{-15}x + 2A^{-18}x^2 - A^{-18} - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} - A^9x^3 + 2A^9x - A^6x^2 + A^6 + A^5x$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^2 - A^{-7}x^3 + 2A^{-7}x + 2A^{-10}x^2 - A^{-10} + A^{-11}x^3 - A^{-11}x - A^{-14}x^2 - A^{-15}x^3 + A^{-15}x + A^{-18} + A^{-19}x$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - 2A^{14}x^2 + A^{14} - A^{13}x^3 + 2A^{13}x + 2A^{10}x^2 - A^{10} + A^9x^3 - A^9x - A^6 - A^5x^3 + A^5x - A^2x^2 + A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x + A^{-3}x^2 - A^{-3} - A^{-4}x^3 - A^{-7}x^2 + A^{-8}x + A^{-11}$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x - Ax^2 + A + x^3 + x + A^{-3}x^2 - A^{-4}x - A^{-7}$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^2 + A^{-3}x^3 - 2A^{-3}x - 2A^{-6}x^2 + A^{-6} - 2A^{-7}x^3 + 3A^{-7}x + A^{-10}x^2 + A^{-11}x^3 - A^{-11}x - A^{-14} - A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 3A^{10}x^2 - 2A^{10} + 2A^9x^3 - 4A^9x - A^6x^2 - 2A^5x^3 + 2A^5x - A^2x^2 + 2A^2 + Ax$
$\mathcal{S}_{2,\infty}(5_{85}) = -A^{-15}x$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{13}x^3 - 2A^{13}x + A^{10}x^2 - A^{10} - A^9x - A^5x^3 + A^5x - A^2x^2 + A^2 + Ax$

## Knots in $L(7,2)$

$\mathcal{S}_{2,\infty}(3_2) = A^6x^2 - A^2x^2 + A^{-2}x^3 - 2A^{-2}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{3_2}) = A^2x^2 - A^2 - A^{-2}x^2 + A^{-6}x^3 - 2A^{-6}x + A^{-6} - A^{-10}x$
$\mathcal{S}_{2,\infty}(3_5) = A^2x^2 - 2A^{-2}x^2 + A^{-2} + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x - A^{-10}x - A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{3_5}) = -A^{10} + 2A^6x^2 - A^6 - 2A^2x^2 + A^2 + A^{-2}x^3 - 2A^{-2}x + A^{-2} - A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{3_6}) = A^{10}x^2 - A^{10} + A^2x^3 - A^2x^2 - 2A^2x + A^2 - A^{-2}x$
$\mathcal{S}_{2,\infty}(4_1) = -A^{-6}x^2 + A^{-6} + A^{-10}x^3 - 2A^{-10}x + A^{-10} - A^{-14}x^3 + A^{-14}x + A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{4_1}) = -A^{14}x^2 + A^{14} + A^{10}x^2 - A^6x^3 + 2A^6x + A^2x^3 - A^2x^2 - A^2x + A^2 - A^{-2}x$
$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x - x^3 + 2x + A^{-4}x^3 - A^{-4}x^2 - A^{-4}x + A^{-4} - A^{-8}x + A^{-8}$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x - A^{-8}x^3 + A^{-8}x + A^{-12}x^3 - A^{-12}x^2 - A^{-12}x + A^{-12} - A^{-16}x + A^{-16}$
$\mathcal{S}_{2,\infty}(4_4) = A^2x^2 - 3A^{-2}x^2 + 2A^{-2} + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-6} - A^{-10}x^3 + A^{-10}x - A^{-10} + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{4_4}) = -A^{10}x^2 + 2A^6x^2 - A^2x^3 - 2A^2x^2 + 2A^2x + A^2 + A^{-2}x^3 - A^{-2}x + A^{-2} - A^{-6}x$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x - 2A^{-4}x^3 + 2A^{-4}x + A^{-8}x^3 - A^{-8}x^2 + A^{-8} - A^{-12}x + A^{-12}$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - A^{-4}x^3 + A^{-8}x^3 - A^{-8}x^2 - A^{-8}x + A^{-8} - A^{-12}x + A^{-12}$
$\mathcal{S}_{2,\infty}(4_{12}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x - A^{-10}x^3 - A^{-10}x^2 + A^{-10}x + A^{-14}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{4_{12}}) = A^{14} - 2A^{10}x^2 + A^{10} + 3A^6x^2 - A^6 - A^2x^3 - 2A^2x^2 + 2A^2x + A^{-2}x^3 - A^{-2}x + A^{-2} - A^{-6}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x - 2A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-8}x^2 + A^{-8} - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(4_{15}) = -A^6x^2 + 2A^2x^2 - A^2 - A^{-2}x^3 - 2A^{-2}x^2 + 2A^{-2}x + A^{-6}x^3 + A^{-6}x^2 - A^{-6}x - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{4_{15}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x - A^{-6}x^3 + A^{-6}x - A^{-6} + A^{-10}x$

$\mathcal{S}_{2,\infty}(4_{17}) = A^6x^2 - 2A^2x^2 + A^2 + A^{-2}x^3 - 2A^{-2}x + A^{-2} - A^{-6}x^3 + A^{-6}x + A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{4_{17}}) = -A^6x^2 + A^6 + A^2x^2 - A^{-2}x^3 - A^{-2}x^2 + 2A^{-2}x + A^{-6}x^3 - A^{-6}x + A^{-6} - A^{-10}x$

$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + A^{-12}x^3 - A^{-12}x^2 + A^{-12} - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x - x^3 + x + A^{-4}x^3 - A^{-4}x^2 - A^{-4}x + A^{-4} - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(4_{19}) = -A^{10}x^2 + 2A^6x^2 - A^6 - A^2x^3 - A^2x^2 + 2A^2x - A^2 + A^{-2}x^3 - A^{-2}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{4_{19}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x - A^{-10}x^3 + A^{-10}x - A^{-10} + A^{-14}x$

$\mathcal{S}_{2,\infty}(4_{20}) = -A^{-6}x^2 + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x - A^{-10} - A^{-14}x^3 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{4_{20}}) = -A^{14}x^2 + 2A^{10}x^2 - 2A^{10} - A^6x^3 + 2A^6x - A^6 + A^2x^3 - A^2x^2 - A^2x + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 - A^{-4}x^3 + A^{-4}x + A^{-8}x^3 - A^{-8}x^2 - A^{-8}x + A^{-8} - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(4_{24}) = -A^{-12}x^3 + 2A^{-12}x + A^{-16}x^3 - A^{-16}x^2 - A^{-16}x + A^{-16} - A^{-20}x + A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + x^3 - x^2 - x + 1 - A^{-4}x + A^{-4}$

$\mathcal{S}_{2,\infty}(5_2) = A^{-6} - 2A^{-10}x^2 + A^{-10} + A^{-14}x^3 + A^{-14}x^2 - 2A^{-14}x - A^{-18}x^3 + A^{-18}x + A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_2}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^3 - A^{10}x^2 + 2A^{10}x + A^6x^3 - A^6x^2 - A^6x + 2A^6 - A^2x$

$\mathcal{S}_{2,\infty}(5_4) = A^{14}x^2 - A^{14} - A^6x^2 + A^6 + A^2x^3 - 2A^2x - A^{-2}x$

$\mathcal{S}_{2,\infty}(\overline{5_4}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-6} + A^{-10}x^3 - A^{-10}x^2 - 2A^{-10}x + A^{-10} + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x - A^{-12}x^3 + A^{-12}x + 2A^{-16}x^3 - A^{-16}x^2 - 3A^{-16}x + A^{-16} - A^{-20}x^3 + A^{-20}x^2 + A^{-24}x - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x + A^8x^3 - 2A^8x - A^4x^3 + A^4x^2 + A^4x - A^4 + x^3 - x^2 - A^{-4}x + A^{-4}$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^2 - A^{10} - A^6x^3 + 2A^6x + A^2x^3 - A^2x^2 - A^2x + A^2 - A^{-2}x^3 + A^{-2}x^2 + A^{-2}x - A^{-2} + A^{-6}x^3 - A^{-6}x^2 - A^{-10}x + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = -A^{-6}x^3 + 2A^{-6}x + A^{-10}x^3 - A^{-10}x - A^{-14}x^3 + A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x^3 - A^{-18}x^2 - A^{-22}x + A^{-22}$

$\mathcal{S}_{2,\infty}(5_7) = -2A^{-6}x^2 + A^{-6} + A^{-10}x^3 + 2A^{-10}x^2 - 2A^{-10}x - A^{-10} - 2A^{-14}x^3 + 3A^{-14}x - 2A^{-14} + A^{-18}x^3 - A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_7}) = A^{18}x^2 - A^{18} - 2A^{14}x^2 + A^{10}x^3 + 2A^{10}x^2 - 2A^{10}x - A^{10} - 2A^6x^3 + 3A^6x - A^6 + A^2x^3 - A^2x^2 + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x - 2A^{-8}x^3 + A^{-8}x + 2A^{-12}x^3 - A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x + 3A^4x^3 - 4A^4x - 2x^3 + x^2 + 2x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_9) = 2A^6x^2 - A^6 - A^2x^3 - 2A^2x^2 + 2A^2x + 2A^{-2}x^3 - 3A^{-2}x + A^{-2} - 2A^{-6}x^3 + A^{-6}x^2 + 2A^{-6}x - A^{-6} + A^{-10}x^3 - A^{-10}x^2 + A^{-10}x - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^3 - A^{-2}x^2 + 4A^{-2}x - A^{-2} + 2A^{-6}x^3 - 2A^{-6}x + A^{-6} - A^{-10}x^3 + A^{-10}x^2 - A^{-10} + A^{-14}x^3 - A^{-14}x^2 - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{11}) = A^2x^2 - 4A^{-2}x^2 + 2A^{-2} + A^{-6}x^3 + 3A^{-6}x^2 - 2A^{-6}x - 2A^{-10}x^3 - A^{-10}x^2 + 3A^{-10}x - A^{-10} + A^{-14}x^3 + A^{-14} - A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{11}}) = A^{14}x^2 - 3A^{10}x^2 + A^{10} + A^6x^3 + 3A^6x^2 - 2A^6x - 2A^2x^3 - 2A^2x^2 + 3A^2x + A^{-2}x^3 + A^{-2} - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x - 4A^{-8}x^3 + 3A^{-8}x + 3A^{-12}x^3 - A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 - A^{-16}x + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x + 4A^4x^3 - 4A^4x - 2x^3 + x^2 + x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{13}) = -A^6x^2 + 3A^2x^2 - 2A^2 - A^{-2}x^3 - 3A^{-2}x^2 + 2A^{-2}x + 2A^{-6}x^3 + A^{-6}x^2 - 3A^{-6}x + A^{-6} - A^{-10}x^3 - A^{-10} + A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{13}}) = -A^{10}x^2 + 3A^6x^2 - A^6 - A^2x^3 - 3A^2x^2 + 2A^2x + 2A^{-2}x^3 + A^{-2}x^2 - 3A^{-2}x - A^{-6}x^3 - A^{-6} + A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x - 3A^{-4}x^3 + 2A^{-4}x + 3A^{-8}x^3 - A^{-8}x^2 - 3A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 - A^{-12}x + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x + 3x^3 - 2x - 2A^{-4}x^3 + A^{-4}x^2 + A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 4A^6x^2 - A^6 - 2A^2x^3 - 4A^2x^2 + 4A^2x + A^2 + 3A^{-2}x^3 + A^{-2}x^2 - 4A^{-2}x + A^{-2} - 2A^{-6}x^3 + A^{-6}x^2 + A^{-6}x - 2A^{-6} + A^{-10}x^3 - A^{-10}x^2 + A^{-10}x - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 4A^2x^2 - 2A^2 - 2A^{-2}x^3 - 4A^{-2}x^2 + 4A^{-2}x + A^{-2} + 3A^{-6}x^3 + A^{-6}x^2 - 4A^{-6}x + 2A^{-6} - 2A^{-10}x^3 + A^{-10}x^2 + A^{-10}x - 2A^{-10} + A^{-14}x^3 - A^{-14}x^2 + A^{-14}x - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x + 4x^3 - 3x - 3A^{-4}x^3 + A^{-4}x^2 + 2A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 + A^{-8}x - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x - 3A^{-4}x^3 + 2A^{-4}x + 2A^{-8}x^3 - A^{-8}x^2 - A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(5_{17}) = A^{14}x^2 - 2A^{10}x^2 + A^6x^3 + 3A^6x^2 - 2A^6x - A^6 - A^2x^3 - 2A^2x^2 + A^2x + A^{-2}x^3 - A^{-2}x + A^{-2} - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{5_{17}}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 3A^{-6}x^2 - 2A^{-6}x - A^{-6} - A^{-10}x^3 - A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}x^3 - A^{-14}x + A^{-14} - A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{18}) = -2A^{-6}x^2 + A^{-6} + A^{-10}x^3 + 2A^{-10}x^2 - 2A^{-10}x - 2A^{-14}x^3 - A^{-14}x^2 + 3A^{-14}x + A^{-18}x^3 + A^{-18} - A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{18}}) = A^{18}x^2 - 3A^{14}x^2 + 2A^{14} + A^{10}x^3 + 2A^{10}x^2 - 2A^{10}x - 2A^6x^3 + 3A^6x - A^6 + A^2x^3 - A^2x^2 + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x - 2A^{-4}x^3 + A^{-4}x + 2A^{-8}x^3 - A^{-8}x^2 - 2A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x + 3x^3 - 3x - 2A^{-4}x^3 + A^{-4}x^2 + A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{20}) = 2A^2x^2 - A^2 - A^{-2}x^3 - 3A^{-2}x^2 + 2A^{-2}x + A^{-2} + 2A^{-6}x^3 + A^{-6}x^2 - 3A^{-6}x + A^{-6} - 2A^{-10}x^3 + A^{-10}x^2 + 2A^{-10}x - 2A^{-10} + A^{-14}x^3 - A^{-14}x^2 + A^{-14}x - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 3A^6x^2 - A^6 - 2A^2x^3 - 2A^2x^2 + 4A^2x + 2A^{-2}x^3 - 2A^{-2}x + A^{-2} - A^{-6}x^3 + A^{-6}x^2 - A^{-6} + A^{-10}x^3 - A^{-10}x^2 - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x + 3A^4x^3 - 3A^4x - 2x^3 + x^2 + 2x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 3A^{-8}x^3 + 3A^{-8}x + 2A^{-12}x^3 - A^{-12}x^2 - A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(5_{22}) = -A^6x^2 + 2A^2x^2 - A^2 - A^{-2}x^3 - 2A^{-2}x^2 + 2A^{-2}x + A^{-2} + A^{-6}x^3 - A^{-6}x + 2A^{-6} - A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{5_{22}}) = A^6x^2 + A^6 - 3A^2x^2 + 2A^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x - A^{-6}x^3 + A^{-6}x - A^{-6} + A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{24}) = A^{-6} + A^{-10}x^3 - A^{-10}x^2 - 2A^{-10}x + A^{-10} + A^{-14}x^2 - A^{-14}x - A^{-14} - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{24}}) = -A^{18} + A^{14}x^2 - A^{14} + A^6 + A^2x^3 - A^2x^2 - 2A^2x + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_{25}) = -A^{-6}x^2 + 2A^{-10}x^2 - 4A^{-14}x^2 + 2A^{-14} + A^{-18}x^3 + 3A^{-18}x^2 - 2A^{-18}x - A^{-22}x^3 - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}x + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{25}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - A^{18} - A^{14}x^3 - 3A^{14}x^2 + 2A^{14}x + A^{14} + A^{10}x^3 + A^{10}x^2 - A^{10}x + A^{10} - A^6x^2 - A^6x$

$\mathcal{S}_{2,\infty}(5_{28}) = A^{14}x^2 - 2A^{10}x^2 + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x + A^{-2} - A^{-6}x - A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{28}}) = -A^6 + 2A^2x^2 - 3A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x - 2A^{-10}x^2 - A^{-10}x + A^{-14}x^2$

$\mathcal{S}_{2,\infty}(5_{34}) = -A^6x^2 + 3A^2x^2 - A^2 - 3A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x - A^{-10}x^2 - A^{-10}x - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{34}}) = A^{14} - A^{10}x^2 - A^{10} + 3A^6x^2 - A^6 - 3A^2x^2 + A^2 + A^{-2}x^3 + 2A^{-2}x^2 - 2A^{-2}x - A^{-6}x^2 - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{35}) = -A^{-6}x^2 + 2A^{-10}x^2 - A^{-10} - 3A^{-14}x^2 + A^{-18}x^3 + 3A^{-18}x^2 - 2A^{-18}x - A^{-18} - A^{-22}x^3 - A^{-22}x^2 + A^{-22}x - A^{-22} + A^{-26}x + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{35}}) = A^{26} - 2A^{22}x^2 + 4A^{18}x^2 - 2A^{18} - A^{14}x^3 - 2A^{14}x^2 + 2A^{14}x - A^{14} + A^{10}x^3 + A^{10}x^2 - A^{10}x - A^6x^2 - A^6x$

$\mathcal{S}_{2,\infty}(5_{38}) = -A^{-2} + A^{-6}x^2 - A^{-6} - 2A^{-10}x^2 + A^{-14}x^3 + A^{-14}x^2 - 2A^{-14}x - A^{-18}x^3 + A^{-18}x + A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{38}}) = -A^{18}x^2 + A^{18} + 2A^{14}x^2 - A^{14} - A^{10}x^3 - A^{10}x^2 + 2A^{10}x - A^{10} + A^6x^3 - A^6x - A^2x - A^2$

$\mathcal{S}_{2,\infty}(5_{41}) = A^{18} + A^{10}x^2 - A^{10} - A^6x^2 + A^2x^3 - 2A^2x - A^{-2}x$

$\mathcal{S}_{2,\infty}(\overline{5_{41}}) = A^{-2}x^2 - A^{-2} - A^{-6}x^2 + A^{-10}x^3 - 2A^{-10}x - A^{-14}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{43}) = A^6x^2 - 2A^2x^2 + A^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x - A^{-6}x^3 + A^{-6}x - A^{-6} + A^{-10}x^3 - A^{-10}x - A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{43}}) = A^{10}x^2 - A^{10} - A^6x^2 + A^2x^3 + A^2x^2 - 2A^2x - A^{-2}x^3 - A^{-2}x^2 + A^{-2}x + A^{-6}x^3 - A^{-6}x + A^{-6} - A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 3A^{-8}x^3 + 2A^{-8}x + 3A^{-12}x^3 - A^{-12}x^2 - 3A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 - A^{-16}x + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - 2x^3 + x^2 + 2x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{45}) = -A^{10}x^2 + 3A^6x^2 - A^6 - A^2x^3 - 3A^2x^2 + 2A^2x + A^2 + 2A^{-2}x^3 - 3A^{-2}x + 2A^{-2} - A^{-6}x^3 + A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{5_{45}}) = -A^6x^2 + A^6 + 2A^2x^2 - A^{-2}x^3 - 3A^{-2}x^2 + 2A^{-2}x + A^{-2} + 2A^{-6}x^3 + A^{-6}x^2 - 3A^{-6}x + A^{-6} - A^{-10}x^3 - A^{-10} + A^{-14}x$

$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x - 4A^{-8}x^3 + 4A^{-8}x + 3A^{-12}x^3 - A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 - A^{-16}x + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x + 3A^4x^3 - 2A^4x - 2x^3 + x^2 + 2x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 4A^6x^2 - 2A^6 - 2A^2x^3 - 3A^2x^2 + 4A^2x + 3A^{-2}x^3 - 4A^{-2}x + 2A^{-2} - 2A^{-6}x^3 + A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}x^3 - A^{-10}x^2 + A^{-10}x - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 3A^2x^2 - A^2 - 2A^{-2}x^3 - 3A^{-2}x^2 + 4A^{-2}x + 3A^{-6}x^3 + A^{-6}x^2 - 4A^{-6}x + A^{-6} - 2A^{-10}x^3 + A^{-10}x^2 + A^{-10}x - 2A^{-10} + A^{-14}x^3 - A^{-14}x^2 + A^{-14}x - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x + 3A^{-4}x^3 - 3A^{-4}x - 3A^{-8}x^3 + A^{-8}x^2 + 3A^{-8}x - A^{-8} + A^{-12}x^3 - A^{-12}x^2 + A^{-12}x - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - 2x^3 + 2x + 2A^{-4}x^3 - A^{-4}x^2 - 2A^{-4}x + A^{-4} - A^{-8}x^3 + A^{-8}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{49}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + A^6x^3 + 2A^6x^2 - 2A^6x - A^2x^3 - A^2x^2 + A^2x - A^2 + A^{-2}x^3 - A^{-2}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{5_{49}}) = A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x - A^{-10}x^3 - A^{-10}x^2 + A^{-10}x + A^{-14}x^3 - A^{-14}x + A^{-14} - A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{53}) = A^{-2}x^2 - 2A^{-6}x^2 + A^{-6} + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x - A^{-14}x^2 - A^{-14}x + A^{-18}x^2 - A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{53}}) = -A^{22} + A^{18}x^2 - A^{14}x^2 + 2A^{10}x^2 - A^{10} - 2A^6x^2 + A^6 + A^2x^3 - 2A^2x + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_{55}) = -A^{18}x^2 + 3A^{14}x^2 - A^{14} - A^{10}x^3 - 3A^{10}x^2 + 2A^{10}x + A^{10} + A^6x^3 + A^6x^2 - A^6x + A^6 - A^2x^2 - A^2x + A^{-2}$

$\mathcal{S}_{2,\infty}(\overline{5_{55}}) = A^2 - A^{-2}x^2 + 2A^{-6}x^2 - 4A^{-10}x^2 + 2A^{-10} + A^{-14}x^3 + 2A^{-14}x^2 - 2A^{-14}x - A^{-18}x^3 + A^{-18}x - A^{-18} + A^{-22}x$

$\mathcal{S}_{2,\infty}(5_{56}) = A^{14}x^2 - 2A^{10}x^2 + A^{10} + 2A^6x^2 - 2A^2x^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x - A^{-6}x - A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{56}}) = -A^6 + 2A^2x^2 - A^2 - 2A^{-2}x^2 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-6} - 2A^{-10}x^2 - A^{-10}x + A^{-10} + A^{-14}x^2$

$\mathcal{S}_{2,\infty}(5_{59}) = A^2x^2 - 3A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 3A^{-6}x^2 - 2A^{-6}x - A^{-10}x^3 - 2A^{-10}x^2 + A^{-10}x + A^{-14}x^3 + A^{-14}x^2 - A^{-14}x - A^{-18}x - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{59}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 3A^{10}x^2 + A^{10} + A^6x^3 + 3A^6x^2 - 2A^6x - A^2x^3 - 2A^2x^2 + A^2x + A^{-2}x^3 - A^{-2}x + A^{-2} - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{61}) = A^2x^2 - 4A^{-2}x^2 + A^{-2} + A^{-6}x^3 + 4A^{-6}x^2 - 2A^{-6}x - A^{-6} - 2A^{-10}x^3 - 2A^{-10}x^2 + 3A^{-10}x - A^{-10} + A^{-14}x^3 + A^{-14}x^2 - A^{-18}x - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{61}}) = -A^{18} + 2A^{14}x^2 - A^{14} - 4A^{10}x^2 + A^{10} + A^6x^3 + 4A^6x^2 - 2A^6x - A^6 - 2A^2x^3 - 2A^2x^2 + 3A^2x - A^2 + A^{-2}x^3 + A^{-2} - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x - 4A^{-4}x^3 + 2A^{-4}x + 3A^{-8}x^3 - A^{-8}x^2 - 2A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 - A^{-12}x + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x + 5x^3 - 4x - 3A^{-4}x^3 + A^{-4}x^2 + A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 + A^{-8}x - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{63}) = -A^6x^2 + 3A^2x^2 - A^2 - A^{-2}x^3 - 4A^{-2}x^2 + 2A^{-2}x + A^{-2} + 2A^{-6}x^3 + 2A^{-6}x^2 - 3A^{-6}x + A^{-6} - A^{-10}x^3 - A^{-10}x^2 + A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{63}}) = A^{14} - 2A^{10}x^2 + A^{10} + 4A^6x^2 - A^6 - A^2x^3 - 4A^2x^2 + 2A^2x + A^2 + 2A^{-2}x^3 + A^{-2}x^2 - 3A^{-2}x + A^{-2} - A^{-6}x^3 - A^{-6} + A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x - 4A^{-4}x^3 + 3A^{-4}x + 3A^{-8}x^3 - A^{-8}x^2 - 2A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 - A^{-12}x + A^{-16}x - A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x + 4x^3 - 2x - 3A^{-4}x^3 + A^{-4}x^2 + 2A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 + A^{-8}x - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 5A^2x^2 - 2A^2 - 2A^{-2}x^3 - 6A^{-2}x^2 + 4A^{-2}x + A^{-2} + 4A^{-6}x^3 + 3A^{-6}x^2 - 6A^{-6}x + A^{-6} - 3A^{-10}x^3 + 2A^{-10}x - 2A^{-10} + A^{-14}x^3 - A^{-14}x^2 + 2A^{-14}x + A^{-14} - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 7A^6x^2 - 3A^6 - 3A^2x^3 - 5A^2x^2 + 6A^2x + 4A^{-2}x^3 + A^{-2}x^2 - 5A^{-2}x + 2A^{-2} - 2A^{-6}x^3 + A^{-6}x^2 - 2A^{-6} + A^{-10}x^3 - A^{-10}x^2 + A^{-10}x - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{66}) = A^{10}x^2 - 2A^6x^2 + A^6 + A^2x^3 + 2A^2x^2 - 2A^2x - A^{-2}x^3 - 2A^{-2}x^2 + A^{-2}x + A^{-6}x^3 + A^{-6}x^2 - A^{-6}x - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{66}}) = -A^{10} + 2A^6x^2 - A^6 - 3A^2x^2 + A^2 + A^{-2}x^3 + 2A^{-2}x^2 - 2A^{-2}x - A^{-6}x^3 - A^{-6}x^2 + A^{-6}x + A^{-10}x^3 - A^{-10}x + A^{-10} - A^{-14}x$

$\mathcal{S}_{2,\infty}(5_{67}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x - A^{-14}x^3 + A^{-14}x - A^{-14} + A^{-18}x^3 - A^{-18}x - A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{67}}) = A^{18}x^2 - A^{18} - A^{14}x^2 + A^{10}x^3 + A^{10}x^2 - 2A^{10}x - A^6x^3 + A^6x + A^2x^3 - A^2x^2 - A^2x + A^2 - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x - x^3 + x + 2A^{-4}x^3 - A^{-4}x^2 - 3A^{-4}x + A^{-4} - A^{-8}x^3 + A^{-8}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x + 2A^{-4}x^3 - 3A^{-4}x - 2A^{-8}x^3 + A^{-8}x^2 + 2A^{-8}x - A^{-8} + A^{-12}x^3 - A^{-12}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^2 - A^{-2} - A^{-6}x^3 - A^{-6}x^2 + 2A^{-6}x + 2A^{-10}x^3 - 3A^{-10}x + A^{-10} - 2A^{-14}x^3 + A^{-14}x^2 + 2A^{-14}x - A^{-14} + A^{-18}x^3 - A^{-18}x^2 + A^{-18}x - A^{-22}x + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^2 - A^{10} - 2A^6x^3 + 4A^6x - A^6 + 2A^2x^3 - A^2x^2 - 2A^2x + A^2 - A^{-2}x^3 + A^{-2}x^2 - A^{-2} + A^{-6}x^3 - A^{-6}x^2 - A^{-10}x + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x - 2x^3 + x^2 + 3x - 1 + A^{-4}x^3 - A^{-4}x^2 - A^{-8}x + A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + 2A^{-12}x^3 - A^{-12}x^2 - 2A^{-12}x + A^{-12} - A^{-16}x^3 + A^{-16}x^2 + A^{-20}x - A^{-20}$

$\mathcal{S}_{2,\infty}(5_{71}) = -A^6x^3 + 2A^6x + A^2x^3 - A^2x^2 - A^2x + A^{-2}x^2 - A^{-2}x - A^{-2} - A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{71}}) = -A^6 + A^2x^2 - A^2 - A^{-2} - A^{-6}x^2 + A^{-6} + A^{-10}x^3 - A^{-10}x^2 - 2A^{-10}x + A^{-10} - A^{-14}x^3 + A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x - 2A^{-12}x^3 + 3A^{-12}x + 2A^{-16}x^3 - A^{-16}x^2 - 2A^{-16}x + A^{-16} - A^{-20}x^3 + A^{-20}x^2 + A^{-24}x - A^{-24}$

$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x + A^8x^3 - A^8x - A^4x^3 + A^4x^2 + A^4x - A^4 + x^3 - x^2 - A^{-4}x + A^{-4}$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^2 - A^2x^3 - A^2x^2 + 2A^2x + A^{-2}x^3 - A^{-2}x - A^{-6}x^3 + A^{-6}x^2 + A^{-6}x - A^{-6} + A^{-10}x^3 - A^{-10}x^2 - A^{-14}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^2 - A^2 - A^{-2}x^3 - A^{-2}x^2 + 2A^{-2}x + A^{-6}x^3 - A^{-6}x + A^{-6} - A^{-10}x^3 + A^{-10}x^2 + A^{-10}x - A^{-10} + A^{-14}x^3 - A^{-14}x^2 - A^{-18}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{74}) = A^{-14}x^3 - A^{-14}x^2 - 2A^{-14}x + A^{-14} + A^{-18}x^2 - A^{-18}x - A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{74}}) = -A^{22} + A^{18}x^2 + A^6x^3 - A^6x^2 - 2A^6x + A^6 - A^2x$

$\mathcal{S}_{2,\infty}(\overline{5_{75}}) = A^2 - A^{-6}x^2 + A^{-6} + A^{-10}x^3 - A^{-10}x^2 - 2A^{-10}x + A^{-10} - A^{-14}x^3 + A^{-14}x^2 + A^{-14}x - A^{-14} + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{77}) = -A^6x^2 + A^6 + 2A^2x^2 - A^2 - A^{-2}x^2 + A^{-6}x^3 - 2A^{-6}x + A^{-6} - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{77}}) = -A^{10} + A^6x^2 - A^2x^2 + A^{-2}x^3 + A^{-2}x^2 - 2A^{-2}x - A^{-6}x^2 - A^{-6}x + A^{-6}$

$\mathcal{S}_{2,\infty}(5_{79}) = -A^{-6}x^2 + A^{-6} + A^{-10}x^2 - 2A^{-14}x^2 + A^{-14} + A^{-18}x^3 + A^{-18}x^2 - 2A^{-18}x + A^{-18} - A^{-22}x^3 + A^{-22}x - A^{-22} + A^{-26}x$

$\mathcal{S}_{2,\infty}(\overline{5_{79}}) = -A^{22}x^2 + 2A^{18}x^2 - A^{14}x^3 - A^{14}x^2 + 2A^{14}x + A^{10}x^3 - A^{10}x + A^{10} - A^6x^2 - A^6x + A^6$

$\mathcal{S}_{2,\infty}(5_{80}) = A^{14}x^2 - A^{14} - A^{10}x^2 + A^6x^2 - A^2x^2 + A^{-2}x^3 - 2A^{-2}x + A^{-2} - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{5_{80}}) = A^2x^2 - A^{-2}x^2 + A^{-6}x^3 - 2A^{-6}x + A^{-6} - A^{-10}x^2 - A^{-10}x + A^{-14}x^2 - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{81}) = A^{-14}x^3 - A^{-14}x^2 - 2A^{-14}x + 2A^{-18}x^2 - A^{-18}x - A^{-18} - A^{-22}x^2 + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{81}}) = A^{26} - A^{22}x^2 + 2A^{18}x^2 - A^{18} - A^{14} + A^6x^3 - A^6x^2 - 2A^6x + A^6 - A^2x$
$\mathcal{S}_{2,\infty}(5_{82}) = -A^{-6}x^2 + A^{-10}x^3 + 2A^{-10}x^2 - 2A^{-10}x - A^{-10} - A^{-14}x^3 - A^{-14}x^2 + A^{-14}x + A^{-18}x^3 - A^{-18}x + A^{-18} - A^{-22}x$
$\mathcal{S}_{2,\infty}(\overline{5_{82}}) = A^{18}x^2 - 2A^{14}x^2 + A^{14} + A^{10}x^3 + 2A^{10}x^2 - 2A^{10}x - A^{10} - A^6x^3 + A^6x - A^6 + A^2x^3 - A^2x^2 - A^2x + A^2 - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x - 2A^{-4}x^3 + 2A^{-4}x + 2A^{-8}x^3 - A^{-8}x^2 - 2A^{-8}x + A^{-8} - A^{-12}x^3 + A^{-12}x^2 + A^{-16}x - A^{-16}$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - 2A^{-4}x^3 + A^{-4}x^2 + 2A^{-4}x - A^{-4} + A^{-8}x^3 - A^{-8}x^2 - A^{-12}x + A^{-12}$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^2 - A^{-6}x^3 - 2A^{-6}x^2 + 2A^{-6}x + A^{-6} + 2A^{-10}x^3 + A^{-10}x^2 - 3A^{-10}x - 2A^{-14}x^3 + A^{-14}x^2 + 2A^{-14}x - 2A^{-14} + A^{-18}x^3 - A^{-18}x^2 + A^{-18}x - A^{-22}x + A^{-22}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 3A^{10}x^2 - 2A^{10} - 2A^6x^3 - A^6x^2 + 4A^6x + 2A^2x^3 - A^2x^2 - 2A^2x + 2A^2 - A^{-2}x^3 + A^{-2}x^2 - A^{-2} + A^{-6}x^3 - A^{-6}x^2 - A^{-10}x + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{85}) = -A^{-18}x^3 + A^{-18}x^2 + 2A^{-18}x - A^{-18} + A^{-22}x^3 - A^{-22}x^2 - A^{-26}x + A^{-26}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = -A^{10}x^3 + A^{10}x^2 + 2A^{10}x - A^{10} + A^6x + A^{-2}x^3 - A^{-2}x^2 - A^{-2}x - A^{-6}x + A^{-6}$

## Knots in $L(8,1)$

$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x + A^2x^3 - 2A^2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x + A^{-6}x^3 - 2A^{-6}x - A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x + A^{-2}x^3 - 2A^{-2}x - A^{-4}x^3 - A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x + A^{-2}x^3 - 2A^{-2}x - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x + A^{-2}x^3 - 2A^{-2}x - A^{-4}x^3 - A^{-4}x - A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x - A^{-8}x^3 - A^{-10}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x + A^2x^3 - 2A^2x - x - A^{-2}x$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 + A^{-2}x^3 - 2A^{-2}x - A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^6x^3 - 2A^6x + A^4x^3 - 2A^4x - A^2x$
$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x + A^{-10}x^3 - 2A^{-10}x - A^{-12}x - A^{-14}x^3 + A^{-14}x + A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x - A^{10}x^3 + 2A^{10}x + A^6x^3 - A^6x + A^4x^3 - 2A^4x - A^2x$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 - A^2x^2 + A^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x + A^{-6}x^3 - 2A^{-6}x - A^{-8}x^3 - A^{-8}x - A^{-10}x^3 + A^{-10}x + A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x - A^6x^3 + 2A^6x + 2A^4x^3 - 2A^4x + A^2x^3 - A^2x - x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 + A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - A^{-6}x^2 + 2A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-6}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x + A^{-6}x^3 - 2A^{-6}x - 3A^{-8}x^3 + A^{-8}x - A^{-10}x^3 + A^{-10}x + A^{-12}x^3 + A^{-12}x + A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x - A^6x^3 + 2A^6x + 3A^4x^3 - 2A^4x + A^2x^3 - A^2x - 2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x + A^{-2}x^3 - 2A^{-2}x - 2A^{-4}x^3 - A^{-6}x^3 + A^{-6}x + A^{-8}x^3 + A^{-10}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x - A^2x^3 + 2A^2x + 2x^3 + A^{-2}x^3 - A^{-2}x - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 + A^{-2}x^4 - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 + A^{-6}x^4 + A^{-6} - A^{-10}x^2 + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x - A^2x^3 + 2A^2x + 3x^3 - x + A^{-2}x^3 - A^{-2}x - A^{-4}x^3 - A^{-4}x - A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x + A^{-2}x^3 - 2A^{-2}x - 2A^{-4}x^3 - A^{-6}x^3 + A^{-6}x + 2A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x + A^{-2}x^3 - 2A^{-2}x - A^{-4}x^3 - A^{-4}x - A^{-6}x^3 + A^{-6}x + A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x - A^2x^3 + 2A^2x + 2x^3 - x + A^{-2}x^3 - A^{-2}x - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + A^{-6}x^4 - A^{-6}x^2 + A^{-6} - A^{-10}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x - A^6x^3 + 2A^6x + 2A^4x^3 - A^4x + A^2x^3 - A^2x - x - A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-6}x^3 - 2A^{-6}x - 2A^{-8}x^3 + A^{-8}x - A^{-10}x^3 + A^{-10}x + 2A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-6}x^3 - 2A^{-6}x - 2A^{-8}x^3 - A^{-10}x^3 + A^{-10}x + A^{-12}x^3 + A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x - A^6x^3 + 2A^6x + 2A^4x^3 - A^4x + A^2x^3 - A^2x - x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x - 3A^{-8}x^3 + 2A^{-8}x - A^{-10}x^3 + A^{-10}x + A^{-12}x^3 + A^{-12}x + A^{-14}x - A^{-16}x$
$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x - A^6x^3 + 2A^6x + 2A^4x^3 + A^2x^3 - A^2x - x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 + A^{-2}x^4 - A^{-2}x^2 + A^{-2} - A^{-6}x^2 + 2A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-10}x^2 + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x - A^{-2}x^3 + 2A^{-2}x + 2A^{-4}x^3 - A^{-4}x + A^{-6}x^3 - A^{-6}x - A^{-8}x^3 - A^{-10}x + A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x + A^2x^3 - 2A^2x - x^3 - A^{-2}x^3 + A^{-2}x + A^{-4}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x + A^{-2}x^3 - 2A^{-2}x - 3A^{-4}x^3 - A^{-6}x^3 + A^{-6}x + A^{-8}x^3 + A^{-8}x + A^{-10}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x - A^2x^3 + 2A^2x + 4x^3 - 2x + A^{-2}x^3 - A^{-2}x - A^{-4}x^3 - 2A^{-4}x - A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x + A^{-2}x^3 - 2A^{-2}x - 3A^{-4}x^3 + A^{-4}x - A^{-6}x^3 + A^{-6}x + A^{-8}x^3 + A^{-8}x + A^{-10}x - A^{-12}x$
$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x - A^2x^3 + 2A^2x + 3x^3 + A^{-2}x^3 - A^{-2}x - A^{-4}x^3 - A^{-4}x - A^{-6}x + A^{-8}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 2A^{-6}x^4 - 3A^{-10}x^2 + 2A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 + A^{-2}x^4 + A^{-2}x^2 - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x + A^2x^3 - 2A^2x - x - A^{-2}x^3 + A^{-2}x + A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x - A^{-2}x^3 + 2A^{-2}x + A^{-4}x^3 - A^{-4}x + A^{-6}x^3 - A^{-6}x - A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} + A^{-10}x^4 - 2A^{-10}x^2 + A^{-10} - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 + A^{-2}$
$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x - A^6x^3 + 2A^6x + A^4x^3 + A^2x^3 - A^2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x - A^{-8}x^3 - A^{-10}x^3 + A^{-10}x + A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x + A^{-10}x^3 - 2A^{-10}x - A^{-12}x^3 + A^{-12}x - A^{-14}x^3 + A^{-14}x + A^{-16}x + A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x - A^{10}x^3 + 2A^{10}x + A^8x + A^6x^3 - A^6x + A^4x^3 - 2A^4x - A^2x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x + A^{-2}x^3 - 2A^{-2}x - A^{-4}x^3 - A^{-6}x^3 + A^{-6}x + A^{-8}x + A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x - A^2x^3 + 2A^2x + x^3 + x + A^{-2}x^3 - A^{-2}x - A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 + A^{-10}x^4 - A^{-10}x^2 - A^{-14}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 + A^2 + A^{-2}$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^6x^4 + 2A^6x^2 + A^2$

## Knots in $L(8,3)$

$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x - x^3 + 2x + A^{-2}x^3 - 2A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x - A^{-8}x^3 + A^{-8}x + A^{-10}x^3 - 2A^{-10}x + A^{-12}x^3 - 2A^{-12}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x - 2A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x + A^{-8}x^3 - A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - A^{-4}x^3 + A^{-6}x^3 - 2A^{-6}x + A^{-8}x^3 - 2A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x - 2A^{-4}x^3 + A^{-4}x + A^{-6}x^3 - 2A^{-6}x + A^{-8}x^3 - A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + A^{-10}x^3 - 2A^{-10}x + A^{-12}x^3 - A^{-12}x - A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x - x^3 + x + A^{-2}x^3 - 2A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 - A^{-4}x^3 + A^{-4}x + A^{-6}x^3 - 2A^{-6}x + A^{-8}x^3 - 2A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(4_{24}) = -A^{-12}x^3 + 2A^{-12}x + A^{-14}x^3 - 2A^{-14}x + A^{-16}x^3 - 2A^{-16}x - A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^2x^3 - 2A^2x + x^3 - 2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x - A^{-12}x^3 + A^{-12}x + A^{-14}x^3 - 2A^{-14}x + 2A^{-16}x^3 - 4A^{-16}x - A^{-18}x^3 + A^{-18}x - A^{-20}x^3 + 2A^{-20}x + A^{-22}x$
$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x + A^8x^3 - 2A^8x - A^6x^3 + 2A^6x - A^4x^3 + 2A^4x + A^2x^3 - A^2x + x^3 - 2x - A^{-2}x$
$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 + A^2x^4 - 3A^2x^2 + A^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-10}x^2 + 2A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 2A^{-10}x^2 - A^{-14}x^4 + 3A^{-14}x^2 - A^{-14} - A^{-22}x^2 + 2A^{-22}$
$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x - 2A^{-8}x^3 + A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 2A^{-12}x^3 - 3A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + 2A^{-16}x + A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x + 3A^4x^3 - 4A^4x - A^2x^3 + 2A^2x - 2x^3 + 3x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 + 2A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-6}x^2 - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} - A^{-18}x^2 + 2A^{-18}$
$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x - 4A^{-8}x^3 + 3A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 3A^{-12}x^3 - 3A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + A^{-16}x + A^{-18}x$
$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x + 4A^4x^3 - 4A^4x - A^2x^3 + 2A^2x - 2x^3 + 2x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x - 3A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 3A^{-8}x^3 - 4A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x + 3x^3 - 2x - A^{-2}x^3 + 2A^{-2}x - 2A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - A^{-6}x + A^{-8}x^3 - 2A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 + 2A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} - A^{-14}x^2 + 2A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} - A^{-18}x^2 + 2A^{-18}$
$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x + 4x^3 - 3x - A^{-2}x^3 + 2A^{-2}x - 3A^{-4}x^3 + 3A^{-4}x + A^{-6}x^3 - A^{-6}x + A^{-8}x^3 - A^{-8}x - A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x - 3A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 2A^{-8}x^3 - 2A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + 2A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x - 2A^{-4}x^3 + A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 2A^{-8}x^3 - 3A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + 2A^{-12}x + A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x + 3x^3 - 3x - A^{-2}x^3 + 2A^{-2}x - 2A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - A^{-6}x +$

$A^{-8}x^3 - 2A^{-8}x - A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + 2A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} - A^{-18}x^2 + 2A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} - A^{-14}x^2 + 2A^{-14}$

$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x + 3A^4x^3 - 3A^4x - A^2x^3 + 2A^2x - 2x^3 + 3x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 3A^{-8}x^3 + 3A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 2A^{-12}x^3 - 2A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + 2A^{-16}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - 3A^{-8}x^3 + 2A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 3A^{-12}x^3 - 4A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + A^{-16}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - A^2x^3 + 2A^2x - 2x^3 + 3x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x - 4A^{-8}x^3 + 4A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 3A^{-12}x^3 - 3A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + A^{-16}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x + 3A^4x^3 - 2A^4x - A^2x^3 + 2A^2x - 2x^3 + 3x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-14}x^2 + 2A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 2A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} - A^{-18}x^2 + 2A^{-18}$

$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x + 3A^{-4}x^3 - 3A^{-4}x - A^{-6}x^3 + 2A^{-6}x - 3A^{-8}x^3 + 4A^{-8}x + A^{-10}x^3 - A^{-10}x + A^{-12}x^3 - A^{-12}x - A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - 2x^3 + 2x + A^{-2}x^3 - 2A^{-2}x + 2A^{-4}x^3 - 3A^{-4}x - A^{-6}x^3 + A^{-6}x - A^{-8}x^3 + 2A^{-8}x + A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x - 4A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 3A^{-8}x^3 - 3A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + A^{-12}x + A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x + 5x^3 - 4x - A^{-2}x^3 + 2A^{-2}x - 3A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - A^{-6}x + A^{-8}x^3 - A^{-8}x - A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x - 4A^{-4}x^3 + 3A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 3A^{-8}x^3 - 3A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + A^{-12}x + A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x + 4x^3 - 2x - A^{-2}x^3 + 2A^{-2}x - 3A^{-4}x^3 + 3A^{-4}x + A^{-6}x^3 - A^{-6}x + A^{-8}x^3 - A^{-8}x - A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 3A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + A^{-14} - A^{-18}x^2 + 2A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} - A^{-14}x^2 + 2A^{-14}$

$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x - x^3 + x + A^{-2}x^3 - 2A^{-2}x + 2A^{-4}x^3 - 4A^{-4}x - A^{-6}x^3 + A^{-6}x - A^{-8}x^3 + 2A^{-8}x + A^{-10}x$

$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x + 2A^{-4}x^3 - 3A^{-4}x - A^{-6}x^3 + 2A^{-6}x - 2A^{-8}x^3 + 3A^{-8}x + A^{-10}x^3 - A^{-10}x + A^{-12}x^3 - 2A^{-12}x - A^{-14}x$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} + 2A^{-10}x^4 - 4A^{-10}x^2 + A^{-10} - A^{-14}x^4 + 2A^{-14}x^2 - A^{-22}x^2 + 2A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 + A^2x^4 - 2A^2x^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-10}x^2 + 2A^{-10}$

$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x - A^2x^3 + 2A^2x - 2x^3 + 4x + A^{-2}x^3 - A^{-2}x + A^{-4}x^3 - 2A^{-4}x - A^{-6}x$

$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x - 2A^{-8}x^3 + 2A^{-8}x + A^{-10}x^3 - 2A^{-10}x + 2A^{-12}x^3 - 3A^{-12}x - A^{-14}x^3 + A^{-14}x - A^{-16}x^3 + 2A^{-16}x + A^{-18}x$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x - 2A^{-12}x^3 + 3A^{-12}x + A^{-14}x^3 - 2A^{-14}x + 2A^{-16}x^3 - 3A^{-16}x - A^{-18}x^3 + A^{-18}x - A^{-20}x^3 + 2A^{-20}x + A^{-22}x$

$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x + A^8x^3 - A^8x - A^6x^3 + 2A^6x - A^4x^3 + 2A^4x + A^2x^3 - A^2x + x^3 - 2x - A^{-2}x$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 + A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} - A^{-14}x^2 + 2A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} - A^{-18}x^2 + 2A^{-18}$

$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x - 2A^{-4}x^3 + 2A^{-4}x + A^{-6}x^3 - 2A^{-6}x + 2A^{-8}x^3 - 3A^{-8}x - A^{-10}x^3 + A^{-10}x - A^{-12}x^3 + 2A^{-12}x + A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - A^{-2}x^3 + 2A^{-2}x - 2A^{-4}x^3 + 3A^{-4}x + A^{-6}x^3 - A^{-6}x + A^{-8}x^3 - 2A^{-8}x - A^{-10}x$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-14}x^4 + 2A^{-14}x^2 - A^{-14} - A^{-22}x^2 + 2A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 + A^2x^4 - 2A^2x^2 + A^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-10}x^2 + 2A^{-10}$

$\mathcal{S}_{2,\infty}(5_{85}) = A^{-14}x^4 - 3A^{-14}x^2 + A^{-14} - A^{-18}x^4 + 3A^{-18}x^2 - A^{-18} - A^{-26}x^2 + 2A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^2x^4 + 3A^2x^2 - A^2 - A^{-6}x^2 + 2A^{-6}$

## Knots in $L(9,1)$

$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x + A^3x^4 - 3A^3x^2 + A^3 - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - A^{-8}x - A^{-9}x^2 + A^{-9}$

$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - A^{-4}x^3 - A^{-5}x^2 + A^{-5} + A^{-8}x$

$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - 2A^{-4}x - A^{-5}x^2 + A^{-5}$

$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - A^{-4}x^3 - A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x$

$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - A^{-8}x^3 - A^{-9}x^2 + A^{-9} + A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x + A^3x^4 - 3A^3x^2 + A^3 - x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - A^{-4}x - A^{-5}x^2 + A^{-5}$

$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x + A^7x^4 - 3A^7x^2 + A^7 + A^4x^3 - 2A^4x - A^3x^2 + A^3$

$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x + A^{-9}x^4 - 3A^{-9}x^2 + A^{-9} - A^{-12}x - A^{-13}x^4 + 2A^{-13}x^2 + A^{-17}x^2 - A^{-17}$

$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x - A^{11}x^4 + 3A^{11}x^2 - A^{11} + A^7x^4 - 2A^7x^2 + A^4x^3 - 2A^4x - A^3x^2 + A^3$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 - A^3x^3 + 2A^3x + A^{-1}x$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-9}x^3 + 2A^{-9}x + A^{-10}x^2 - A^{-10} + A^{-13}x$

$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - A^{-8}x^3 - A^{-8}x - A^{-9}x^4 + 2A^{-9}x^2 + A^{-12}x + A^{-13}x^2 - A^{-13}$

$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x - A^7x^4 + 3A^7x^2 - A^7 + 2A^4x^3 - 2A^4x + A^3x^4 - 2A^3x^2 - x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 - A^{-1}x^3 + 2A^{-1}x + A^{-2}x^4 - A^{-2}x^2 + A^{-5}x - A^{-6}x^2 + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-5}x^3 + 2A^{-5}x + 2A^{-6}x^2 - A^{-6} + A^{-9}x$

$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - 3A^{-8}x^3 + A^{-8}x - A^{-9}x^4 + 2A^{-9}x^2 + A^{-12}x^3 + A^{-12}x + A^{-13}x^2 - A^{-13} - A^{-16}x$

$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x - A^7x^4 + 3A^7x^2 - A^7 + 3A^4x^3 - 2A^4x + A^3x^4 - 2A^3x^2 - 2x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - 2A^{-4}x^3 - A^{-5}x^4 + 2A^{-5}x^2 + A^{-8}x^3 + A^{-9}x^2 - A^{-9} - A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x - A^3x^4 + 3A^3x^2 - A^3 + 2x^3 + A^{-1}x^4 - 2A^{-1}x^2 - 2A^{-4}x - A^{-5}x^2 + A^{-5}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 - A^{-1}x^3 + 2A^{-1}x + A^{-2}x^4 + A^{-2}x^2 - A^{-2} + A^{-5}x - A^{-6}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 - A^{-5}x^3 + 2A^{-5}x + A^{-6}x^4 + A^{-6}x^2 + A^{-9}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x - A^3x^4 + 3A^3x^2 - A^3 + 3x^3 - x + A^{-1}x^4 - 2A^{-1}x^2 - A^{-4}x^3 - A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x$

$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - 2A^{-4}x^3 - A^{-5}x^4 + 2A^{-5}x^2 + 2A^{-8}x + A^{-9}x^2 - A^{-9}$

$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - A^{-4}x^3 - A^{-4}x - A^{-5}x^4 + 2A^{-5}x^2 + A^{-8}x + A^{-9}x^2 - A^{-9}$

$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x - A^3x^4 + 3A^3x^2 - A^3 + 2x^3 - x + A^{-1}x^4 - 2A^{-1}x^2 - 2A^{-4}x - A^{-5}x^2 + A^{-5}$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 - A^{-5}x^3 + 2A^{-5}x + A^{-6}x^4 + A^{-9}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 - A^{-1}x^3 + 2A^{-1}x + 2A^{-2}x^2 - A^{-2} + A^{-5}x$

$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x - A^7x^4 + 3A^7x^2 - A^7 + 2A^4x^3 - A^4x + A^3x^4 - 2A^3x^2 - x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - 2A^{-8}x^3 + A^{-8}x - A^{-9}x^4 + 2A^{-9}x^2 + 2A^{-12}x + A^{-13}x^2 - A^{-13}$

$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - 2A^{-8}x^3 - A^{-9}x^4 + 2A^{-9}x^2 + A^{-12}x^3 + A^{-13}x^2 - A^{-13} - A^{-16}x$

$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x - A^7x^4 + 3A^7x^2 - A^7 + 2A^4x^3 - A^4x + A^3x^4 - 2A^3x^2 - x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - 3A^{-8}x^3 + 2A^{-8}x - A^{-9}x^4 + 2A^{-9}x^2 + A^{-12}x^3 + A^{-12}x + A^{-13}x^2 - A^{-13} - A^{-16}x$

$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x - A^7x^4 + 3A^7x^2 - A^7 + 2A^4x^3 + A^3x^4 - 2A^3x^2 - x - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 - A^{-1}x^3 + 2A^{-1}x + A^{-2}x^4 + A^{-5}x - A^{-6}x^2 + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-5}x^3 + 2A^{-5}x + A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-9}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x - A^{-1}x^4 + 3A^{-1}x^2 - A^{-1} + 2A^{-4}x^3 - A^{-4}x + A^{-5}x^4 - 2A^{-5}x^2 - A^{-8}x^3 - A^{-9}x^2 + A^{-9} + A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x + A^3x^4 - 3A^3x^2 + A^3 - x^3 - A^{-1}x^4 + 2A^{-1}x^2 + A^{-4}x + A^{-5}x^2 - A^{-5}$

$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - 3A^{-4}x^3 - A^{-5}x^4 + 2A^{-5}x^2 + A^{-8}x^3 + A^{-8}x + A^{-9}x^2 - A^{-9} - A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x - A^3x^4 + 3A^3x^2 - A^3 + 4x^3 - 2x + A^{-1}x^4 - 2A^{-1}x^2 - A^{-4}x^3 - 2A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x$

$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - 3A^{-4}x^3 + A^{-4}x - A^{-5}x^4 + 2A^{-5}x^2 + A^{-8}x^3 + A^{-8}x + A^{-9}x^2 - A^{-9} - A^{-12}x$

$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x - A^3x^4 + 3A^3x^2 - A^3 + 3x^3 + A^{-1}x^4 - 2A^{-1}x^2 - A^{-4}x^3 - A^{-4}x - A^{-5}x^2 + A^{-5} + A^{-8}x$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-5}x^3 + 2A^{-5}x + 2A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-9}x - 3A^{-10}x^2 + A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 - A^{-1}x^3 + 2A^{-1}x + A^{-2}x^4 + 2A^{-2}x^2 -$

$A^{-2} + A^{-5}x - A^{-6}x^2$

$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x + A^3x^4 - 3A^3x^2 + A^3 - x - A^{-1}x^4 + 2A^{-1}x^2 + A^{-5}x^2 - A^{-5}$

$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x - A^{-1}x^4 + 3A^{-1}x^2 - A^{-1} + A^{-4}x^3 - A^{-4}x + A^{-5}x^4 - 2A^{-5}x^2 - A^{-8}x - A^{-9}x^2 + A^{-9}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} - A^{-9}x^3 + 2A^{-9}x + A^{-10}x^4 - A^{-10}x^2 + A^{-13}x - A^{-14}x^2 + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 - A^3x^3 + 2A^3x + A^2x^2 - A^2 + A^{-1}x$

$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x - A^7x^4 + 3A^7x^2 - A^7 + A^4x^3 + A^3x^4 - 2A^3x^2 - A^{-1}x^2 + A^{-1}$

$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x + A^{-5}x^4 - 3A^{-5}x^2 + A^{-5} - A^{-8}x^3 - A^{-9}x^4 + 2A^{-9}x^2 + A^{-12}x + A^{-13}x^2 - A^{-13}$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x + A^{-9}x^4 - 3A^{-9}x^2 + A^{-9} - A^{-12}x^3 + A^{-12}x - A^{-13}x^4 + 2A^{-13}x^2 + A^{-16}x + A^{-17}x^2 - A^{-17}$

$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x - A^{11}x^4 + 3A^{11}x^2 - A^{11} + A^8x + A^7x^4 - 2A^7x^2 + A^4x^3 - 2A^4x - A^3x^2 + A^3$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 - A^{-1}x^3 + 2A^{-1}x + A^{-2}x^2 - A^{-2} + A^{-5}x$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-5}x^3 + 2A^{-5}x + A^{-6}x^2 + A^{-9}x$

$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x + A^{-1}x^4 - 3A^{-1}x^2 + A^{-1} - A^{-4}x^3 - A^{-5}x^4 + 2A^{-5}x^2 + A^{-8}x + A^{-9}x^2 - A^{-9}$

$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x - A^3x^4 + 3A^3x^2 - A^3 + x^3 + x + A^{-1}x^4 - 2A^{-1}x^2 - A^{-4}x - A^{-5}x^2 + A^{-5}$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 - A^{-9}x^3 + 2A^{-9}x + A^{-10}x^4 - A^{-10} + A^{-13}x - A^{-14}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 - A^3x^3 + 2A^3x + A^2x^2 + A^{-1}x$

$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^7x^3 + 2A^7x - A^6x^4 + 3A^6x^2 - A^6 + A^3x$

## Knots in $L(9,2)$

$\mathcal{S}_{2,\infty}(4_2) = -A^8x^3 + A^8x + A^4x^3 - A^4x - x^4 + 3x^2 - 1 + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{4_2}) = -x^3 + 2x + A^{-4}x^3 - A^{-4}x - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8}x - A^{-8} + A^{-12}x^2 - A^{-12}$

$\mathcal{S}_{2,\infty}(4_5) = -A^4x^3 + 2x^3 - x - A^{-4}x^4 - A^{-4}x^3 + 3A^{-4}x^2 - A^{-4} + A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{4_5}) = A^8x - 2A^4x^3 + 2A^4x + 2x^3 - x - A^{-4}x^4 + 3A^{-4}x^2 - 2A^{-4}x - A^{-4} + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(4_{14}) = A^8x - 2A^4x^3 + A^4x + 3x^3 - 2x - A^{-4}x^4 - A^{-4}x^3 + 3A^{-4}x^2 - A^{-4}x - A^{-4} + A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(4_{18}) = -x^3 + x + 2A^{-4}x^3 - 2A^{-4}x - A^{-8}x^4 - A^{-8}x^3 + 3A^{-8}x^2 - A^{-8} + A^{-12}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{4_{18}}) = A^{12}x - 2A^8x^3 + 2A^8x + 2A^4x^3 - 2A^4x - x^4 + 3x^2 - x - 1 + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(4_{21}) = -A^4x^3 + A^4x + x^3 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4}x - A^{-4} + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{4_{24}}) = -A^{12}x^3 + 2A^{12}x - A^4x^4 + A^4x^3 + 3A^4x^2 - 2A^4x - A^4 + x^2 - 1$

$\mathcal{S}_{2,\infty}(5_5) = A^{-8}x^3 - 2A^{-8}x - A^{-12}x^4 + 3A^{-12}x^2 - A^{-12}x - A^{-12} + A^{-16}x^4 - 2A^{-16}x^2 - A^{-20}x^2 + A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_5}) = A^{16}x^3 - 2A^{16}x - A^{12}x^3 + A^{12}x + A^8x^4 - 3A^8x^2 + A^8 - A^4x^4 + A^4x^3 + 2A^4x^2 - 2A^4x + x^2 - 1$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 + A^2x^4 - 3A^2x^2 + A^2 - A^{-2}x^4 + A^{-2}x^3 + 2A^{-2}x^2 - 2A^{-2}x + A^{-6}x^2 - A^{-6}x - A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 2A^{-10}x^2 - A^{-14}x^4 + A^{-14}x^3 + 2A^{-14}x^2 - 2A^{-14}x + A^{-18}x^2 - A^{-18}x - A^{-18}$

$\mathcal{S}_{2,\infty}(5_8) = -x^3 + x + 3A^{-4}x^3 - 4A^{-4}x - A^{-8}x^4 - A^{-8}x^3 + 3A^{-8}x^2 - A^{-8}x - A^{-8} + A^{-12}x^4 - 2A^{-12}x^2 + A^{-12}x - A^{-16}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_8}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + A^8x + A^4x^4 + 2A^4x^3 - 3A^4x^2 - 2A^4x + A^4 - x^4 + 2x^2 - x + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 + 2A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} - A^{-6}x^4 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-6} + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-6}x^2 - A^{-10}x^4 +$

$A^{-10}x^3 + 2A^{-10}x^2 - 2A^{-10}x + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{12}) = -x^3 + 4A^{-4}x^3 - 4A^{-4}x - A^{-8}x^4 - 3A^{-8}x^3 + 3A^{-8}x^2 + A^{-8}x - A^{-8} + A^{-12}x^4 + A^{-12}x^3 - 2A^{-12}x^2 + A^{-12}x - A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{12}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 3A^8x + A^4x^4 + 3A^4x^3 - 3A^4x^2 - 2A^4x + A^4 - x^4 + 2x^2 - 2x + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(5_{14}) = -A^4x^3 + 3x^3 - 2x - A^{-4}x^4 - 2A^{-4}x^3 + 3A^{-4}x^2 - A^{-4} + A^{-8}x^4 + A^{-8}x^3 - 2A^{-8}x^2 - A^{-12}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{14}}) = -A^{12}x + 2A^8x^3 - 2A^8x - 3A^4x^3 + 2A^4x + x^4 + 2x^3 - 3x^2 + 1 - A^{-4}x^4 + 2A^{-4}x^2 - 2A^{-4}x + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 + 2A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{16}) = A^8x^3 - 3A^4x^3 + 2A^4x + x^4 + 3x^3 - 3x^2 - x + 1 - A^{-4}x^4 - A^{-4}x^3 + 2A^{-4}x^2 - A^{-4}x + A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{16}}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 3x - A^{-4}x^4 - 2A^{-4}x^3 + 3A^{-4}x^2 - A^{-4} + A^{-8}x^4 - 2A^{-8}x^2 + 2A^{-8}x - A^{-12}x^2 + A^{-12}$

$\mathcal{S}_{2,\infty}(5_{19}) = -A^4x^3 + 3x^3 - 3x - A^{-4}x^4 - A^{-4}x^3 + 3A^{-4}x^2 - A^{-4}x - A^{-4} + A^{-8}x^4 - 2A^{-8}x^2 + A^{-8}x - A^{-12}x^2 + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{19}}) = A^8x^3 - A^8x - 2A^4x^3 + A^4x + x^4 + 2x^3 - 3x^2 - x + 1 - A^{-4}x^4 + 2A^{-4}x^2 - 2A^{-4}x + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + 2A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} - A^{-10}x^4 + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(5_{21}) = A^{12}x^3 - 3A^8x^3 + 3A^8x + A^4x^4 + 2A^4x^3 - 3A^4x^2 - A^4x + A^4 - x^4 + 2x^2 - x + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{5_{21}}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - A^{-8}x^4 - 2A^{-8}x^3 + 3A^{-8}x^2 + A^{-8}x - A^{-8} + A^{-12}x^4 - 2A^{-12}x^2 + 2A^{-12}x - A^{-16}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(5_{44}) = -x^3 + x + 3A^{-4}x^3 - 3A^{-4}x - A^{-8}x^4 - 2A^{-8}x^3 + 3A^{-8}x^2 - A^{-8} + A^{-12}x^4 + A^{-12}x^3 - 2A^{-12}x^2 - A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{44}}) = -A^{16}x + 2A^{12}x^3 - 2A^{12}x - 3A^8x^3 + 2A^8x + A^4x^4 + 2A^4x^3 - 3A^4x^2 - A^4x + A^4 - x^4 + 2x^2 - x + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(5_{46}) = -x^3 + x + 3A^{-4}x^3 - 2A^{-4}x - A^{-8}x^4 - 3A^{-8}x^3 + 3A^{-8}x^2 + 2A^{-8}x - A^{-8} + A^{-12}x^4 + A^{-12}x^3 - 2A^{-12}x^2 + A^{-12}x - A^{-16}x^2 - A^{-16}x + A^{-16}$

$\mathcal{S}_{2,\infty}(\overline{5_{46}}) = -A^{16}x + 2A^{12}x^3 - A^{12}x - 4A^8x^3 + 4A^8x + A^4x^4 + 2A^4x^3 - 3A^4x^2 + A^4 - x^4 + 2x^2 - x + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-6} + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 2A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + A^{-10}x^3 + A^{-10}x^2 - 2A^{-10}x + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{48}) = A^4x^3 - A^4x - 2x^3 + 2x + A^{-4}x^4 + 2A^{-4}x^3 - 3A^{-4}x^2 - A^{-4}x + A^{-4} - A^{-8}x^4 - A^{-8}x^3 + 2A^{-8}x^2 + A^{-12}x^2 + A^{-12}x - A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{48}}) = A^{12}x - 2A^8x^3 + 2A^8x + 3A^4x^3 - 3A^4x - x^4 - x^3 + 3x^2 - 1 + A^{-4}x^4 - 2A^{-4}x^2 + A^{-4}x - A^{-8}x^2 + A^{-8}$

$\mathcal{S}_{2,\infty}(5_{62}) = A^8x - 2A^4x^3 + 5x^3 - 4x - A^{-4}x^4 - 3A^{-4}x^3 + 3A^{-4}x^2 - A^{-4} + A^{-8}x^4 + A^{-8}x^3 - 2A^{-8}x^2 + A^{-8}x - A^{-12}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{62}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 2A^4x + x^4 + 4x^3 - 3x^2 - 2x + 1 - A^{-4}x^4 - A^{-4}x^3 + 2A^{-4}x^2 - 2A^{-4}x + A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{64}) = A^8x - 2A^4x^3 + A^4x + 4x^3 - 2x - A^{-4}x^4 - 3A^{-4}x^3 + 3A^{-4}x^2 + A^{-4}x - A^{-4} + A^{-8}x^4 + A^{-8}x^3 - 2A^{-8}x^2 + A^{-8}x - A^{-12}x^2 - A^{-12}x + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{64}}) = -A^{12}x + 2A^8x^3 - A^8x - 4A^4x^3 + 3A^4x + x^4 + 3x^3 - 3x^2 + 1 - A^{-4}x^4 - A^{-4}x^3 + 2A^{-4}x^2 - A^{-4}x +$

$A^{-8}x^2 + A^{-8}x - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 3A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + A^{-10}x^3 - A^{-10}x^2 - 2A^{-10}x + A^{-10} + A^{-14}x^2 - A^{-14}x$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + A^{-6}x^3 + A^{-6}x^2 - 2A^{-6}x + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(5_{68}) = -A^8x^3 + A^8x + 2A^4x^3 - 3A^4x - x^4 + 3x^2 - x - 1 + A^{-4}x^4 - 2A^{-4}x^2 - A^{-8}x^2 + A^{-8}$

$\mathcal{S}_{2,\infty}(\overline{5_{68}}) = A^4x^3 - 2A^4x - x^3 + x + A^{-4}x^4 + A^{-4}x^3 - 3A^{-4}x^2 - A^{-4}x + A^{-4} - A^{-8}x^4 + 2A^{-8}x^2 - A^{-8}x + A^{-12}x^2 - A^{-12}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} + 2A^{-10}x^4 - 4A^{-10}x^2 + A^{-10} - A^{-14}x^4 + A^{-14}x^3 + A^{-14}x^2 - 2A^{-14}x + A^{-14} + A^{-18}x^2 - A^{-18}x - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 + A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^3 + 2A^{-2}x^2 - 2A^{-2}x + A^{-6}x^2 - A^{-6}x - A^{-6}$

$\mathcal{S}_{2,\infty}(5_{70}) = A^{12}x^3 - A^{12}x - 2A^8x^3 + 2A^8x + A^4x^4 + A^4x^3 - 3A^4x^2 + A^4 - x^4 + 2x^2 + A^{-4}x^2 - A^{-4}$

$\mathcal{S}_{2,\infty}(\overline{5_{70}}) = -x^3 + 2x + 2A^{-4}x^3 - 2A^{-4}x - A^{-8}x^4 - A^{-8}x^3 + 3A^{-8}x^2 - A^{-8} + A^{-12}x^4 - 2A^{-12}x^2 + A^{-12}x - A^{-16}x^2 + A^{-16}$

$\mathcal{S}_{2,\infty}(5_{72}) = A^{-8}x^3 - A^{-8}x - A^{-12}x^4 - A^{-12}x^3 + 3A^{-12}x^2 + A^{-12}x - A^{-12} + A^{-16}x^4 - 2A^{-16}x^2 + A^{-16}x - A^{-20}x^2 + A^{-20}$

$\mathcal{S}_{2,\infty}(\overline{5_{72}}) = A^{16}x^3 - A^{16}x - 2A^{12}x^3 + 3A^{12}x + A^8x^4 - 3A^8x^2 + A^8x + A^8 - A^4x^4 + A^4x^3 + 2A^4x^2 - 2A^4x + x^2 - 1$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 + A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + A^{-6}x^3 + 2A^{-6}x^2 - 2A^{-6}x + A^{-10}x^2 - A^{-10}x - A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + A^{-10}x^3 + 2A^{-10}x^2 - 2A^{-10}x + A^{-14}x^2 - A^{-14}x - A^{-14}$

$\mathcal{S}_{2,\infty}(5_{83}) = -A^4x^3 + A^4x + 2x^3 - x - A^{-4}x^4 - A^{-4}x^3 + 3A^{-4}x^2 - A^{-4} + A^{-8}x^4 - 2A^{-8}x^2 + A^{-8}x - A^{-12}x^2 + A^{-12}$

$\mathcal{S}_{2,\infty}(\overline{5_{83}}) = A^8x^3 - A^8x - 2A^4x^3 + 2A^4x + x^4 + x^3 - 3x^2 + x + 1 - A^{-4}x^4 + 2A^{-4}x^2 - A^{-4}x + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-14}x^4 + A^{-14}x^3 + A^{-14}x^2 - 2A^{-14}x + A^{-18}x^2 - A^{-18}x - A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 + A^2x^4 - 2A^2x^2 + A^2 - A^{-2}x^4 + A^{-2}x^3 + 2A^{-2}x^2 - 2A^{-2}x + A^{-6}x^2 - A^{-6}x - A^{-6}$

$\mathcal{S}_{2,\infty}(5_{85}) = A^{-14}x^4 - 3A^{-14}x^2 + A^{-14} - A^{-18}x^4 + A^{-18}x^3 + 2A^{-18}x^2 - 2A^{-18}x + A^{-22}x^2 - A^{-22}x - A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^2x^4 + A^2x^3 + 2A^2x^2 - 2A^2x + A^{-2}x^2 - A^{-2}x - A^{-2}$

## Knots in $L(10,1)$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 - A^4x^4 + 3A^4x^2 - A^4 + x^2 - 1$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} + A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12}$

$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 - x^4 + 3x^2 - 1 + A^{-2}x^4 - A^{-2}x^2 + A^{-4}x^2 - A^{-4} - A^{-6}x^2 + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + 2A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 - x^4 + 3x^2 - 1 + A^{-2}x^4 + A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + A^{-6}x^4 + A^{-6}x^2 + A^{-8}x^2 - A^{-8} - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + A^{-6}x^4 + A^{-8}x^2 - A^{-8} - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 - x^4 + 3x^2 - 1 + 2A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4}$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 - x^4 + 3x^2 - 1 + A^{-2}x^4 + A^{-4}x^2 - A^{-4} - A^{-6}x^2 + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - A^{-10}x^2$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + 2A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - 3A^{-10}x^2 + A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 - x^4 + 3x^2 - 1 + A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6}x^2$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} + A^{-10}x^4 - A^{-10}x^2 + A^{-12}x^2 - A^{-12} - A^{-14}x^2 + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 - A^4x^4 + 3A^4x^2 - A^4 + A^2x^2 - A^2 + x^2 - 1$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 - x^4 + 3x^2 - 1 + A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4}$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} + A^{-6}x^2 + A^{-8}x^2 - A^{-8}$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} + A^{-10}x^4 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 - A^4x^4 + 3A^4x^2 - A^4 + A^2x^2 + x^2 - 1$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^8x^4 + 3A^8x^2 - A^8 - A^6x^4 + 3A^6x^2 - A^6 + A^4x^2 - A^4$

## Knots in $L(10,3)$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 + A^2x^4 - 3A^2x^2 + A^2 - x^4 + 3x^2 - 1 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6} + A^{-10}$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^4 - 2A^{-10}x^2 - A^{-12}x^4 + 3A^{-12}x^2 - A^{-12} - A^{-14}x^4 + 3A^{-14}x^2 - A^{-14} + A^{-16}x^2 - A^{-16} - A^{-18} + A^{-22}$
$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 + 2A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-6}x^4 - A^{-6}x^2 - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 + 2A^{-2}x^4 - 2A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + 2A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^4 - A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 2A^{-6}x^4 - 2A^{-6}x^2 - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14} + A^{-18}$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + 3A^{-6}x^4 - 2A^{-6}x^2 - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + A^{-12}x^2 - A^{-12} + A^{-18}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 + 2A^{-2}x^4 - A^{-2}x^2 - A^{-4}x^4 +$

$3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} + 2A^{-10}x^4 - 4A^{-10}x^2 + A^{-10} - A^{-12}x^4 + 3A^{-12}x^2 - A^{-12} - A^{-14}x^4 + 2A^{-14}x^2 + A^{-16}x^2 - A^{-16} - A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 + A^2x^4 - 2A^2x^2 - x^4 + 3x^2 - 1 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 + A^{-2}x^4 - 2A^{-2}x^2 - A^{-4}x^4 + 3A^{-4}x^2 - A^{-4} - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} + A^{-8}x^2 - A^{-8} - A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-8}x^4 + 3A^{-8}x^2 - A^{-8} - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} + A^{-12}x^2 - A^{-12} - A^{-14} + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-12}x^4 + 3A^{-12}x^2 - A^{-12} - A^{-14}x^4 + 2A^{-14}x^2 - A^{-14} + A^{-16}x^2 - A^{-16} - A^{-18} + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 + A^2x^4 - 2A^2x^2 + A^2 - x^4 + 3x^2 - 1 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-4}x^2 - A^{-4} - A^{-6} + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{85}) = A^{-14}x^4 - 3A^{-14}x^2 + A^{-14} - A^{-16}x^4 + 3A^{-16}x^2 - A^{-16} - A^{-18}x^4 + 3A^{-18}x^2 - A^{-18} + A^{-20}x^2 - A^{-20} - A^{-22} + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^4x^4 + 3A^4x^2 - A^4 - A^2x^4 + 3A^2x^2 - A^2 + x^2 - 1 - A^{-2} + A^{-6}$

## Knots in $L(11,1)$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 - A^5x^5 + 4A^5x^3 - 3A^5x + Ax^3 - 2Ax$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-7}x^5 + 4A^{-7}x^3 - 3A^{-7}x + A^{-10}x^2 - A^{-10} + A^{-11}x^3 - 2A^{-11}x$

$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 - Ax^5 + 4Ax^3 - 3Ax + A^{-2}x^4 - A^{-2}x^2 + A^{-3}x^3 - 2A^{-3}x - A^{-6}x^2 + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + 2A^{-6}x^2 - A^{-6} + A^{-7}x^3 - 2A^{-7}x$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 - Ax^5 + 4Ax^3 - 3Ax + A^{-2}x^4 + A^{-2}x^2 - A^{-2} + A^{-3}x^3 - 2A^{-3}x - A^{-6}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + A^{-6}x^4 + A^{-6}x^2 + A^{-7}x^3 - 2A^{-7}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + A^{-6}x^4 + A^{-7}x^3 - 2A^{-7}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 - Ax^5 + 4Ax^3 - 3Ax + 2A^{-2}x^2 - A^{-2} + A^{-3}x^3 - 2A^{-3}x$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 - Ax^5 + 4Ax^3 - 3Ax + A^{-2}x^4 + A^{-3}x^3 - 2A^{-3}x - A^{-6}x^2 + A^{-6}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-7}x^3 - 2A^{-7}x - A^{-10}x^2$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + 2A^{-6}x^4 + A^{-6}x^2 - A^{-6} + A^{-7}x^3 - 2A^{-7}x - 3A^{-10}x^2 + A^{-10} + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 - Ax^5 + 4Ax^3 - 3Ax + A^{-2}x^4 + 2A^{-2}x^2 - A^{-2} + A^{-3}x^3 - 2A^{-3}x - A^{-6}x^2$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} - A^{-7}x^5 + 4A^{-7}x^3 - 3A^{-7}x + A^{-10}x^4 - A^{-10}x^2 + A^{-11}x^3 - 2A^{-11}x - A^{-14}x^2 + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 - A^5x^5 + 4A^5x^3 - 3A^5x + A^2x^2 -$

$A^2 + Ax^3 - 2Ax$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 - Ax^5 + 4Ax^3 - 3Ax + A^{-2}x^2 - A^{-2} + A^{-3}x^3 - 2A^{-3}x$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-3}x^5 + 4A^{-3}x^3 - 3A^{-3}x + A^{-6}x^2 + A^{-7}x^3 - 2A^{-7}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 - A^{-7}x^5 + 4A^{-7}x^3 - 3A^{-7}x + A^{-10}x^4 - A^{-10} + A^{-11}x^3 - 2A^{-11}x - A^{-14}x^2$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 - A^5x^5 + 4A^5x^3 - 3A^5x + A^2x^2 + Ax^3 - 2Ax$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^9x^5 + 4A^9x^3 - 3A^9x - A^6x^4 + 3A^6x^2 - A^6 + A^5x^3 - 2A^5x$

## Knots in $L(11,2)$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 + A^2x^5 - 4A^2x^3 + 3A^2x - A^{-2}x^3 + 2A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-10}x^5 - 4A^{-10}x^3 + A^{-10}x^2 + 3A^{-10}x - A^{-10} - A^{-14}x^3 + 2A^{-14}x$
$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 + A^{-2}x^5 + A^{-2}x^4 - 4A^{-2}x^3 - A^{-2}x^2 + 3A^{-2}x - A^{-6}x^3 - A^{-6}x^2 + 2A^{-6}x + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-6}x^5 - 4A^{-6}x^3 + 2A^{-6}x^2 + 3A^{-6}x - A^{-6} - A^{-10}x^3 + 2A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 + A^{-2}x^5 + A^{-2}x^4 - 4A^{-2}x^3 + A^{-2}x^2 + 3A^{-2}x - A^{-2} - A^{-6}x^3 - A^{-6}x^2 + 2A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 + A^{-6}x^5 + A^{-6}x^4 - 4A^{-6}x^3 + A^{-6}x^2 + 3A^{-6}x - A^{-10}x^3 - A^{-10}x^2 + 2A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 + A^{-6}x^5 + A^{-6}x^4 - 4A^{-6}x^3 + 3A^{-6}x - A^{-10}x^3 - A^{-10}x^2 + 2A^{-10}x$
$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 + A^{-2}x^5 - 4A^{-2}x^3 + 2A^{-2}x^2 + 3A^{-2}x - A^{-2} - A^{-6}x^3 + 2A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 + A^{-2}x^5 + A^{-2}x^4 - 4A^{-2}x^3 + 3A^{-2}x - A^{-6}x^3 - A^{-6}x^2 + 2A^{-6}x + A^{-6}$
$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + A^{-6}x^5 + A^{-6}x^4 - 4A^{-6}x^3 + A^{-6}x^2 + 3A^{-6}x - A^{-6} - A^{-10}x^3 - A^{-10}x^2 + 2A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} + A^{-6}x^5 + 2A^{-6}x^4 - 4A^{-6}x^3 + A^{-6}x^2 + 3A^{-6}x - A^{-6} - A^{-10}x^3 - 3A^{-10}x^2 + 2A^{-10}x + A^{-10} + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 + A^{-2}x^5 + A^{-2}x^4 - 4A^{-2}x^3 + 2A^{-2}x^2 + 3A^{-2}x - A^{-2} - A^{-6}x^3 - A^{-6}x^2 + 2A^{-6}x$
$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} + A^{-10}x^5 + A^{-10}x^4 - 4A^{-10}x^3 - A^{-10}x^2 + 3A^{-10}x - A^{-14}x^3 - A^{-14}x^2 + 2A^{-14}x + A^{-14}$
$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 + A^2x^5 - 4A^2x^3 + A^2x^2 + 3A^2x - A^2 - A^{-2}x^3 + 2A^{-2}x$
$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 + A^{-2}x^5 - 4A^{-2}x^3 + A^{-2}x^2 + 3A^{-2}x - A^{-2} - A^{-6}x^3 + 2A^{-6}x$
$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 + A^{-6}x^5 - 4A^{-6}x^3 + A^{-6}x^2 + 3A^{-6}x - A^{-10}x^3 + 2A^{-10}x$
$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 + A^{-10}x^5 + A^{-10}x^4 - 4A^{-10}x^3 + 3A^{-10}x - A^{-10} - A^{-14}x^3 - A^{-14}x^2 + 2A^{-14}x$
$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 + A^2x^5 - 4A^2x^3 + A^2x^2 + 3A^2x - A^{-2}x^3 + 2A^{-2}x$
$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} + A^6x^5 - A^6x^4 - 4A^6x^3 + 3A^6x^2 + 3A^6x - A^6 - A^2x^3 + 2A^2x$

## Knots in $L(11,3)$

$\mathcal{S}_{2,\infty}(5_6) = A^{10}x^4 - 2A^{10}x^2 - A^6x^4 + 2A^6x^2 - A^3x^5 + 4A^3x^3 - 3A^3x + A^2x^4 - 3A^2x^2 + A^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-5}x^3 - A^{-5}x - A^{-6} - A^{-9}x + A^{-10}$

$\mathcal{S}_{2,\infty}(\overline{5_6}) = A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 - A^{-9}x^5 + 4A^{-9}x^3 - 3A^{-9}x + A^{-10}x^4 - 2A^{-10}x^2 - A^{-14}x^4 + 3A^{-14}x^2 - A^{-14} + A^{-17}x^3 - A^{-17}x - A^{-18} - A^{-21}x + A^{-22}$

$\mathcal{S}_{2,\infty}(5_9) = A^6x^4 - A^6x^2 - 2A^2x^4 + 3A^2x^2 - A^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + 2A^{-2}x^4 - 4A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_9}) = -A^6x^2 + A^6 + 2A^2x^4 - 4A^2x^2 + A^2 - 2A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + A^{-6}x^4 - A^{-6}x^2 - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} + A^{-13}x^3 - A^{-13}x - A^{-14} - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{15}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 + A^6 - 3A^2x^4 + 3A^2x^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + 2A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{15}}) = -A^6x^2 + 2A^2x^4 - 2A^2x^2 - 3A^{-2}x^4 + 3A^{-2}x^2 - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + 2A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-13}x^3 - A^{-13}x - A^{-14} - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{20}) = A^2x^4 - A^2x^2 - 2A^{-2}x^4 + 2A^{-2}x^2 - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + 2A^{-6}x^4 - 3A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-13}x^3 - A^{-13}x - A^{-14} - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{20}}) = -A^{10}x^2 + 2A^6x^4 - 3A^6x^2 + A^6 - 2A^2x^4 + 2A^2x^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{47}) = -A^{10}x^2 + 2A^6x^4 - 2A^6x^2 - 3A^2x^4 + 4A^2x^2 - A^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + 2A^{-2}x^4 - 3A^{-2}x^2 + A^{-2} - A^{-6}x^4 + 2A^{-6}x^2 + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{47}}) = -A^6x^2 + A^6 + 2A^2x^4 - 3A^2x^2 + A^2 - 3A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + 2A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + 2A^{-10}x^2 - A^{-10} + A^{-13}x^3 - A^{-13}x - A^{-14} - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{65}) = -A^6x^2 + 2A^2x^4 - A^2x^2 - 4A^{-2}x^4 + 4A^{-2}x^2 - A^{-2} - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + 3A^{-6}x^4 - 2A^{-6}x^2 - A^{-10}x^4 + A^{-13}x^3 - A^{-13}x - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(\overline{5_{65}}) = A^{14} - 3A^{10}x^2 + A^{10} + 3A^6x^4 - 2A^6x^2 - 4A^2x^4 + 4A^2x^2 - A^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + 2A^{-2}x^4 - A^{-2}x^2 - A^{-6}x^4 + 2A^{-6}x^2 - A^{-6} + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(5_{69}) = A^{-2}x^4 - 2A^{-2}x^2 - 2A^{-6}x^4 + 4A^{-6}x^2 - A^{-6} - A^{-9}x^5 + 4A^{-9}x^3 - 3A^{-9}x + 2A^{-10}x^4 - 4A^{-10}x^2 + A^{-10} - A^{-14}x^4 + 2A^{-14}x^2 + A^{-17}x^3 - A^{-17}x - A^{-18} - A^{-21}x + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{69}}) = -A^{14}x^2 + A^{14} + 2A^{10}x^4 - 4A^{10}x^2 + A^{10} - 2A^6x^4 + 4A^6x^2 - A^6 - A^3x^5 + 4A^3x^3 - 3A^3x + A^2x^4 - 2A^2x^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-5}x^3 - A^{-5}x - A^{-6} - A^{-9}x + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{73}) = A^6x^4 - 2A^6x^2 + A^6 - A^2x^4 + A^2x^2 - A^{-1}x^5 + 4A^{-1}x^3 - 3A^{-1}x + A^{-2}x^4 - 2A^{-2}x^2 - A^{-6}x^4 + 3A^{-6}x^2 - A^{-6} + A^{-9}x^3 - A^{-9}x - A^{-10} - A^{-13}x + A^{-14}$

$\mathcal{S}_{2,\infty}(\overline{5_{73}}) = A^2x^4 - 2A^2x^2 - A^{-2}x^4 + A^{-2}x^2 - A^{-5}x^5 + 4A^{-5}x^3 - 3A^{-5}x + A^{-6}x^4 - 2A^{-6}x^2 + A^{-6} - A^{-10}x^4 + 3A^{-10}x^2 - A^{-10} + A^{-13}x^3 - A^{-13}x - A^{-14} - A^{-17}x + A^{-18}$

$\mathcal{S}_{2,\infty}(5_{84}) = A^{-2}x^4 - 2A^{-2}x^2 + A^{-2} - 2A^{-6}x^4 + 3A^{-6}x^2 - A^{-9}x^5 + 4A^{-9}x^3 - 3A^{-9}x + 2A^{-10}x^4 - 3A^{-10}x^2 - A^{-14}x^4 + 2A^{-14}x^2 - A^{-14} + A^{-17}x^3 - A^{-17}x - A^{-18} - A^{-21}x + A^{-22}$

$\mathcal{S}_{2,\infty}(\overline{5_{84}}) = -A^{14}x^2 + 2A^{10}x^4 - 3A^{10}x^2 - 2A^6x^4 + 3A^6x^2 - A^3x^5 + 4A^3x^3 - 3A^3x + A^2x^4 - 2A^2x^2 + A^2 - A^{-2}x^4 + 3A^{-2}x^2 - A^{-2} + A^{-5}x^3 - A^{-5}x - A^{-6} - A^{-9}x + A^{-10}$

$\mathcal{S}_{2,\infty}(5_{85}) = -A^{-13}x^5 + 4A^{-13}x^3 - 3A^{-13}x + A^{-14}x^4 - 3A^{-14}x^2 + A^{-14} - A^{-18}x^4 + 3A^{-18}x^2 - A^{-18} + A^{-21}x^3 - A^{-21}x - A^{-22} - A^{-25}x + A^{-26}$

$\mathcal{S}_{2,\infty}(\overline{5_{85}}) = A^{14}x^4 - 3A^{14}x^2 + A^{14} - A^7x^5 + 4A^7x^3 - 3A^7x - A^2x^4 + 3A^2x^2 - A^2 + A^{-1}x^3 - A^{-1}x - A^{-2} - A^{-5}x + A^{-6}$

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# Daljši povzetek

## Uvod in osnovne definicije

Kljub temu, da je teorija vozlov precej razvita veda, se je v zadnjih desetletjih v okviru te teorije porodilo kar nekaj novosti. V poznih 90. letih je M. Khovanov z nizom predavanj uveljavil tako imenovano homologijo Hovanova, ki kategorificira Jonesov polinom in je močna invarianta spletov v $\mathbb{R}^3$ [32]. Izredno moč te invariante kaže dejstvo, da homologija Hovanova zazna nevozel in poda pozitiven odgovor na Milnorjevo domnevo. Podobno sta leta 2002 P. Ozsváth in Z. Szabó kategorificirala Alexandrov polinom s tako imenovano vozelno Floerovo homologijo.

Sistematičen študij vozlov, ki so vloženi v kakšen drug 3-dimenzionalni prostor, kot sta $\mathbb{R}^3$ ali $S^3$, se je pričel nekje proti koncu 20. stoletja, predvsem s teorijo premenjalnih modulov, ki sta jih leta 1987 uvedla J. H. Przytycki in V. G. Turaev [55, 42]. Pomemben korak je naredila tudi Ju. V. Drobotukhina, ki je leta 1991 klasificirala splete v projektivnem prostoru [17]. Od tedaj naprej zajeten del sodobne teorije vozlov posveča pozornost študiju vozlov v poljubnih 3-mnogoterostih.

Za vse nadaljne konstrukcije predpostavljamo, da se nahajamo v gladki kategoriji topoloških mnogoterosti – s tem se bomo izognili nevšečnostim, kot so na primer divji vozli.

Običajno je splet vložitev disjunkte unije $n$ krožnic v $S^3$ (ali $\mathbb{R}^3$). Pri $n = 1$ takšni vložitvi pravimo vozel. Z besedo "vozel" bomo včasih označili tudi sliko vložitve. Dva vozla (ali spleta) sta ekvivalentna, če med njima obstaja ambientna izotopija.

Slika 1: Diagram vozla.

Če vozel projiciramo na ravnino, kjer označimo tudi pod- in nadvoze, taki projekciji pravimo diagram vozla (slika 1). Zahtevamo, da je projekcija regularna, to je, zadostuje naslednjim pogojem (glej sliko 2):

- tri točke se nikoli ne projicirajo v eno samo točko,
- vse dvojne točke se sekajo transverzalno,
- ni vogalov.

(a) trojna točka (b) ni transverzalno (c) vogal

Slika 2: Nedovoljene projekcije v regularnem diagramu.

Regularnost lahko vedno dosežemo z majhno izotopijo vložitve.

Dvojnim točkam projekcije pravimo križišča.

Diagrame vozlov spremljajo trije Reidemeistovi premiki: R-I, R-II in R-III. Vsak od teh premikov predstavlja majhno lokalno transformacijo diagrama, kot je prikazano na sliki 3.

(a) R-I (b) R-II (c) R-III

Slika 3: Reidemeistrovi premiki.

**Izrek** (Reidemeister [49]). *Dva diagrama predstavljata ekvivalentna vozla natanko takrat, kadar obstaja končno zaporedje Reidemeistrovih premikov, ki pripeljejo prvi splet v drugega.*

Vozel, ki dopušča diagram brez križišč imenujemo nevozel oziroma trivialen vozel, označimo ga z $O$. Če vozel usmerimo, takemu vozli pravimo orientiran vozel. Križišču orientiranega vozla lahko priredimo število $+1$ ali $-1$ glede na sliko 4, temu številu pravimo predznak križišča. Vsoto vseh predznakov križišč diagrama $D$ imenujemo zvitost $wr(D)$ diagrama $D$.

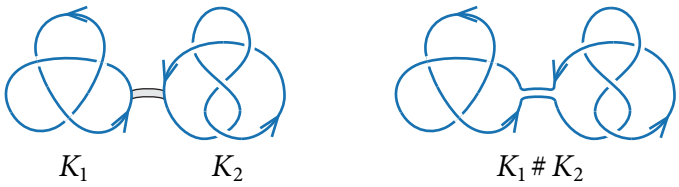
(a) pozitivno križišče (b) negativno križišče

Slika 4: Predznak križišča.

Povezana vsota orientiranih vozlov $K_1$ in $K_2$ je vozel $K = K_1 \# K_2$, ki nastane tako, da vsakemu izmed vozlov $K_1$ in $K_2$ odstranimo majhen lok ter ju povežemo z robom traku na tak način, da se nova orientacija ujema z orientacijama prvotnih vozlov (slika 5). Če vozel $K$ ni mogoče zapisati kot povezano vsoto dveh netrivialnih vozlov, takemu vozlu pravimo pravozel. V kolikor vozla nista orientirana, operacija povezane vsote ni dobro definirana – v splošnem obstajata dva ekvivalenčna razreda povezanih vsot (glede na relativno izbiro orientecije, ki jo izberemo, ko želimo vozla povezati).

Če namesto krožnice vložimo kolobar $S^1 \times I$, taki vložitvi pravimo ogroden vozel (slika 6). Dva diagrama ogrodenih vozlov predstavljata ekvivalentna ogrodena vozla, če sta povezana s končnim zaporedjem Reidemeistrovih premikov R-II in R-III [30].

Če vozel zrcalimo čez ravnino, dobimo zrcalni vozel. Vozel je akiralen, če je ekvivalenten svoji zrcalni sliki.




Slika 5: Konstrukcija povezane vsote vozlov.

Slika 6: Ogroden vozel.

Vozlu, vloženemu v neko 3-mnogoterost, pravimo afin, če leži znotraj 3-krogle, sicer je neafin.

### Tabele vozlov

Klasifikacijo vozlov navadno podamo s tabelo pravozlov do dolečenega števila križišč, pri katerem vozle označujemo s simbolom $n_k$, kjer je $n$ najmanjše število križišč, ki jih ima lahko diagram vozla, $k$ pa je zaporedna številka takšnega vozla (glede na neko ureditev). Prvo tabelo vozlov v $S^3$ je leta 1885 konstruiral P. G. Tait. Kot smo že omenili, so bili poleg $S^3$, vozli klasificirani še v projektivnem prostoru $\mathbb{R}\mathrm{P}^3$ [17].

## Premenjalni moduli

Z naraščanjem zanimanja za študij vozlov v neevklidskih 3-mogoterostih, se je povečala tudi potreba po izdelavi novih invariant, ki v teh prostorih bolje zaznajo neekvivalentne vozle. Večina klasičnih invariant, če so že dobro definirane v teh prostorih, je namreč zelo šibkih. Med glavnimi invariantami, ki bolje zaznajo vozle v 3-mnogoterostih, spadajo premenjalni moduli. Na žalost je narava premenjalnih modulov takšna, da je modul sicer lahko definirati, izračunati pa ga je potrebno za vsako mnogoterost posebej.

Naj bo $M$ 3-mnogoterost in naj bo $R$ komutativen kolobar z enoto. Naj bo $\mathcal{L}(M)$ množica izotopnih razredov spletov v $M$ (pri tej definiciji lahko vzamemo tudi orientirane in/ali ogrodene splete) in naj bo $R\mathcal{L}(M)$ prosti $R$-modul generiran z $\mathcal{L}(M)$. Naj bo $\mathcal{S}(M;\mathcal{R})$ podmodul $R\mathcal{L}(M)$ generiran z množico $\mathcal{R}$ končnih izrazov oblike

$$r_0L_0 + r_1L_1 + \cdots + rL_\infty,$$

kjer $r, r_0, r_1, r_2, \ldots \in R$ in so $L_0, L_1, L_2 \ldots$ razredi spletov, ki so identični, razen v neki majhni okolici, kot prikazuje slika 7.

Premenjalni modul $\mathcal{S}(M;R,\mathcal{R})$ je $R\mathcal{L}(M)$ modulo $\mathcal{S}(M;\mathcal{R})$ [44]:

$$\mathcal{S}(M;R,\mathcal{R}) = R\mathcal{L}(M)/\mathcal{S}(M,\mathcal{R}).$$

(a) $L_0$ (b) $L_1$ (c) $L_2$ (d) $L_3$ (e) $L_\infty$

Slika 7: Premenjalna množica.

V nadaljevanju bomo predpostavili, da je kolobar $R$ fiksen in ga bomo izpuščali v označbah.

Eden izmed najbolj študiranih premenjalnih modulov je Kauffmanov oklepajski premenjalni modul [39].

## Kauffmanov oklepajski premenjalni modul

Naj bo $R$ komutativen kolobar z enoto in naj bo $A \in R$ obrnljiv element v $R$. Naj bo $\mathcal{L}_{\mathrm{fr}}(M)$ množica ekvivalenčnih razredov ogrodenih spletov v $M$, v katero dodamo tudi prazen splet $[\varnothing]$. Naj bo $\mathcal{S}_{\mathrm{fr}}(M)$ podmodul modula $R\mathcal{L}_{\mathrm{fr}}(M)$, generan z izrazi oblike

$$L \sqcup \bigcirc - (-A^2 - A^{-2})L.$$

Kauffmanov oklepajski premenjalni modul $\mathcal{S}_{2,\infty}(M)$ je $R\mathcal{L}_{\mathrm{fr}}(M)$ modulo zgornje relacije:

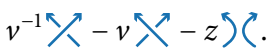
$$\mathcal{S}_{2,\infty}(M) = R\mathcal{L}_{\mathrm{fr}}(M)/\mathcal{S}_{\mathrm{fr}}(M).$$

**Izrek** ([55]). *Kauffmanov oklepajski premenjalni modul polnega torus $T$ je prosto generiran z bazo $\{x^n\}_{n=0}^{\infty}$, kjer je $x^n$ splet sestavljen iz $n$ vzporednih longitud torusa $T$.*

**Izrek** ([24]). *Kauffmanov oklepajski premenjalni modul lečastega prostora $L(p,q)$ je prosto generiran z bazo $\{x^n\}_{n=0}^{\lfloor p/2 \rfloor}$, kjer je $x^n$ splet sestavljen iz $n$ vzporednih longitud torusa $T$, ki je standardno vložen v $L(p,q)$ (podrobnejši opis te vložitve sledi v nadaljevanju).*

## HOMFLYPT premenjalni modul

Naj kolobar $R$ vsebuje dva obrnljiva elementa $v, z \in R$ in naj bo $\mathcal{L}_{\mathrm{or}}(M)$ množica ekvivalenčnih razredov orientiranih spletov v $M$, skupaj s praznim spletom $[\varnothing]$. Naj bo $R\mathcal{L}_{\mathrm{or}}(M)$ prost $R$-modul generiran z $\mathcal{L}_{\mathrm{or}}(M)$. Vzemimo podmodul $\mathcal{S}_{\mathrm{or}}(M)$ modula $R\mathcal{L}_{\mathrm{or}}(M)$ generanega z izrazi oblike



Temu dodamo tudi izraz, ki vključuje prazen splet:

$$v^{-1}\varnothing - v\varnothing - zO.$$

HOMFLYPT premenjalni modul $\mathcal{S}_3(M)$ prostora $M$ je $R\mathcal{L}_{\mathrm{or}}(M)$ modulo zgornje relacije:

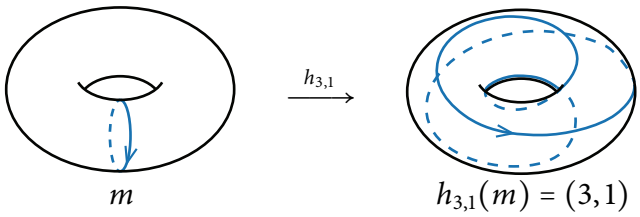
$$\mathcal{S}_3(M) = R\mathcal{L}_{\mathrm{or}}(M)/\mathcal{S}_{\mathrm{or}}(M).$$

## Diagrami spletov v $L(p,q)$

Naj bo $T = S^1 \times D$ poln torus in naj bo $x_0 \in \partial T$ točka na robu torusa. Naj bosta $l$ in $m$ orientirani longituda oziroma meridiana $T$ in naj bosta $p, q \in \mathbb{N}$ taki tuji si naravni števili, da velja $0 < q < p$. Krožnico vloženo v $\partial T$, ki ima homotopski tip $p[l] + q[m] \in \pi_1(T, x_0)$, imenujemo $(p,q)$-krivulja.

Naj bosta $T_1$ and $T_2$ polna torusa in naj bo $h_{p,q} : \partial T_1 \to \partial T_2$ homeomorfizem, ki meridiano $m \subset \partial T_1$ preslika v $(p,q)$-krivuljo na $\partial T_2$, kot je prikazano na sliki 8. Tri-dimenzionalen lečast prostor $L(p,q)$ je zlepek torusov po robu preko $h_{p,q}$:

$$L(p,q) = T_1 \cup_{h_{p,q}} T_2.$$




Slika 8: Robni homeomorfizem $h_{3,1} : \partial T_1 \to \partial T_2$.

**Izrek** (Reidemeister, 1935 [48]). *Lečasta prostora $L(p,q)$ in $L(p',q')$ sta homeomorfna natanko takrat, kadar velja $p = p'$ in $q \equiv \pm q'^{\pm 1} \pmod{p}$.*

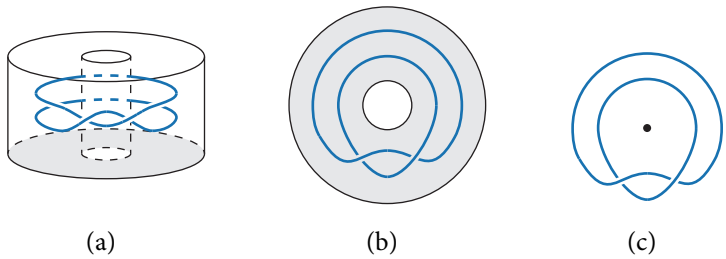
V nadaljevanju bomo predstavili dvoje vrst diagramov vozlov v polnem torusu.

Ker lahko vsak vozel $K \in L(p,q)$ z izotopijo premaknemo v prvo komponento $T_1$ zlepka $T_1 \cup_{h_{p,q}} T_2 = L(p,q)$, diagram vozla v torusu sovpada z diagramom vozla v lečastem prostoru. Razlika nastopi z dodatnim Reidemeistrovim premikom, ki se ustvari z lepljenjem drugega torusa v zlepku lečastega prostora. Prvo vrsto diagrama (diagram na preluknjenem disku) bomo uporabili pri klasifikaciji vozlov v $L(p,q)$, z uporabo drugega diagrama (puščični diagram) pa bomo izračunali HOMFLYPT premenjalni modul za $L(p,1)$.

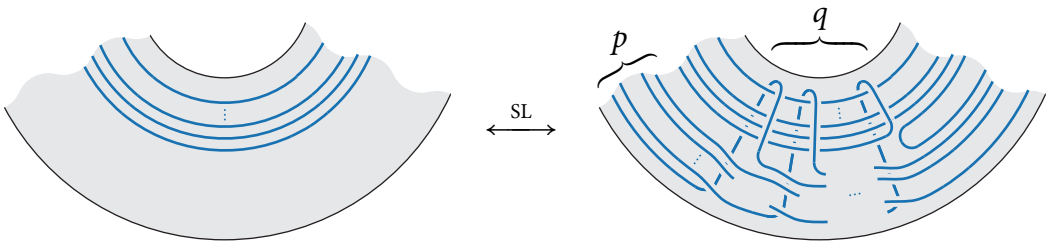
## Diagram vozla na preluknjenem disku

Naj bo $K$ vozel vložen v poln torus $T = A \times I$, kjer je $A$ kolobar (slika 9(a)). Diagram vozla $K$ je regularna projekcija $K$ na $A$, kjer ohranimo infomacijo o pod- in nadvozih(slika 9(b)). Diagram poenostavimo tako, da ne rišemo robnih komponent $\partial A$, temveč s piko označimo območje diagrama, ki vsebuje notranjo komponento roba $\partial T$, neomejeno območje diagrama pa sovpada z območjem, ki vsebuje zunanjo komponento $\partial T$ (slika 9(c)).

Za torus so Reidemeistrovi premiki enaki kot pri klasičnih diagramih, le da premikov ne smemo narediti čez območje, ki vsebuje piko. V kolikor predstavljamo diagram vozla v $L(p,q)$, imamo še dodaten Reidemestrov premik, tako imenovani drsni premik SL, ki je prikazan na sliki 10.



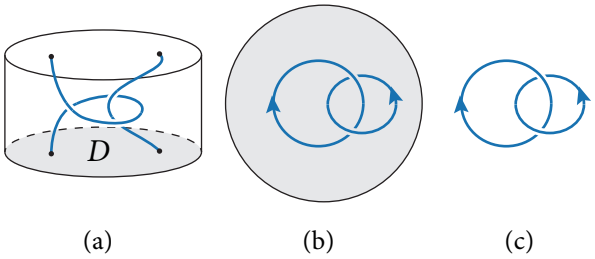

Slika 9: Diagram vozla na preluknjenem disku.




Slika 10: Drsni premik v diagramih vozlov v $L(p,q)$.

## Puščični diagram

Naj bo $K$ vozel vložen v poln torus $T$. Če $T$ prerežemo vzdolž meridianskega diska $D$, dobimo cilinder $D \times I$. Puščični diagram vozla $K$ je projekcija vozla na disk $D \times \{0\}$, pri čemer ohranimo informacijo o pod- in nadvozih. Na mestih kjer $K$ seka $M$, v diagramu narišemo puščice, ki kažejo k $D \times \{0\}$ in stran od $D \times \{1\}$. Zahtevamo tudi, da v bližini puščic ni dvojnih točk (v kolikor so, to popravimo z izotopijo vložitve) in da rob diska $D$, na katerega projiciramo, leži v neomejenem območju diagrama $D$ (slika 11).




Slika 11: Konstrukcija puščičnega diagrama vozla v polnem torusu.

Po dogovoru označimo $n$ zaporednih puščic na loku z eno samo puščico s pripadajočim naravnim številom $n$. Pri $n < 0$ je smer puščic obrnjena.

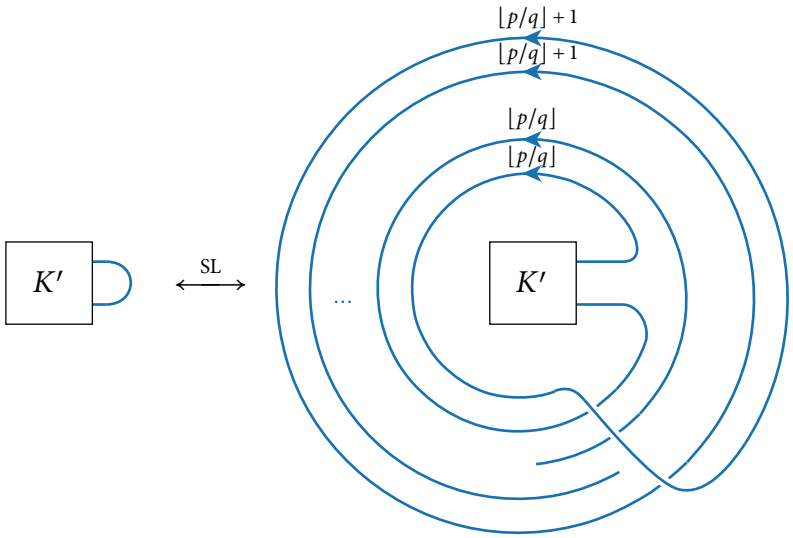
Poleg klasičnih Reidemeistovih premikov, imamo na puščičnem diagramu polnega torusa še dva dodatna premika R-IV in R-V. Gre za premika, ki vključujeta puščici, torej delujeta preko meridianskega diska $D$ (slika 12).

Podobno kot v primeru diagrama na preluknjenem disku, je diagram spleta v lečastem prostoru $L(p,q)$ enak diagramu spleta v torusu, le da imamo še dodaten drsni premik SL, ki se v

(a) R-IV (b) R-V

Slika 12: Dodatna Reidemeistrova premika v puščičnem diagramu.

puščičnem diagramu izraža kot premik prikazan na sliki 13.

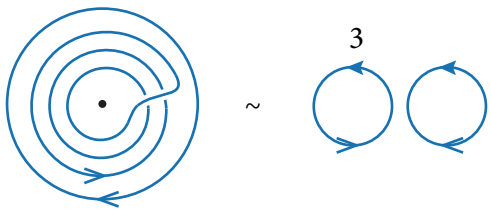



Slika 13: Drsni premik v puščičnem diagramu.

## HOMFLYPT premenjalni modul $L(p,1)$

V [23, 55] je izračunan HOMFLYPT premenjalni modul $\mathcal{S}_3(T)$ polnega torusa $T$.

**Izrek.** *$\mathcal{S}_3(T)$ je prosto generiran z elementi $\mathcal{B} = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, i_1, \dots, i_s \in \mathbb{N}\} \cup \{\varnothing\}$, kjer je $\varnothing$ prazen vozel. Element $t_k$, $k > 0$, je orientiran vozel, ki predstavlja $k$ v $\pi_1(T) \cong \mathbb{Z}$. Za $k < 0$, je $t_k$ enak $t_{|k|}$ z obrnjeno orientacijo (glej zgled na sliki 14).*




Slika 14: Generator $t_3 t_{-1}$ v $\mathcal{S}_3(T)$ (preluknjeni disk levo in puščični diagram desno).

Naj bo $\mathcal{B}_p = \{t_{k_1}^{i_1} \dots t_{k_s}^{i_s} : s \in \mathbb{N}, k_1 < \dots < k_s \in \mathbb{Z} \setminus \{0\}, -\frac{p}{2} < k_1 < \dots < k_s \leq \frac{p}{2}, i_1, \dots, i_s \in \mathbb{N}\} \cup \{\varnothing\}$ in naj bo $\mathcal{D}$ množica puščičnih diagramov spletov v $L(p,1)$.

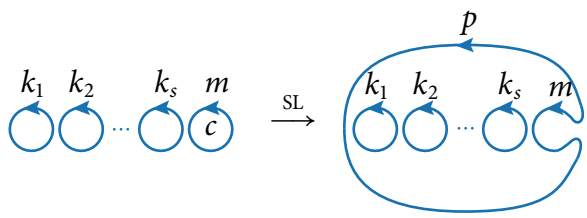
V disertaciji skonstruiramo preslikavo $H : \mathcal{D} \to R\mathcal{B}_p$, za katero pokažemo, da inducira izomorfizem

$$H_* : \mathcal{S}_3(L(p,1)) \longrightarrow R\mathcal{B}_p.$$

Naj bo $g$ standarden diagram spleta $t_{k_1}^{i_1} \dots t_{k_s}^{i_s} \in \mathcal{B}$ (kot na sliki 14) in naj bo $m = \max\{|k_i|\}_i$. Naj bo $H_T(g')$ izraz v standardni bazi $\mathcal{S}_3(T)$, ki pripada spletu diagrama $g'$. Tedaj je preslikava $H : \mathcal{D} \to R\mathcal{B}_p$ definirana rekurzivno s formulo:

$$H(g) = \begin{cases} g, & g \in \mathcal{B}_p, \\ H(H_T(g')), & \text{sicer}, \end{cases}$$

kjer $g'$ označuje diagram, ki ga dobimo, če naredimo drsni premik SL na komponenti $t_{-m}$ diagrama $g$, oziroma na komponenti $t_m$, če komponenta $t_{-m}$ ne obstaja (slika 15).




(a) $D$ (b) $D'$

Slika 15: Standardni drsni premik.

Daljši račun pokaže, da ima $H(H_T(g'))$ manj puščic, ali pa enako število puščic in manj slabih puščic kot $g$ (puščica je slaba, če kaže v nasprotno smer orientacije). Torej se rekurzivni postopek konča.

**Izrek.** *Preslikava $H$ inducira izomorfizem $H^* : \mathcal{S}_3(L(p,1)) \to R\mathcal{B}_p$. Torej je $\mathcal{S}_3(L(p,1))$ prost $R$-modul nad $\mathcal{B}_p$.*

Ideja dokaza je sledeča. Preslikava $H_T$ je invariantna glede na Reidemeistrove premike, razen za drsni premik SL, torej je tudi $H$ invariantna za te premike. Preslikava $H_T$ spoštuje HOMFLYPT relacije, torej jih tudi $H$. Dokaz, da je $H$ invariantna glede na vse drsne premike oz. je definicija neodvisna od izbire komponente nad katerim izvedeno drsni premik, poteka z z dvojno indukcijo na $n$ in $l$, kjer je $n$ število puščic diagrama, $l$ pa število puščic diagrama izven komponente na kateri premik izvedemo.

## Kategorifikacija Kauffmanovega oklepajskega premenjalnega modula

Homologija Hovanova je invarianta spletov v $S^3$. Izračunamo jo tako, da spletu kombinatorično priredimo dvojno stopničast (Hovanov) verižni kompleks, homologija tega verižnega kompleksa je dvojno stopničasta grupa (poleg običajnega homološkega stopničenja imamo še dodatno višinsko stopničenje). Izkaže se, da je ta homologija invarianta spleta, ki kategorificira Jonesov polinom v smislu, da je stopničasta Eulerjeva karakteristika enaka Jonesovemu

polinomu. Omeniti velja še, da vsebuje homologija Hovanova več informacij o spletu, kot jo vsebuje Jonesov polinom [4, 32].

Do sedaj je nastalo več posplošitev homologije Hovanova. Asaeda, Przytycki and Sikora so posplošili konstrukcijo homologije Hovanova tako, da so kategorificirali Kauffmanov oklepajski premenjalni modul $I$-svežnjev nad ploskvami [3]. Njihova konstrukicja deluje za vse ploskve, razen za projektivno ravnino $\mathbb{R}\mathrm{P}^2$. Pri tej ploskvi nastopi problem tako imenovane $1 \to 1$ bifurkacije.

Nekoliko drugače je homologijo Hovanova posplošil Manturov, ki je uspel kategorificirati Jonesov polinom virtualnih spletov [37]. Ta posplošitev zajema tudi homologijo Hovanova Jonesovega polinoma spletov v $\mathbb{R}\mathrm{P}^3$.

V disertaciji razrešimo odprti problem v [3]: pokažemo kako je mogoče kategorificirati Kauffmanov oklepajski premenjalni modul projektivnega prostora $L(2,1) = \mathbb{R}\mathrm{P}^3$ ($\mathbb{R}\mathrm{P}^2 \widetilde{\times} I \approx \mathbb{R}\mathrm{P}^3 \setminus \{*\}$). Z razrešitvijo tega problema pridelamo tudi močno invarianto spletov v projektivnem prostoru.

Naj bo $D$ diagram spleta $L$. Posamezno križišče diagrama $D$ spleta $L$ lahko zgladimo na način tipa 0 ali na način tipa 1, kot prikazuje slika 16. Naj bo $\mathcal{X}$ množica križišč v $D$, tedaj vsak element diskretne kocke $\{0,1\}^{\mathcal{X}}$ predstavlja neko popolno zladitev diagrama $D$. Elementu $s \in \{0,1\}^{\mathcal{X}}$ pravimo stanje diagrama $D$, z besedo stanje označimo tudi popolno zgladitev, ki temu stanju pripada.

(a) zgladitev tipa 0 (b) zgladitev tip 1

Slika 16: Dvoje vrst zgladitev.

Konstrukcija verižnega kompleksa poteka podobno kot v [3]: vsaki krožnici $c$ v nekem stanju $s$ diagrama $D$ spleta $L$ priredimo dvojno stopničasto grupo $V = \langle 1, X\rangle$ ali pa $\overline{V} = \langle \overline{1}, \overline{X}\rangle$, kjer so 1, $X$, $\overline{1}$ in $\overline{X}$ elementi s stopnjami $\deg 1 = (-2,0)$, $\deg X = (2,0)$, $\deg \overline{1} = (0,1)$ in $\deg \overline{X} = (0,-1)$. Natančneje, če je krožnica afina, ji priredimo $V$, če je neafina, pa ji priredimo $\overline{V}$. Nato stanju $s$ priredimo stopničasto grupo, kjer zmnožimo vse grupe krožnic stanja $s$ preko vnanjega produkta (tu nastopi razlika s homologijo definirano v [3], kjer se uporabi tenzorski produkt):

$$C_s = \begin{cases} (V_1 \wedge V_2 \wedge \cdots \wedge V_n)\{\#0(s) - \#1(s)\}, & \text{če so vse krožnice v } s \text{ afine,} \\ (V_1 \wedge V_2 \wedge \cdots \wedge V_{n-1} \wedge \overline{V}_n)\{\#0(s) - \#1(s)\}, & \text{sicer,} \end{cases}$$

kjer $\{\#0(s) - \#1(s)\}$ predstavlja zamik prvega stopničenja za razliko med številom ničel v $s$ in številom enic v $s$.

Grupa $i$-verig je sedaj definirana kot naslednja direktna vsota vseh stanj diagrama $D$:

$$\mathcal{C}_i = \bigoplus_{\substack{s \in \{0,1\}^{\mathcal{X}} \\ \#0(s) - \#1(s) = i}} C_s.$$

Nekoliko več dela je z definicijo diferenciala verižnega kompleksa, saj mora biti dobro definiran tudi v primeru, ko projiciramo na neorientabilno projektivno ploskev. Diferencial definiramo kombinatorično glede na orientacije in vrstni red krožnic pri stanjih.

Pokažemo, da je

$$0 \longrightarrow \mathcal{C}_n \longrightarrow \mathcal{C}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-n+2} \longrightarrow \mathcal{C}_{-n} \longrightarrow 0$$

dobro definiran (Hovanov) verižni kompleks.

Pokažemo tudi, da je homologija tega verižnega kompleksa neodvisna od izbir orientacij in vrstnega reda krožnic ter neodvisna od izbire diagrama $D$ spleta $L$.

Izkaže se, da je stopničasta Eulerjeva karakteristika

$$\chi(\mathcal{H}) = \sum_{i,j,k} (-1)^{\frac{j-i}{2}} A^j z^k \operatorname{rank}(\mathcal{H}_i)_{j,k}$$

ob izvedbi enostavne substitucije enaka Kauffmanovemu oklepajskemu premenjalnemu modulu spleta v standardni bazi.

## Klasifikacija

Računalniški algoritem, ki smo ga napisali za klasifikacijo vozlov v polnem torusu in v lečastih prostorih, ki hkrati omogoča tudi izračun Kauffmanovega oklepajskega premenjalnega modula in HOMFLYPT premenjalnega modula, je dostopen na [10].

Prvo si oglejmo na kakšen način vozle shranimo v računalniškem programu. Vozli so v programu shranjeni v razširjeni Gaussovi kodi. Gaussova beseda dolžine $n$ je beseda na abecedi $\pm 1, \pm 2, \ldots, \pm n$, kjer se vsaka črka ponovi natanko enkrat. Gaussovo besedo, ki pripada diagramu $D$ spleta $L$, dobimo po naslednjem postopku:

1. Označimo križišča od 1 do $n$ v $D$.
2. V kolikor diagram ni orientiram, ga orientiramo in izberemo izhodiščno točko.
3. Pričnemo s prazno besedo in potujemo po diagramu iz izhodiščne točke v smeri orientacije, vsakič ko naletimo na križišče, Gaussovi besedi dodamo črko križišča $\pm k$, kjer ima črka pozitiven predznak, če gremo čez nadvoz in negativen predznak, če gremo čez podvoz.

Da lahko vozel v celoti rekonstruiramo iz Gaussove besede, moramo Gaussovi besedi dodati tudi predznake križišč. Takemu paru pravimo Gaussova koda.

Gaussova koda je nerealizabilna, če ne pripada nobenemu diagramu spleta.

V kolikor želimo z Gaussovo kodo zakodirati splet v polnem torusu, moramo kodi dodati tudi območje s piko in neomejeno območje. To storimo tako, da označimo loke od 0 do $2n-1$, kjer z 0 označimo prvi lok, ki ga srečamo po prehodu prvega križišča iz izhodiščne točke. Gaussovi kodi, ki ji dodamo tudi zaporedje lokov območja s piko in neomejenega območja, rečemo razširjena Gaussova koda.

Na množici razširjenih Gaussovih kod uvedemo relacijo linearne urejenosti, ki je leksikografska, kjer po vrsti upoštevamo:

1. dolžino Gaussove besede $n$,
2. leksikografsko ureditev $1 < -1 < 2 < -2 < \ldots < n < -n$ Gaussove besede,
3. leksikografkso ureditev $+ < -$ predznakov križišč,
4. leksikografsko ureditev $0 < 1 < 2 < \ldots < 2n-1$ lokov območja s piko,
5. leksikografsko ureditev $0 < 1 < 2 < \ldots < 2n-1$ lokov neomejenega območja.

Če neka Gaussova beseda $W$ predstavlja isti vozel, kot Gaussova koda $W'$ in velja $W' < W$, potem pravimo, da je $W'$ redukcija kode $W$. Dve Gaussovi kodi $W$ in $W'$ predstavljata isti neorientiran diagram, če med njima obstaja končno zaporedje naslednjih operacij:

- preimenovanje črk v Gaussovi kodi (ustreza preoštevilčenju križišč diagrama),
- ciklični premik Gaussove kode (ustreza drugačni izbiri začetne točke),
- obrat Gaussove kode (ustreza spremembi orientacije diagrama),

Gaussovi kodi pravimo, da je v kanonični obliki, če je ni mogoče reducirati s pomočjo zgoraj naštetih operacij.

Algoritem, ki klasificira splete do $n$ križišč v polnem torusu je sledeč:

1. Po vrsti (glede na linearno urejenost) skonstruiraj vse realizabilne razširjene Gaussove besede dolžine največ $n$.
2. Razširjene Gaussove besede razdeli na particije glede na dovolj močno invarianto (HOMFLYPT premenjalni modul).
3. Odstrani vse Gaussove besede, ki predstavljajo povezano vsoto.
4. V vsaki particiji označi prvi vozel, glede na linearno urejenost, kot pravozel.
5. V vsaki particiji poskušaj reducirati vse Gaussove besede, ki niso označene kot pravozel.

Prvo točko optimiziramo tako, da odstranimo diagrame, ki niso v svoji kanonični obliki. V kolikor smo izbrali dovolj močno invarianto v 2. točki, mora algoritem vrniti seznam pravozlov do $n$ križišč.

Število Gaussovih kod za $n \leq 7$ je prikazano v tabeli 1. Prva vrstica je podana s formulo $N = (2n)! \cdot 2^n$, ostale vrstice so določene eksperimentalno.

Ker lahko diagram vozla v lečastem prostoru $L(p,q)$ predstavimo kot diagram vozla v torusu, je algoritem za klasifikacijo vozlov v $L(p,q)$ enak kot pri torusu, le da moramo pri redukciji upoštevati tudi dodaten drsni premik SL. V algoritmu naredimo še to spremembo, da v prvi točki algoritma Gaussovih kod ne generiramo, ampak vzamemo kar (urejeno) množico pravozlov v polnem torusu.

Pri obeh algoritmih je potrebno veliko pozornosti posvetiti optimizaciji zadnje točke (iskanje redukcij), kjer sistematično izvajamo Reidemeistrove premike. Pomagamo si z iskanjem Rei-

Tabela 1: Število Gaussovih kod do dolžine 7.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Gaussove kode | 4 | 96 | 5760 | 645120 | $1.16 \cdot 10^8$ | $3.07 \cdot 10^{10}$ | $1.12 \cdot 10^{13}$ |
| Realizabilne | 4 | 24 | 432 | 13344 | $7.06 \cdot 10^5$ | $5.18 \cdot 10^7$ | $4.88 \cdot 10^9$ |
| Kanonične realizabilne | 2 | 6 | 36 | 278 | $2.94 \cdot 10^3$ | $3.59 \cdot 10^4$ | $4.84 \cdot 10^5$ |
| Razširjene realizable | 36 | 384 | 10800 | 480384 | $3.46 \cdot 10^7$ | $3.31 \cdot 10^9$ | $3.95 \cdot 10^{11}$ |
| Kanon. Razš. realizable | 10 | 68 | 714 | 9392 | $1.41 \cdot 10^5$ | $2.29 \cdot 10^6$ | $3.91 \cdot 10^7$ |

demeistrovih premikov v širino z uporabo binarnih iskalnih dreves. Postopek optimiziramo tudi tako, da na vsakem koraku ne izvedemo Reidemeistrovega premika, ki bi povečal število križišč.

Tabela 2 prikazuje število neafinih pravozlov v polnem torusu do pet križišč. Pri $n = 5$ se pojavi par vozlov, ki ju nobena od zgoraj naštetih invariant ne loči. Predstavimo argumente, ki kažejo na to, da sta si vozla različna.

Tabela 2: Število neafinih pravozlov v polnem torusu.

| $n$ | Število pravozlov | |
|---|---|---|
| **0** | 1 | (0 zrcalnih parov + 1 akiralen) |
| **1** | 2 | (1 zrcalni par + 0 akiralnih) |
| **2** | 5 | (2 zrcalna para + 1 akiralen) |
| **3** | 16 | (8 zrcalnih parov + 0 akiralnih) |
| **4** | 50 | (23 zrcalnih parov + 4 akiralni) |
| **5** | 190* | (95* zrcalnih parov + 0 akiralnih) |

* Ob domnevi, da sta vozla $5_{26}$ in $5_{27}$ različna.

Tabela 3 prikazuje število neafinih pravozlov v lečastih prostorih $L(p, q)$ do pet križišč. Število pravozlov je v posameznem lečastemu prostoru manjše ali kvečjemu enako številu vozlov v torusu, saj se določeni vozli s pomočjo SL operacije reducirajo, postanejo afini ali pa postanejo povezane vsote. Omenimo še to, da v preluknjenih diagramih v lečastem prostoru nimamo dobro definirane operacije zrcaljenja, zato v tabeli upoštevamo obe zrcalni različici.

Tabela 3: Število neafinih pravozlov v lečastih prostorih $L(p,q)$ do 5 križišč.

| $n$ | $L(2,1)$ | $L(3,1)$ | $L(4,1)$ | $L(5,1)$ | $L(5,2)$ | $L(6,1)$ | $L(7,1)$ |
|---|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 4 | 5 | 4 | 4 | 5 | 5 |
| **3** | 12 | 13 | 14 | 16 | 16 | 15 | 15 |
| **4** | 40 | 42 | 45 | 46 | 49 | 49 | 49 |
| **5** | $155^{\star}$ | $163^{\star}$ | $176^{\star}$ | $183^{\star}$ | $189^{\star\diamond}$ | $184^{\star}$ | $187^{\star}$ |

| $n$ | $L(7,2)$ | $L(8,1)$ | $L(8,3)$ | $L(9,1)$ | $L(9,2)$ | $L(10,1)$ | $L(10,3)$ |
|---|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 15 | 16 | 16 | 16 | 16 | 16 | 16 |
| **4** | 50 | 49 | 50 | 49 | 49 | 50 | 50 |
| **5** | $189^{\star\diamond}$ | $188^{\star}$ | $190^{\star\diamond}$ | $189^{\star}$ | $190^{\star\diamond}$ | $189^{\star}$ | $190^{\star\diamond}$ |

| $n$ | $L(11,1)$ | $L(11,2)$ | $L(11,3)$ | $L(12,1)$ | $L(12,5)$ | $L(p,q),\ p \geq 13$ |
|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 | 1 | 1 |
| **1** | 2 | 2 | 2 | 2 | 2 | 2 |
| **2** | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 16 | 16 | 16 | 16 | 16 | 16 |
| **4** | 50 | 50 | 50 | 50 | 50 | 50 |
| **5** | $189^{\star}$ | $189^{\star\diamond}$ | $190^{\star\diamond}$ | $190^{\star}$ | $190^{\star\diamond}$ | $190^{\star\diamond}$ |

$^{\star}$ Ob domnevi, da sta para vozlov $5_{26}$, $5_{27}$ in $\overline{5_{26}}$, $\overline{5_{27}}$ različna v $L(p,q)$.
$^{\diamond}$ Ob domnevi, da so pari vozlov $4_{23}$, $\overline{5_1}$; $\overline{4_{23}}$, $5_1$ in $5_{76}$, $\overline{5_{76}}$ različni v $L(p,q)$, $q > 1$.